Last class

- Schema theorem
- Markov chain modeling
- Convergence
- Running time complexity
- Expectation and tail inequalities
- Example of running time analysis

南京大学人工智能学院
SCHOOL OF ARTIFICIAL INTELLIGENCE, NANJING UNIVERSITY

南京大學

Heuristic Search and Evolutionary Algorithms

Lecture 10: Running Time Analysis of EAs

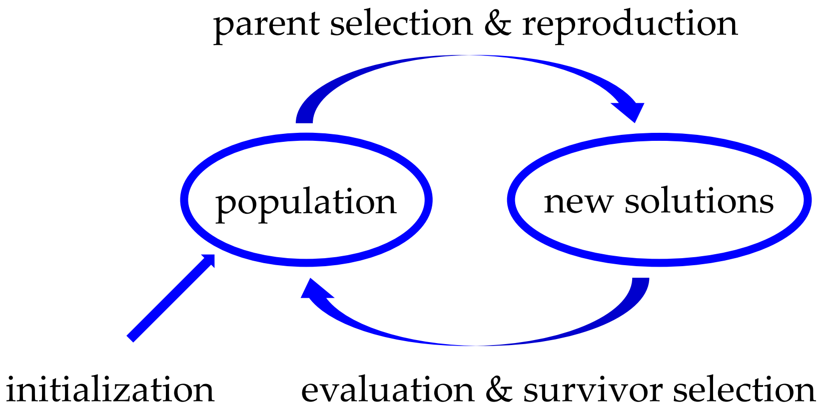
Chao Qian （钱超）

Associate Professor, Nanjing University, China

Email: qianc@nju.edu.cn
Homepage: http://www.lamda.nju.edu.cn/qianc/

Markov chain modeling

EA:

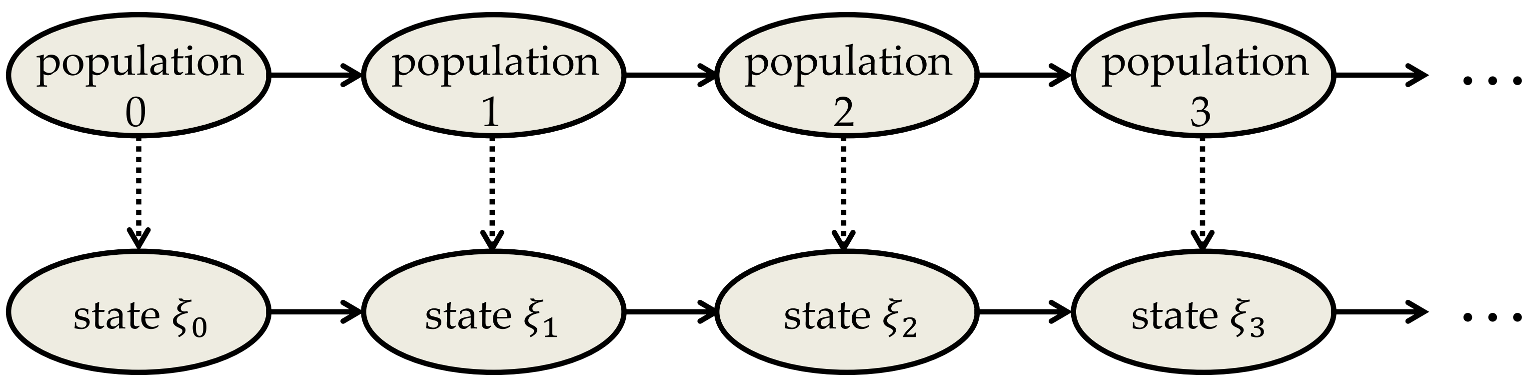


Running time analysis

$\tau = \min \{t \geq 0 \mid \xi_t \in \mathcal{X}^*\}$

a random variable

- $\mathrm{E}[\tau]$
- $P(\tau \leq T)$



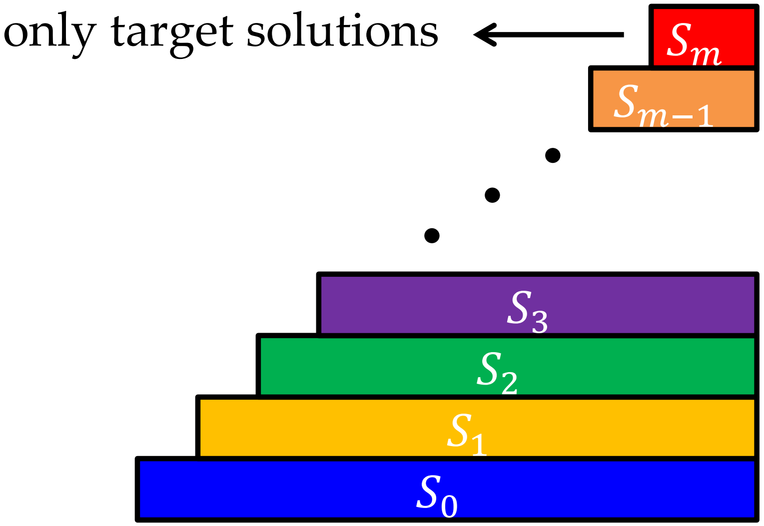
Markov chain: $P(\xi_t | \xi_{t-1}, \dots, \xi_1, \xi_0) = P(\xi_t | \xi_{t-1})$

Fitness level method

The basic idea [Droste et al., TCS'02]:

1. Divide the solution space S into $m+1$ subspaces $S_0, S_1, \dots, S_m$

- $\forall i \neq j: S_i \cap S_j = \emptyset,\ \bigcup_{i=0}^{m} S_i = S$
- $\forall i < j, x \in S_i, y \in S_j: f(x) < f(y)$



Fitness level method

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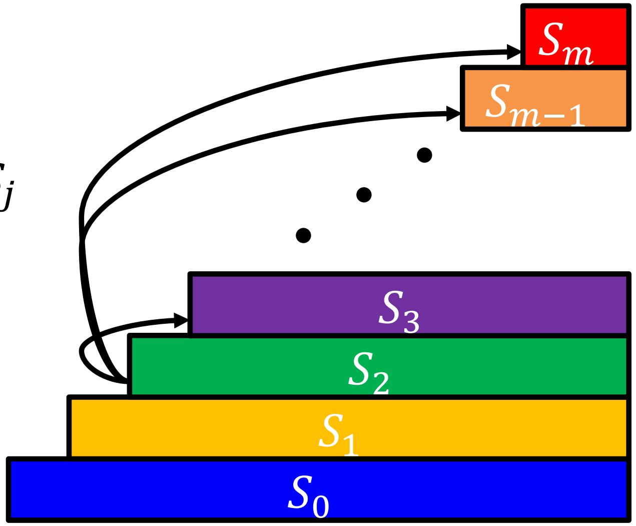
   - $\forall i < j, x \in S_i, y \in S_j: f(x) < f(y)$

2. Bounds on the probability of leaving S_i to higher S_j
   - $P(\xi_{t+1} \in \bigcup_{j=i+1}^{m} S_j | \xi_t \in S_i) \geq v_i$
   - $P(\xi_{t+1} \in \bigcup_{j=i+1}^{m} S_j | \xi_t \in S_i) \leq u_i$

The best solution in ξ_t belongs to S_i



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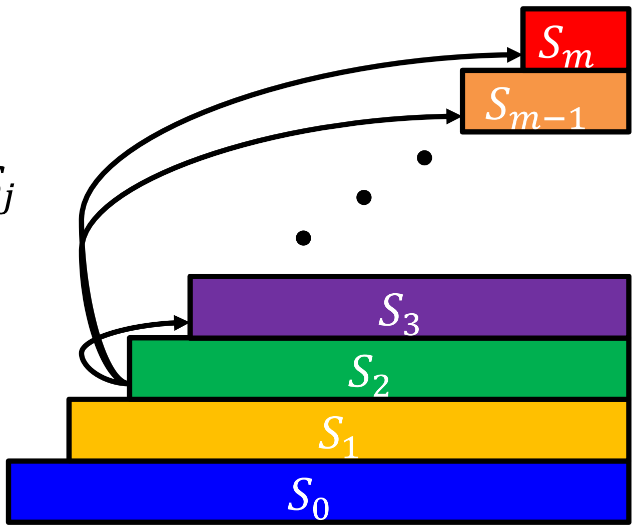

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Expected running time

Upper bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot \sum_{j=i}^{m-1} \frac{1}{v_j}$

the initial distribution

Fitness level method

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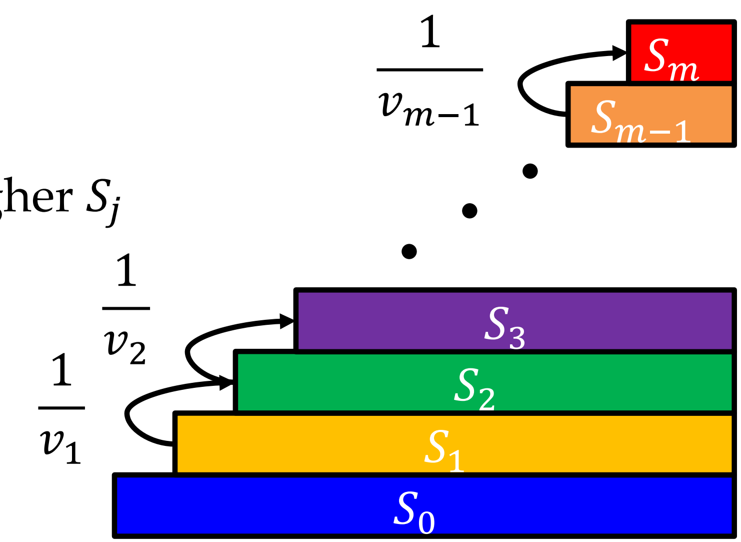

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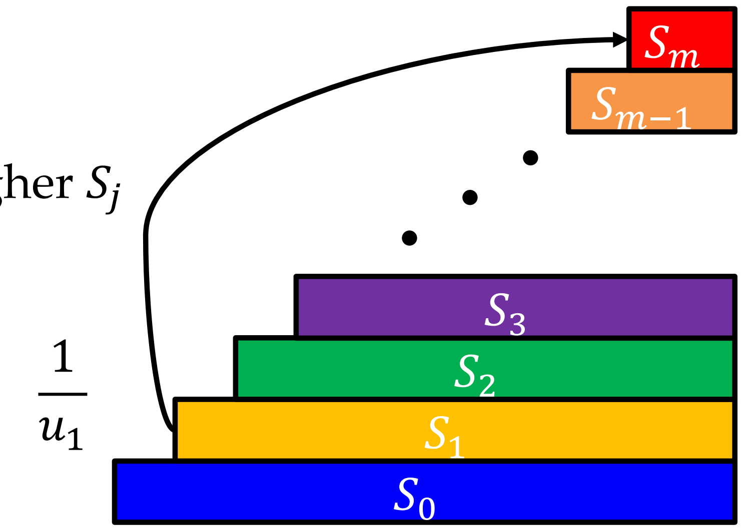


Expected running time

Upper bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot \sum_{j=i}^{m-1} \frac{1}{v_j}$

Lower bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot \frac{1}{u_i}$

Application illustration: (1+1)-EA for OneMax

(1+1)-EA:

Given a pseudo-Boolean function f:

1. $\boldsymbol{x} :=$ randomly selected from $\{0,1\}^n$.
2. Repeat until some termination criterion is met
3. $\boldsymbol{x}' :=$ flip each bit of $\boldsymbol{x}$ with probability $1/n$.
4. if $f(\boldsymbol{x}') \geq f(\boldsymbol{x})$
5. $\boldsymbol{x} = \boldsymbol{x}'$.

Bit-wise mutation

OneMax:

$$max_{\boldsymbol{x}\in\{0,1\}^n} \sum_{i=1}^{n} x_i$$

Count the number of 1-bits

Theorem. [Droste et al., TCS'02] The expected running time of the (1+1)-EA solving the OneMax problem is $O(n \log n)$.

Proof

Theorem. [Droste et al., TCS'02] The expected running time of the (1+1)-EA solving the OneMax problem is $O(n \log n)$.

Main idea:

- Divide the solution space $\{0,1\}^n$ into $S_0, S_1, \ldots, S_n$ with $S_i = \{x \in \{0,1\}^n \mid |x| = i\}$
- The probability of jumping to higher S_j from S_i is lower bounded by

$$P(\xi_{t+1} \in \cup_{j=i+1}^{m} S_j | \xi_t \in S_i) \geq \frac{n-i}{n} \cdot \left(1-\frac{1}{n}\right)^{n-1}$$

the number of 1-bits ($|x|$)

flip one of the $n-i$ 0-bits ($\frac{n-i}{n}$)

keep the other bits unchanged ($(1-\frac{1}{n})^{n-1}$)

$$v_j = \frac{n-j}{n} \cdot \left(1-\frac{1}{n}\right)^{n-1}$$

Upper bound on the expected running time:

$$\sum_{i=0}^{n-1} \pi_0(S_i) \cdot \sum_{j=i}^{n-1} \frac{1}{v_j} \leq \sum_{j=0}^{n-1} \frac{1}{v_j}$$

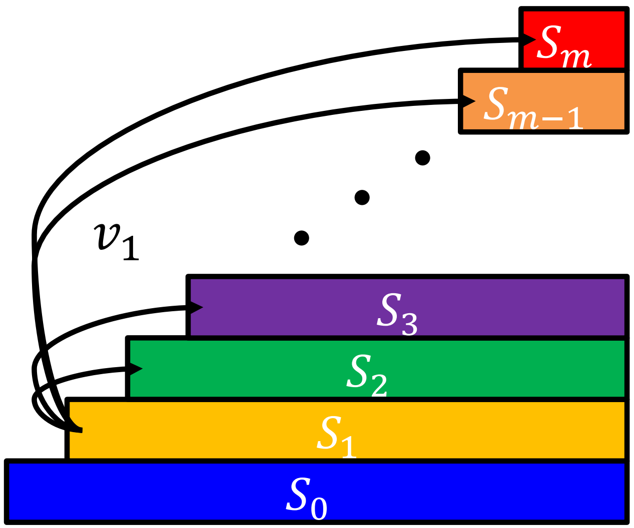
$$= \sum_{j=0}^{n-1} \frac{n}{n-j} \frac{1}{(1-\frac{1}{n})^{n-1}} \leq en \sum_{j=1}^{n} \frac{1}{j} \in O(n \log n)$$

Refined fitness level method

Original [Droste et al., TCS'02]:

$P(\xi_{t+1} \in \cup_{j=i+1}^{m} S_j | \xi_t \in S_i) \geq v_i$

Upper bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot \sum_{j=i}^{m-1} \frac{1}{v_j}$

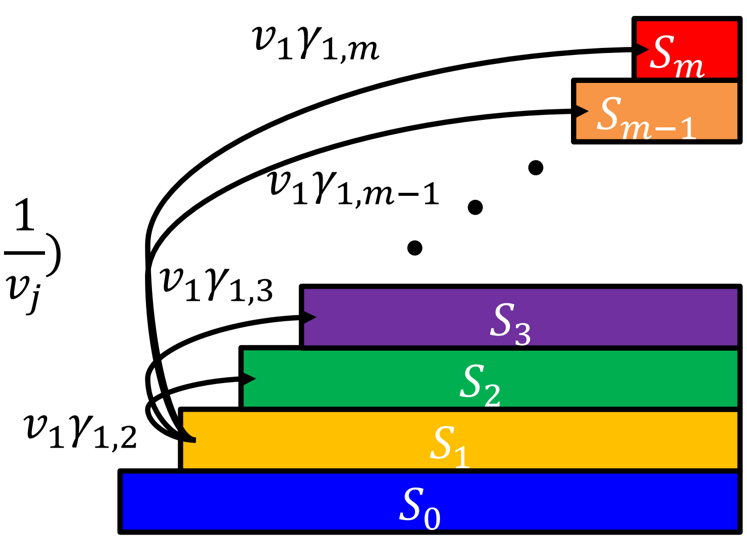


Refined [Sudholt, TEC'13]:

$P(\xi_{t+1} \in S_j | \xi_t \in S_i) \geq v_i \cdot \gamma_{i,j}$

$\sum_{j=i+1}^{m} \gamma_{i,j} = 1 \qquad \gamma_{i,j} \leq \chi \sum_{k=j}^{m} \gamma_{i,k}$

Upper bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot (\frac{1}{v_i} + \chi \sum_{j=i+1}^{m-1} \frac{1}{v_j})$



Original is a specialization of refined with $\chi = 1$

Similar for lower bound analysis

Refined fitness level method

The above two fitness level methods

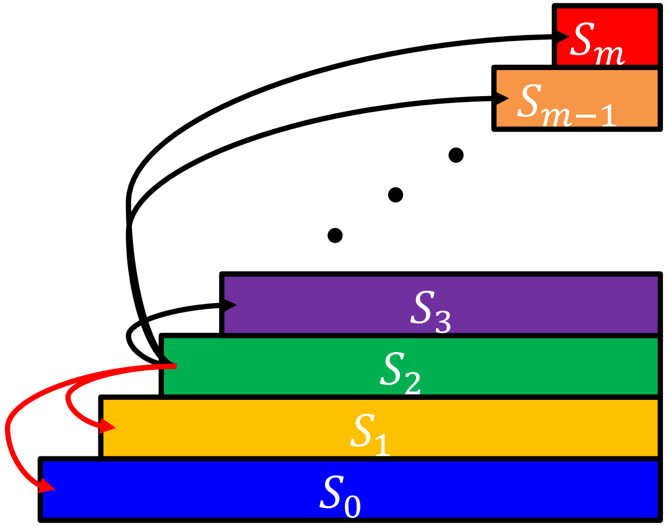
[Droste et al., TCS'02; Sudholt, TEC'13]

- Only consider jumping to higher levels
- Proposed for elitist EAs

Fitness level method for non-elitist EAs

[Dang & Lehre, Algorithmica'16]

- allow jumping to lower levels



Theorem 8 *Given a function $f : \mathcal{X} \to \mathbb{R}$, and an f-based partition $(A_1, \ldots, A_{m+1})$, let T be the number of selection-variation steps until Algorithm 1 with a selection mechanism p_{sel} obtains an element in A_{m+1} for the first time. If there exist parameters $p_0, s_1, \ldots, s_m, s_* \in (0, 1]$, and $\gamma_0 \in (0, 1)$ and $\delta > 0$, such that*

(C1) $p_{\text{mut}}\left(y \in A_j^+ \mid x \in A_j\right) \geq s_j \geq s_*$ *for all* $j \in [m]$,

(C2) $p_{\text{mut}}\left(y \in A_j \cup A_j^+ \mid x \in A_j\right) \geq p_0$ *for all* $j \in [m]$,

(C3) $\beta(\gamma, P) p_0 \geq (1+\delta)\gamma$ *for all* $P \in \mathcal{X}^\lambda$ *and* $\gamma \in (0, \gamma_0]$,

(C4) $\lambda \geq \frac{2}{a} \ln\left(\frac{16m}{ac\varepsilon s_*}\right)$ *with* $a = \frac{\delta^2 \gamma_0}{2(1+\delta)}$, $\varepsilon = \min\{\delta/2, 1/2\}$ *and* $c = \varepsilon^4/24$,

$$\Rightarrow \quad \mathbf{E}[T] \leq \frac{2}{c\varepsilon}\left(m\lambda(1 + \ln(1 + c\lambda)) + \frac{p_0}{(1+\delta)\gamma_0}\sum_{j=1}^{m}\frac{1}{s_j}\right)$$

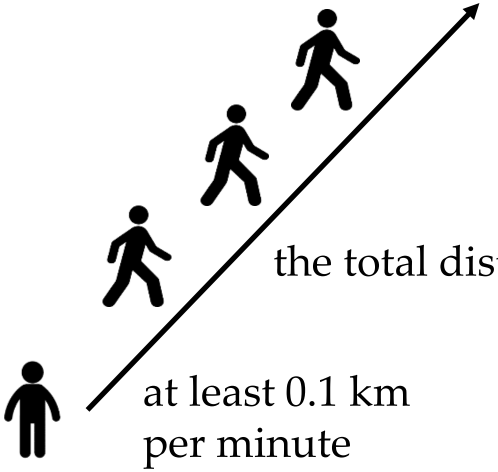
Additive drift analysis

The basic idea [He & Yao, AIJ'01]:

1. Design a distance function $V(x): \mathcal{X} \to \mathbb{R}$ to measure the distance from a state $x \in \mathcal{X}$ to the target state space $\mathcal{X}^*$
   - $\forall x \in \mathcal{X} \setminus \mathcal{X}^*: V(x) > 0$
   - $\forall x \in \mathcal{X}^*: V(x) = 0$

The hotel

2. Bounds on the expected drift in one step
   - $\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq c_l$
   - $\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \leq c_u$



the total distance 3 km

at least 0.1 km per minute

the time $\leq \frac{3}{0.1} = 30$ minutes

Expected running time

Upper bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_l}$

π_0: the initial distribution

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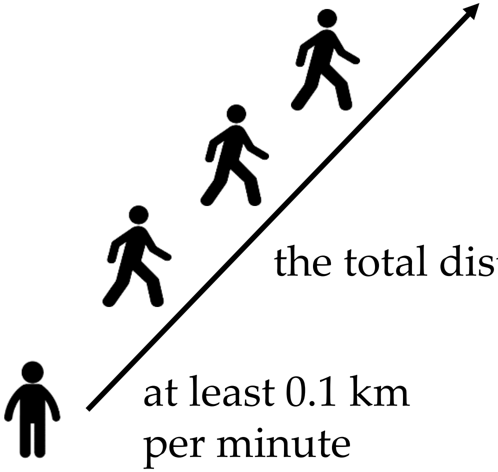
Expected running time

Upper bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_l}$

Lower bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_u}$

The hotel



the total distance 3 km

at least 0.1 km per minute

the time $\leq \frac{3}{0.1} = 30$ minutes

Application illustration: (1+1)-EA for LeadingOnes

(1+1)-EA:

Given a pseudo-Boolean function f:

1. $\boldsymbol{x} :=$ randomly selected from $\{0,1\}^n$.
2. Repeat until some termination criterion is met
3. $\quad \boldsymbol{x}' :=$ flip each bit of $\boldsymbol{x}$ with probability $1/n$.
4. $\quad$ if $f(\boldsymbol{x}') \geq f(\boldsymbol{x})$
5. $\quad\quad \boldsymbol{x} = \boldsymbol{x}'$.

LeadingOnes:

$$max_{x \in \{0,1\}^n} \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

$\rightarrow LO(\boldsymbol{x})$

$\rightarrow$ Count the number of consecutive 1-bits starting from the left

e.g., $f(11010) = 2$, $f(01111) = 0$

Theorem. [He & Yao, AIJ'01] The expected running time of the (1+1)-EA solving the LeadingOnes problem is $O(n^2)$.

Proof

Theorem. [He & Yao, AIJ'01] The expected running time of the (1+1)-EA solving the LeadingOnes problem is $O(n^2)$.

Main idea:

the number of leading 1-bits

- Design the distance function $\mathrm{V}(x) = n - LO(x)$
- The expected drift from a solution x with $LO(x) = i$:

$$\mathrm{E}[\, V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = x]$$
$$= \mathrm{E}[\, LO(\xi_{t+1}) - i \mid \xi_t = x] \geq 1 \cdot \frac{1}{n} \cdot (1 - \frac{1}{n})^i$$

keep the i leading 1-bits unchanged

$LO(x) = i \ \rightarrow \geq i + 1$

flip the first 0-bit

Proof

Theorem. [He & Yao, AIJ'01] The expected running time of the (1+1)-EA solving the LeadingOnes problem is $O(n^2)$.

Main idea:

the number of leading 1-bits

- Design the distance function $V(x) = n - LO(x)$
- The expected drift from a solution x with $LO(x) = i$ is lower bounded by

$$E[V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = x) \geq 1 \cdot \frac{1}{n} \cdot (1 - \frac{1}{n})^i$$

keep the i leading 1-bits unchanged

$LO(x) = i \rightarrow \geq i + 1$

flip the first 0-bit

$c_l = \frac{1}{en}$

Upper bound on the expected running time: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_l} \leq \frac{n}{c_l}$

$$= n/(\frac{1}{en}) = en^2 \in O(n^2)$$

Multiplicative drift analysis

Additive [He & Yao, AIJ'01]:

$\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq c_l$ → not depend on ξ_t

Upper bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_l}$

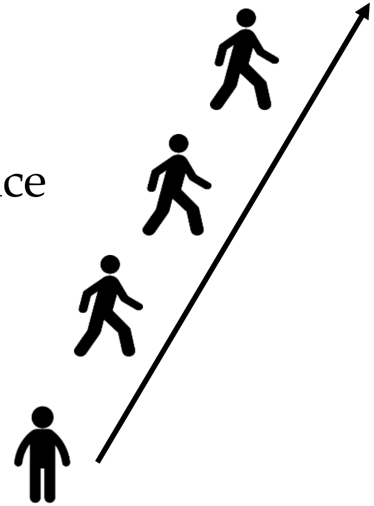
The hotel

Multiplicative [Doerr et al., Algorithmica'12]:

$\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq \delta \cdot V(\xi_t)$ → proportional to the current distance

Upper bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{1 + \ln(V(x)/V_{min})}{\delta}$

V_{min} → $\min\{V(x) \mid V(x) > 0\}$



Multiplicative vs. Additive

For additive and multiplicative drift analysis, which one is stronger?

Theorem. [Doerr et al., Algorithmica'12] Multiplicative drift analysis can be proved by using additive drift analysis. Stronger

Multiplicative drift analysis:

$$\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq \delta \cdot V(\xi_t)$$

Upper bound:

$$\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{1 + \ln\left(V(x)/V_{min}\right)}{\delta}$$

Construct a distance function $U(x)$

$$\mathrm{E}[U(\xi_t) - U(\xi_{t+1}) \mid \xi_t] \geq c_l$$

Apply additive drift analysis

c_l: not depend on ξ_t

Proof

Multiplicative drift analysis:

$$E[\, V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq \delta \cdot V(\xi_t)$$

⇨

Upper bound:

$$\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{1 + \ln\left(V(x)/V_{min}\right)}{\delta}$$

Main idea:

- Design the distance function $U(x) = \begin{cases} 0 & \text{if } V(x) = 0 \\ 1 + \ln(V(x)/V_{min}) & \text{Otherwise} \end{cases}$

- The drift from a state $\xi_t \notin \mathcal{X}^*$ (i.e., $V(\xi_t) > 0$ or $U(\xi_t) > 0$):

If $\xi_{t+1} \in \mathcal{X}^*$, then $U(\xi_t) - U(\xi_{t+1}) = 1 + \ln(V(\xi_t)/V_{min}) \geq 1 = \dfrac{V(\xi_t) - V(\xi_{t+1})}{V(\xi_t)}$

If $\xi_{t+1} \notin \mathcal{X}^*$, then $U(\xi_t) - U(\xi_{t+1}) = \ln(V(\xi_t)/V(\xi_{t+1})) \geq \dfrac{V(\xi_t) - V(\xi_{t+1})}{V(\xi_t)}$

Proof

$$\ln(V(\xi_t)/V(\xi_{t+1})) \geq \frac{V(\xi_t) - V(\xi_{t+1})}{V(\xi_t)}$$

$$\frac{u}{w} = 1 + \frac{u - w}{w} \leq e^{\frac{u-w}{w}}$$

$$\Downarrow$$

$$\ln\frac{u}{w} \leq \frac{u - w}{w}$$

$$\Downarrow$$

$$\ln\frac{w}{u} \geq \frac{w - u}{w}$$

Proof

Multiplicative drift analysis:

$$\mathrm{E}[\, V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq \ \delta \cdot V(\xi_t)$$

⇨

Upper bound:

$$\sum_{x\in\mathcal{X}} \pi_0(x) \cdot \frac{1 + \ln\,(V(x)/V_{min})}{\delta}$$

Main idea:

- Design the distance function $\mathrm{U}(x) = \begin{cases} 0 & \text{if } V(x) = 0 \\ 1 + \ln(V(x)/V_{min}) & \text{Otherwise} \end{cases}$
- The drift from a state $\xi_t \notin \mathcal{X}^*$ (i.e., $\mathrm{V}(\xi_t) > 0$ or $\mathrm{U}(\xi_t) > 0$):

$$U(\xi_t) - U(\xi_{t+1}) \geq \frac{V(\xi_t) - V(\xi_{t+1})}{V(\xi_t)}$$

- The expected drift from a state $\xi_t \notin \mathcal{X}^*$ (i.e., $\mathrm{V}(\xi_t) > 0$ or $\mathrm{U}(\xi_t) > 0$):

$$E[U(\xi_t) - U(\xi_{t+1}) \mid \xi_t] \geq \frac{E[V(\xi_t) - V(\xi_{t+1}) \mid \xi_t]}{V(\xi_t)}$$

Proof

Multiplicative drift analysis:

$$\mathrm{E}[\,V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq \ \delta \cdot V(\xi_t)$$

Upper bound:

$$\sum_{x\in\mathcal{X}} \pi_0(x) \cdot \frac{1+\ln\left(V(x)/V_{min}\right)}{\delta}$$

Main idea:

- Design the distance function $\mathrm{U}(x) = \begin{cases} 0 & \text{if } V(x) = 0 \\ 1+\ln(V(x)/V_{min}) & \text{Otherwise} \end{cases}$
- The expected drift from a state $\xi_t \notin \mathcal{X}^*$ (i.e., $\mathrm{V}(\xi_t) > 0$ or $\mathrm{U}(\xi_t) > 0$):

$$E[U(\xi_t) - U(\xi_{t+1}) \mid \xi_t] \geq \frac{E[V(\xi_t) - V(\xi_{t+1}) \mid \xi_t]}{V(\xi_t)} \geq \delta$$

Additive drift analysis [He & Yao, AIJ'01]:

$$\mathrm{E}[U(\xi_t) - U(\xi_{t+1}) \mid \xi_t] \geq c_l$$

Upper bound: $\sum_{x\in\mathcal{X}} \pi_0(x) \cdot \dfrac{U(x)}{c_l}$

Multiplicative vs. Additive

For additive and multiplicative drift analysis, which one is stronger?

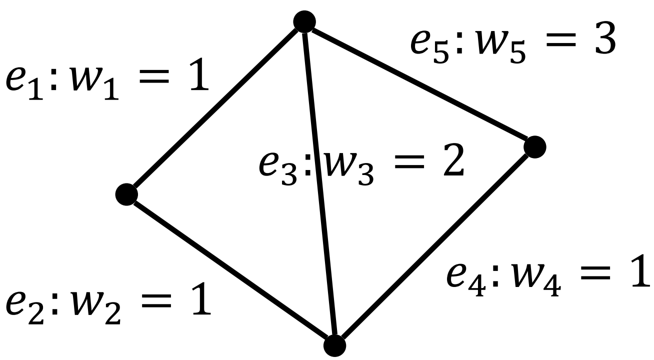
Theorem. [Doerr et al., Algorithmica'12] Multiplicative drift analysis can be proved by using additive drift analysis. → Stronger

Multiplicative drift analysis can be easier to be used when the expected drift under a natural distance function is proportional to the current distance

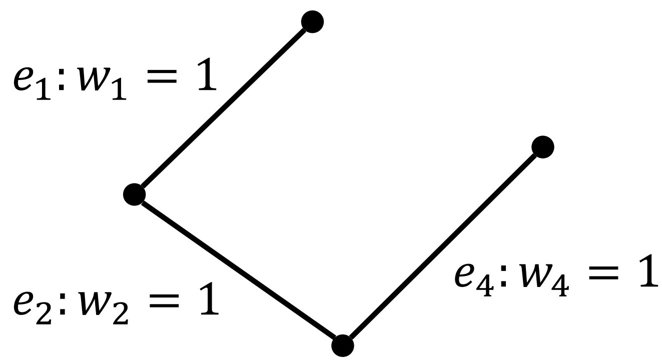
Application illustration: (1+1)-EA for MST

Minimum spanning tree (MST):

- Given: an undirected connected graph $G = (V, E)$ on n vertices and m edges with positive integer weights $w: E \to \mathbb{N}^+$
- The Goal: find a connected subgraph $E' \subseteq E$ with the minimum weight



The original graph



The minimum spanning tree

Application illustration: (1+1)-EA for MST

(1+1)-EA: Given a pseudo-Boolean function f:

1. $\boldsymbol{x}$:= randomly selected from $\{0,1\}^n$.
2. Repeat until some termination criterion is met
3. $\quad \boldsymbol{x}'$:= flip each bit of $\boldsymbol{x}$ with probability $1/n$.
4. $\quad$ if $f(\boldsymbol{x}') \geq f(\boldsymbol{x})$ → $f(\boldsymbol{x}') \leq f(\boldsymbol{x})$
5. $\quad\quad \boldsymbol{x} = \boldsymbol{x}'$.

Solution representation: $\boldsymbol{x} \in \{0,1\}^m \leftrightarrow$ a subgraph

e.g., $\{e_1, e_2, e_4\} \rightarrow 11010$ $\quad x_i = 1$ means that edge e_i is selected

Fitness function:

$$min \quad f(\boldsymbol{x}) = (c(\boldsymbol{x}) - 1) \cdot w_{ub} + \sum_{i:x_i=1} w_i$$

- $c(\boldsymbol{x})$: the number of connected components
- $\sum_{i:x_i=1} w_i$: the sum of edge weights
- w_{max}: the maximum edge weight

$w_{ub} = n^2 \cdot w_{max}$, to make a subgraph with less connected components better

Proof

Theorem. [Neumann & Wegener, TCS'07; Doerr et al., Algorithmica'12] The expected running time of the (1+1)-EA solving the MST problem is $O(m^2(\log n + \log w_{max}))$.

Main idea:

(1) obtain a connected subgraph ⟶ $c(\boldsymbol{x}) = 1$

(2) obtain a minimum spanning tree

The analysis of phase (1): $min \;\; f(\boldsymbol{x}) = (c(\boldsymbol{x}) - 1) \cdot w_{ub} + \sum_{i:x_i=1} w_i$

- $c(\boldsymbol{x})$ cannot increase
- at least $c(\boldsymbol{x}) - 1$ edges, the insertion of which can decrease $c(\boldsymbol{x})$ by 1

⟹ the probability of decreasing $c(\boldsymbol{x})$ by 1 is at least $\frac{c(\boldsymbol{x})-1}{m}(1-\frac{1}{m})^{m-1}$

flip one of those $c(x) - 1$ 0-bits   keep the other bits unchanged

Proof

Theorem. [Neumann & Wegener, TCS'07; Doerr et al., Algorithmica'12] The expected running time of the (1+1)-EA solving the MST problem is $O(m^2(\log n + \log w_{max}))$.

Main idea:

(1) obtain a connected subgraph $\longrightarrow$ $c(\boldsymbol{x}) = 1$

(2) obtain a minimum spanning tree

The analysis of phase (1): $min \ \ f(\boldsymbol{x}) = (c(\boldsymbol{x}) - 1) \cdot w_{ub} + \sum_{i:x_i=1} w_i$

- $c(\boldsymbol{x})$ cannot increase
- at least $c(\boldsymbol{x}) - 1$ edges, the insertion of which can decrease $c(\boldsymbol{x})$ by 1

the probability of decreasing $c(\boldsymbol{x})$ by 1 is at least $\frac{c(\boldsymbol{x})-1}{m}(1-\frac{1}{m})^{m-1}$

fitness level method $\Rightarrow$ The expected running time upper bound: $\sum_{c(x)=n}^{2} \frac{em}{c(x)-1} \in O(m \log n)$

Proof

The analysis of phase (2): $min \;\; f(\boldsymbol{x}) = (c(\boldsymbol{x})-1)\cdot w_{ub} + \sum_{i:x_i=1} w_i$

- it will always be connected, i.e., $c(\boldsymbol{x}) = 1$ always holds
- to analyze $f(\boldsymbol{x}) \;\rightarrow f_{opt}$ the weight of a minimum spanning tree

Using multiplicative drift analysis:

- design the distance function: $V(\boldsymbol{x}) = f(\boldsymbol{x}) - f_{opt}$
- analyze the expected drift:

$$\mathrm{E}[V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = \boldsymbol{x}] = V(\boldsymbol{x}) - \mathrm{E}[V(\xi_{t+1}) \mid \xi_t = \boldsymbol{x}] = f(\boldsymbol{x}) - \mathrm{E}[f(\xi_{t+1}) \mid \xi_t = \boldsymbol{x}]$$

$$\geq f(\boldsymbol{x}) - \left(\sum_{i=1}^{m-(n-1)} f(\boldsymbol{y}^i)\cdot\frac{1}{m}(1-\frac{1}{m})^{m-1} + \sum_{i=1}^{n} f(\boldsymbol{z}^i)\cdot\frac{1}{m^2}(1-\frac{1}{m})^{m-2} + (1-\cdots)f(\boldsymbol{x})\right)$$

there exists a set of $m-(n-1)$ 1-bit flips and a set of n 2-bit flips such that the average weight decrease is at least $(f(\boldsymbol{x}) - f_{opt})/(m+1)$

Proof

- to analyze $f(\boldsymbol{x}) \rightarrow f_{opt}$ the weight of a minimum spanning tree

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$$\geq f(\boldsymbol{x}) - \left(\sum_{i=1}^{m-(n-1)} f(\boldsymbol{y}^{\boldsymbol{i}}) \cdot \frac{1}{m}(1-\frac{1}{m})^{m-1} + \sum_{i=1}^{n} f(\boldsymbol{z}^{\boldsymbol{i}}) \cdot \frac{1}{m^2}(1-\frac{1}{m})^{m-2} + (1-\cdots)f(\boldsymbol{x})\right)$$

there exists a set of $m-(n-1)$ 1-bit flips and a set of n 2-bit flips such that the average weight decrease is at least $(f(\boldsymbol{x}) - f_{opt})/(m+1)$

$$= \sum_{i=1}^{m-(n-1)} \left(f(\boldsymbol{x}) - f(\boldsymbol{y}^{\boldsymbol{i}})\right) \cdot \frac{1}{m}(1-\frac{1}{m})^{m-1} + \sum_{i=1}^{n} \left(f(\boldsymbol{x}) - f(\boldsymbol{z}^{\boldsymbol{i}})\right) \cdot \frac{1}{m^2}(1-\frac{1}{m})^{m-2}$$

Proof

- to analyze $f(\boldsymbol{x})$ $\rightarrow$ f_{opt} the weight of a minimum spanning tree

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$$\geq \frac{1}{m}(1 - \frac{1}{m})^{m-1} \cdot \frac{1}{m} \cdot \left(\sum_{i=1}^{m-(n-1)} \big(f(\boldsymbol{x}) - f(\boldsymbol{y}^{\boldsymbol{i}})\big) \cdot + \sum_{i=1}^{n} \big(f(\boldsymbol{x}) - f(\boldsymbol{z}^{\boldsymbol{i}})\big) \right)$$

Proof

- to analyze $f(\boldsymbol{x}) \rightarrow f_{opt}$ the weight of a minimum spanning tree

Using multiplicative drift analysis:

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there exists a set of $m-(n-1)$ 1-bit flips and a set of n 2-bit flips such that the average weight decrease is at least $(f(\boldsymbol{x}) - f_{opt})/(m+1)$

$$\geq \frac{1}{m}(1-\frac{1}{m})^{m-1} \cdot \frac{1}{m} \cdot \left(\sum_{i=1}^{m-(n-1)} \big(f(\boldsymbol{x}) - f(\boldsymbol{y}^{\boldsymbol{i}})\big) \cdot + \sum_{i=1}^{n} \big(f(\boldsymbol{x}) - f(\boldsymbol{z}^{\boldsymbol{i}})\big)\right)$$

Proof

- to analyze $f(\boldsymbol{x}) \rightarrow f_{opt}$ the weight of a minimum spanning tree

Using multiplicative drift analysis:

- design the distance function: $V(\boldsymbol{x}) = f(\boldsymbol{x}) - f_{opt}$
- analyze the expected drift:

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$$\geq f(\boldsymbol{x}) - \left(\sum_{i=1}^{m-(n-1)} f(\boldsymbol{y}^{\boldsymbol{i}}) \cdot \frac{1}{m}(1-\frac{1}{m})^{m-1} + \sum_{i=1}^{n} f(\boldsymbol{z}^{\boldsymbol{i}}) \cdot \frac{1}{m^2}(1-\frac{1}{m})^{m-2} + (1-\cdots)f(\boldsymbol{x})\right)$$

there exists a set of $m-(n-1)$ 1-bit flips and a set of n 2-bit flips such that the average weight decrease is at least $(f(\boldsymbol{x}) - f_{opt})/(m+1)$

$$\geq \frac{1}{m}(1-\frac{1}{m})^{m-1} \cdot \frac{f(\boldsymbol{x}) - f_{opt}}{m+1} \geq \frac{1}{em(m+1)} V(\boldsymbol{x})$$

Proof

The analysis of phase (2): $min\ f(x) = (c(x) - 1) \cdot w_{ub} + \sum_{i:x_{i=1}} w_i$

- it will always be connected, i.e., $c(x) = 1$ always holds
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Using multiplicative drift analysis:

- design the distance function: $V(\boldsymbol{x}) = f(\boldsymbol{x}) - f_{opt}$
- analyze the expected drift:

$$\mathrm{E}[\, V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = \boldsymbol{x}] \geq \frac{1}{em(m+1)} V(\boldsymbol{x})$$

proportional to the current distance

Upper bound on the expected running time:

$$\sum_{x \in \mathcal{X}} \pi_0(\boldsymbol{x}) \cdot \frac{1 + \ln (V(\boldsymbol{x})/V_{min})}{\delta} \leq em(m+1)(1 + \ln (mw_{max}))$$

$V(\boldsymbol{x}) \leq mw_{max}$ $V_{min} \geq 1$

$$\in O(m^2(\log n + \log w_{max}))$$

Proof

Theorem. [Neumann & Wegener, TCS'07; Doerr et al., Algorithmica'12] The expected running time of the (1+1)-EA solving the MST problem is $O(m^2(\log n + \log w_{max}))$.

Main idea:

(1) obtain a connected subgraph

(2) obtain a minimum spanning tree

The expected running time of phase (1): $O(m \log n)$

The expected running time of phase (2): $O(m^2(\log n + \log w_{max}))$

The total expected running time: $O(m^2(\log n + \log w_{max}))$

Negative drift

Simplified negative drift theorem
for proving exponential lower bounds [Oliveto & Witt, Algorithmica'11]

Theorem 4 (Simplified Drift Theorem) *Let X_t, $t \geq 0$, be the random variables describing a Markov process over a finite state space $S \subseteq \mathbb{R}_0^+$ and denote $\Delta_t(i) := (X_{t+1} - X_t \mid X_t = i)$ for $i \in S$ and $t \geq 0$. Suppose there exist an interval $[a, b]$ in the state space, two constants $\delta, \varepsilon > 0$ and, possibly depending on $\ell := b - a$, a function $r(\ell)$ satisfying $1 \leq r(\ell) = o(\ell/\log(\ell))$ such that for all $t \geq 0$ the following two conditions hold:*

1. $E(\Delta_t(i)) \geq \varepsilon$ *for* $a < i < b$,
2. $\mathrm{Prob}(\Delta_t(i) \leq -j) \leq \frac{r(\ell)}{(1+\delta)^j}$ *for* $i > a$ *and* $j \in \mathbb{N}_0$.

Then there is a constant $c^* > 0$ such that for $T^* := \min\{t \geq 0 \colon X_t \leq a \mid X_0 \geq b\}$ it holds $\mathrm{Prob}(T^* \leq 2^{c^*\ell/r(\ell)}) = 2^{-\Omega(\ell/r(\ell))}$.

The expected drift is negative, i.e., away from the target in expectation

The probability of a drift towards the target decreases exponentially

Exponential running time

negative drift ← 🧍 - - - very unlikely - - →

Negative drift

Simplified negative drift theorem
for proving exponential lower bounds [Oliveto & Witt, Algorithmica'11]

Theorem 4 (Simplified Drift Theorem) *Let X_t, $t \geq 0$, be the random variables describing a Markov process over a finite state space $S \subseteq \mathbb{R}_0^+$ and denote $\Delta_t(i) := (X_{t+1} - X_t \mid X_t = i)$ for $i \in S$ and $t \geq 0$. Suppose there exist an interval $[a, b]$ in the state space, two constants $\delta, \varepsilon > 0$ and, possibly depending on $\ell := b - a$, a function $r(\ell)$ satisfying $1 \leq r(\ell) = o(\ell/\log(\ell))$ such that for all $t \geq 0$ the following two conditions hold:*

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Then there is a constant $c^* > 0$ such that for $T^* := \min\{t \geq 0 \colon X_t \leq a \mid X_0 \geq b\}$ it holds $\text{Prob}(T^* \leq 2^{c^*\ell/r(\ell)}) = 2^{-\Omega(\ell/r(\ell))}$.

a constant

relax

Simplified negative drift with self-loops [Rowe & Sudholt, TCS'14]

Simplified negative drift with scaling [Oliveto & Witt, TCS'14]

Negative drift

For proving exponential lower bounds on the running time of EAs:

To use:

Simplified negative drift [Oliveto & Witt, Algorithmica'11]

easier

Simplified negative drift with self-loops [Rowe & Sudholt, TCS'14]

Simplified negative drift with scaling [Oliveto & Witt, TCS'14]

stronger

easier

Original negative drift [Hajek, 1982]

Switch analysis

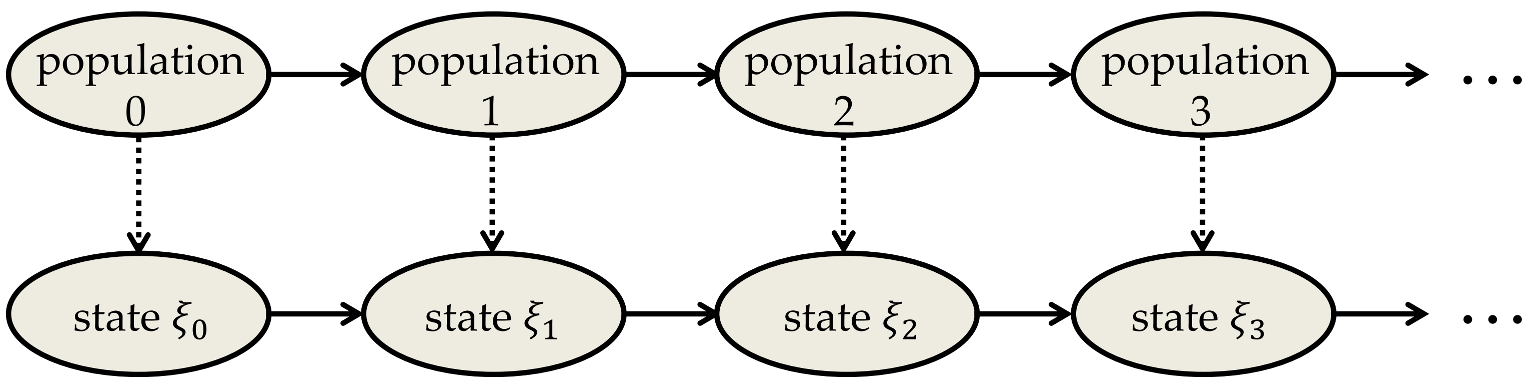
Switch analysis [Yu, Qian & Zhou, TEC'15]

- One step difference with a reference evolutionary process:

$$\forall t: \sum_{x\in\mathcal{X},y\in\mathcal{Y}} \pi_t(x)P(\xi_{t+1}\in\phi^{-1}(y)|\xi_t=x)\mathrm{E}[\tau'|\xi'_0=y]$$
$$-\sum_{u,y\in\mathcal{Y}} \pi_t^{\phi}(u)P(\xi'_1\in y|\xi'_0=u)\mathrm{E}[\tau'|\xi'_1=y]\le\rho_t$$

- Running time upper bound: $\mathrm{E}[\tau|\xi_0\sim\pi_0]\le\mathrm{E}[\tau'|\xi'_0\sim\pi_0^{\phi}]+\sum_{t=0}^{+\infty}\rho_t$

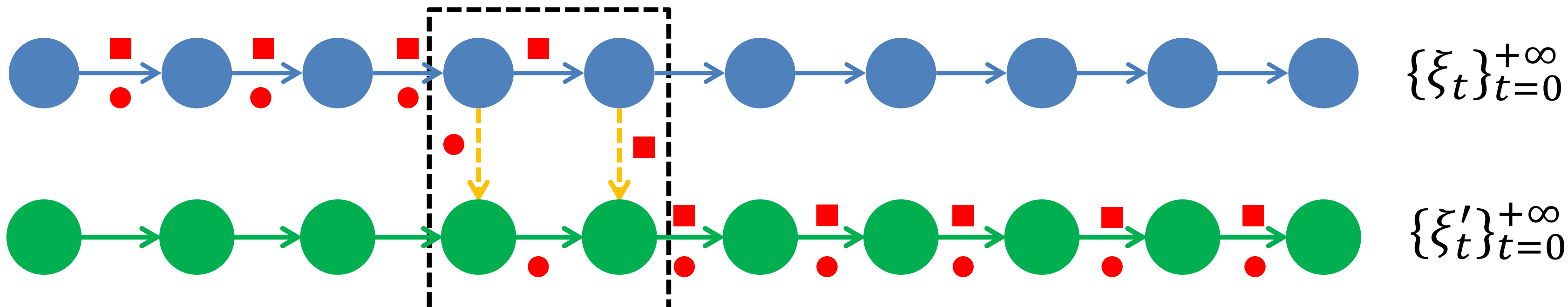
Main idea:



Switch analysis

Main idea [Yu, Qian & Zhou, TEC'15]:

Given EA on the given problem



Reference algorithm on the reference problem

investigate the different behaviors at each step

The expected running time:

$$\mathrm{E}[\tau] \leq (\geq)\ \mathrm{E}[\tau'] + \sum_{t=0}^{+\infty} \rho_t$$

The expected running time of $\{\xi'_t\}_{t=0}^{+\infty}$, easy to analyze

Ability of switch analysis

Fitness level method [Droste et al., TCS'02]

- Jumping probability lower bound:

$$P(\xi_{t+1} \in \cup_{j=i+1}^{m} S_j | \xi_t \in S_i) \geq v_i$$

⇩

- Running time upper bound: $\sum_{i=0}^{m-1} \pi_0(S_i) \cdot \sum_{j=i}^{m-1} \frac{1}{v_j}$

Drift analysis [He & Yao, AIJ'01]

- Expected drift in one step:

$$\mathrm{E}[\, V(\xi_t) - V(\xi_{t+1}) \mid \xi_t] \geq c_l$$

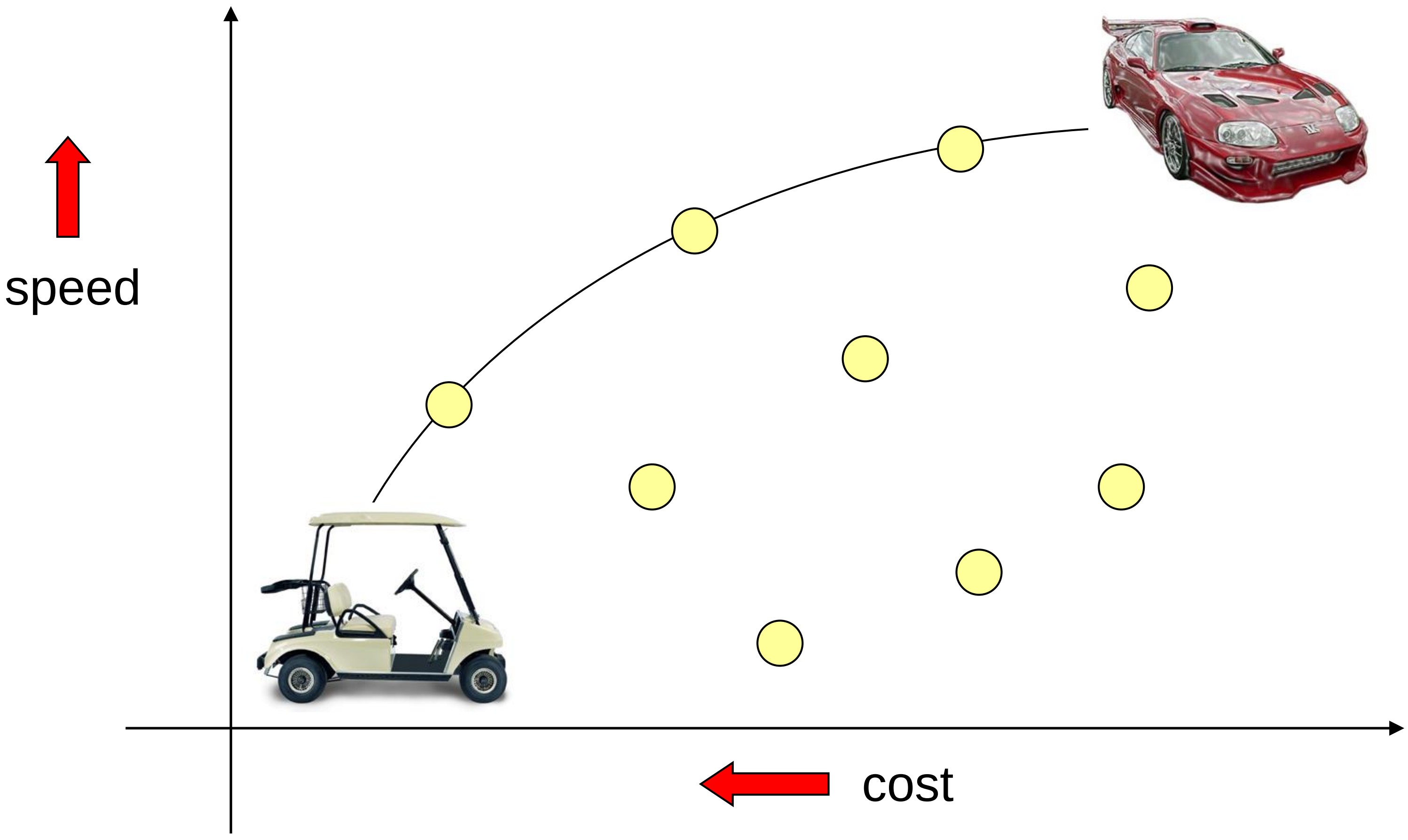
⇩

- Running time upper bound: $\sum_{x \in \mathcal{X}} \pi_0(x) \cdot \frac{V(x)}{c_l}$

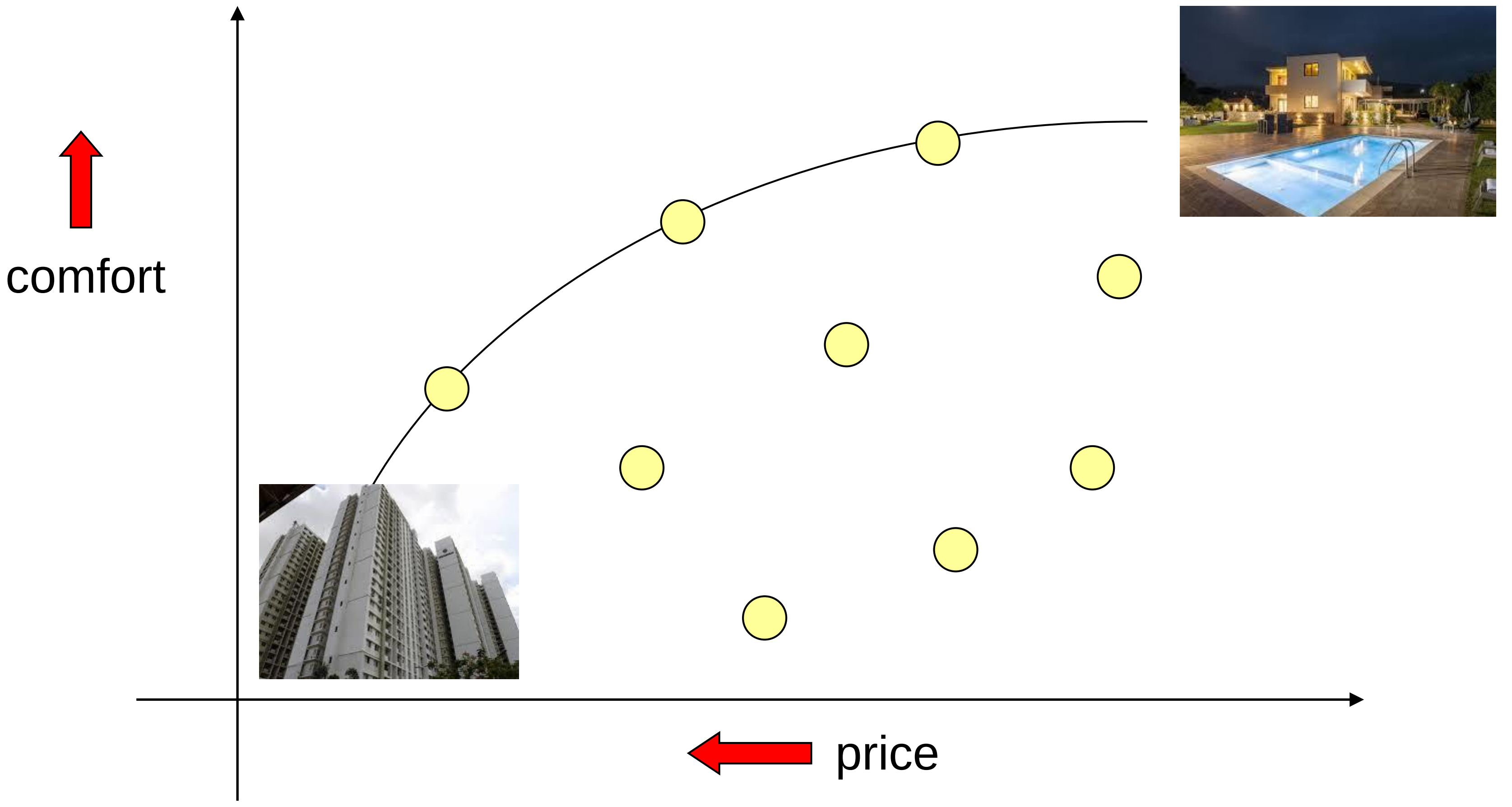
Reducible to **switch analysis** [Yu et al., TEC'15]

Switch analysis can derive at least the same tight bound while requiring no more information

Example of multi-objective optimization



Example of multi-objective optimization



Running time analysis tools

- Fitness level
  - Original fitness level
  - Refined fitness level
- Drift analysis
  - Additive drift
  - Multiplicative drift
  - Negative drift
- Switch analysis

When EAs are applied to solve multi-objective optimization problems, can these approaches be easily used to analyze their running time?

Multi-objective optimization

Multi-objective optimization: optimize multiple objectives (which are usually conflicting) simultaneously

$$max_{x \in \mathcal{X}} \ \big(f_1(x), f_2(x), \ldots, f_m(x)\big)$$

- x weakly dominates y, denoted as $x \succcurlyeq y$, if

$$\forall i \in \{1,2,\ldots,m\}: \ f_i(x) \geq f_i(y)$$

- x dominates y, denoted as $x \succ y$, if

$$\forall i \in \{1,2,\ldots,m\}: \ f_i(x) \geq f_i(y) \text{ and } \exists i \in \{1,2,\ldots,m\}: \ f_i(x) > f_i(y)$$

- x is incomparable with y, if neither $x \succcurlyeq y$ nor $y \succcurlyeq x$

Multi-objective optimization

Multi-objective optimization: optimize multiple objectives (which are usually conflicting) simultaneously

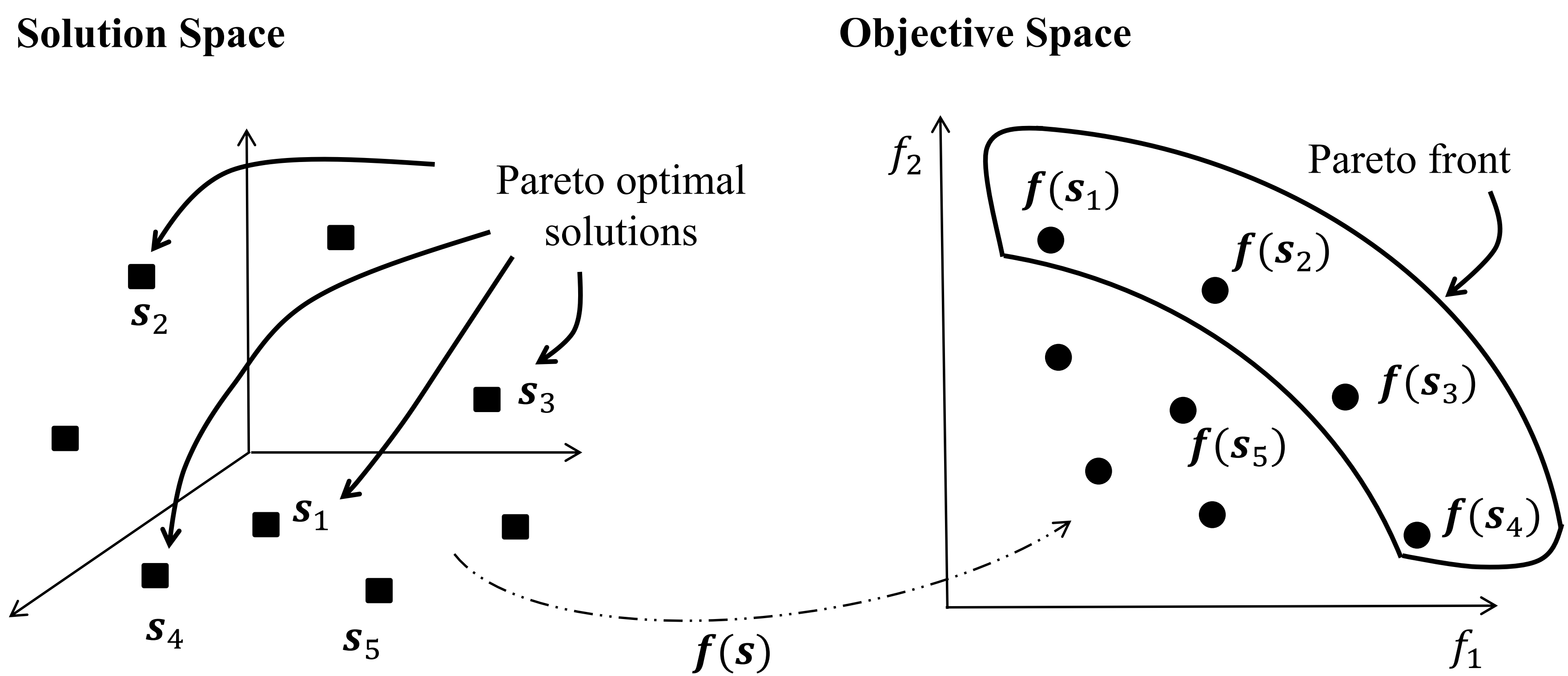
$$max_{x \in \mathcal{X}} \; \big(f_1(x), f_2(x), \ldots, f_m(x)\big)$$

A solution is Pareto optimal if no other solution dominates it

The collection of objective vectors of all Pareto optimal solutions is called the Pareto front

The goal of multi-objective optimization is to find a set of solutions whose objective vectors cover the Pareto front

Example of multi-objective optimization



$s_2 \succcurlyeq s_5$ $\quad$ $s_2 \succ s_5$ $\quad$ s_2 is incomparable with s_3

Assume the solution space contains only these 8 solutions

Example of running time analysis

GSEMO: Given a pseudo-Boolean function vector $\boldsymbol{f}$:

1. $\boldsymbol{x} :=$ randomly selected from $\{0,1\}^n$.
2. $P := \{\boldsymbol{x}\}$.
3. Repeat until some termination criterion is met
4. Choose $\boldsymbol{x}$ from P uniformly at random.
5. $\boldsymbol{x}' :=$ flip each bit of $\boldsymbol{x}$ with probability $1/n$.
6. if $\nexists\, \boldsymbol{z} \in P$ such that $\boldsymbol{z} \succ \boldsymbol{x}'$
7. $P := (P - \{\boldsymbol{z} \in P \mid \boldsymbol{x}' \succcurlyeq \boldsymbol{z}\}) \cup \{\boldsymbol{x}'\}$.

Keep the non-dominated solutions generated so-far

LOTZ: $max_{\boldsymbol{x}\in\{0,1\}^n} \left(\sum_{i=1}^{n} \prod_{j=1}^{i} x_j,\ \sum_{i=1}^{n} \prod_{j=i}^{n} (1 - x_j)\right)$

Count the number of leading 1-bits   Count the number of trailing 0-bits

The Pareto set: $00 \dots 00,\ 10 \dots 00,\ \dots,\ 11 \dots 10,\ 11 \dots 11$

The Pareto front: $(0, n), (1, n-1), \dots, (n-1, 1), (n, 0)$

Proof

Theorem. [Giel, CEC'03] The expected running time of the GSEMO solving the LOTZ problem is $O(n^3)$.

Main idea:

(1) obtain the Pareto optimal solution $11 \ldots 11$

(2) obtain the Pareto set $\{00 \ldots 00, 10 \ldots 00, \ldots, 11 \ldots 10, 11 \ldots 11\}$

The analysis of phase (1):

the first objective $\sum_{i=1}^{n} \prod_{j=1}^{i} x_j$

- consider the largest LO value in the population, which never decreases
- select the solution with the largest LO value, and only flip its first 0 bit
- the probability is at least $\frac{1}{n+1} \frac{1}{n} (1-\frac{1}{n})^{n-1}$

because the population size is no larger than $n+1$, and a solution is uniformly selected at random

Proof

Theorem. [Giel, CEC'03] The expected running time of the GSEMO solving the LOTZ problem is $O(n^3)$.

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The expected running time upper bound $n \cdot en(n+1) \in O(n^3)$

Main idea:

(1) obtain the Pareto optimal solution $11 \dots 11$

(2) obtain the Pareto set $\{00 \dots 00, 10 \dots 00, \dots, 11 \dots 10, 11 \dots 11\}$

The analysis of phase (1):

- consider the largest LO value in the population, which never decreases
- select the solution with the largest LO value, and only flip its first 0 bit
- the probability is at least $\frac{1}{n+1}\frac{1}{n}(1-\frac{1}{n})^{n-1}$

the probability of increasing the largest LO value by 1 is at least $\frac{1}{en(n+1)}$

it is sufficient to increase n times

Proof

The analysis of phase (2):

- the found Pareto optimal solutions will always be kept
- follow the path: $1^n \to 1^{n-1}0 \to \cdots \to 10^{n-1} \to 0^n$

the probability is at least $\frac{1}{n+1} \cdot \frac{1}{n}(1-\frac{1}{n})^{n-1}$

The expected running time upper bound: $n \cdot en(n+1) \in O(n^3)$

The expected running time of phase (1): $O(n^3)$

The total expected running time: $O(n^3)$

Switch analysis for multi-objective optimization

Theorem: $\xi \in \mathcal{X}$ modeling an EA solving a multi-objective optimization problem, a well-defined function $h_{\boldsymbol{\alpha},c}: \mathcal{X} \to \mathbb{N}_0$ and a Markov chain $\xi' \in \mathcal{Y} = \{0,1\}^r$ with $\mathcal{Y}^* = \{1^r\}$ such that $\forall x \notin \mathcal{X}^*, \forall t \geq 0$,

$$\sum\nolimits_{i\in[r]} \mathrm{P}(\min\{h(\xi_{t+1}), r\} = i | \xi_t = x) \mathrm{E}\left[\tau' \mid \xi'_0 = 1^i 0^{r-i}\right]$$
$$\leq \sum\nolimits_{y\in\mathcal{Y}} \mathrm{P}\left(\xi'_1 = y \middle| \xi'_0 = 1^{h(x)} 0^{r-h(x)}\right) \mathrm{E}[\tau' \mid \xi'_1 = y] + \delta,$$

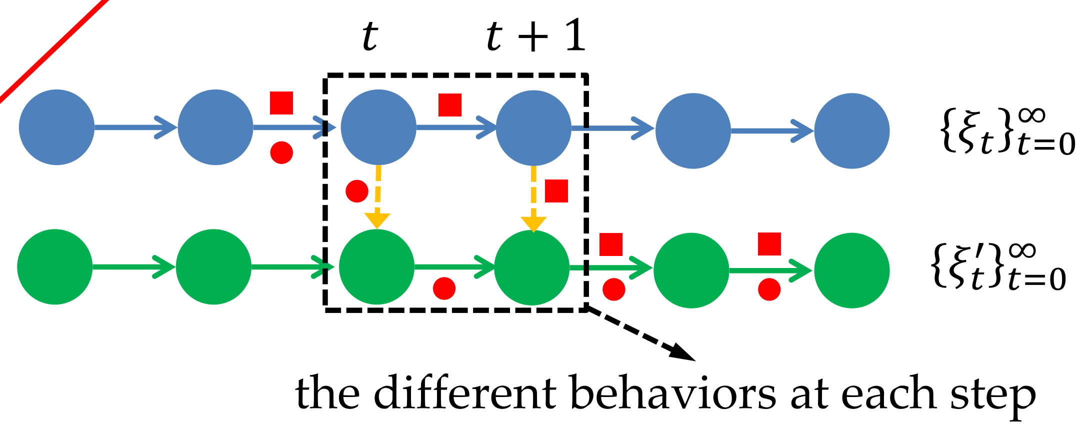
$$\Rightarrow \mathrm{E}[\tau \mid \xi_0 = x_0] \leq \mathrm{E}\left[\tau' \mid \xi'_0 = 1^{\min\{h(x_0),r\}} 0^{r-\min\{h(x_0),r\}}\right] / (1-\delta)$$

Basic idea [Bian et al., IJCAI'18]:

reduce the difficulty of analysis

Given EA solving the given multi-objective problem

A Markov chain modeling a reference single-objective evolutionary process



Application of switch analysis

Application [Bian, Qian and Tang, IJCAI'18]:

GSEMO	Problem	Previous result	Our result
Bi-objective	LOTZ	$O(n^3)$ [Giel, CEC'03]	$\leq 6n^3$
	COCZ	$O(n^2 \log n)$ [Qian et al., AIJ'13]	$\leq 3n^2 \log n$
Many-objective	mCOCZ	$O(n^{m+1})$ [Laumanns et al., TEC'04]	$O(n^m)$ for $m > 4$, $O(n^3 \log n)$ for $m = 4$
Approximate analysis	WOMM	—	$1/n$-approximation: $O(n^2(\log_l n + \log_l(w_n/w_1)))$

→ gives the leading constants

→ is asymptotically tighter than

L. Thiele

Member of Academia Europaea

Switch analysis is general and powerful

Running time analysis tools

- Fitness level
  - Original fitness level
  - Refined fitness level
- Drift analysis
  - Additive drift
  - Multiplicative drift
  - Negative drift
- Switch analysis

Results in single-objective optimization

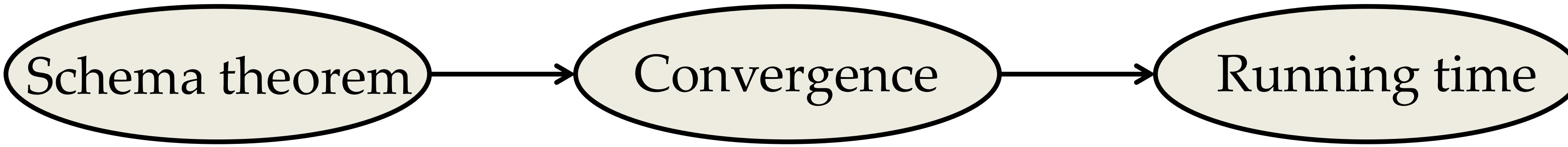
(1+1)-EA	linear function $\Theta(n \log n)$ [Droste et al., TCS'02] minimum spanning tree $O(m^2 \log(n + w_{max}))$ [Neumann & Wegener, TCS'07] partition $O(n^2)$ with $\frac{4}{3}$ approximation [Witt, STACS'05]
(u+1)-EA	OneMax $O(un + n \log n)$; LeadingOnes $O(un \log n + n^2)$ [Witt, ECJ'06] maximum clique $O(un \log n)$ on sparse graphs [Storch, TCS'07] vertex cover $O(un \log n)$ on bipartite graphs [Oliveto et al., TEC'09]
(1+λ)-EA	linear function $O(\lambda n + n \log n)$ [Doerr & Kunnemann, TCS'15] vertex cover *exponential* on bipartite graphs [Oliveto et al., TEC'09]
(N+N)-EA	OneMax $O(Nn \log\log n + n \log n)$; LeadingOnes $O(Nn \log n + n^2)$ [Chen et al., TSMCB'09]

EDA [Chen et al., TEC'10]; ACO [Doerr et al., TCS'11]; PSO [Sudholt & Witt, TCS'10]; GP [Wagner et al., ECJ'15]

Results in multi-objective optimization

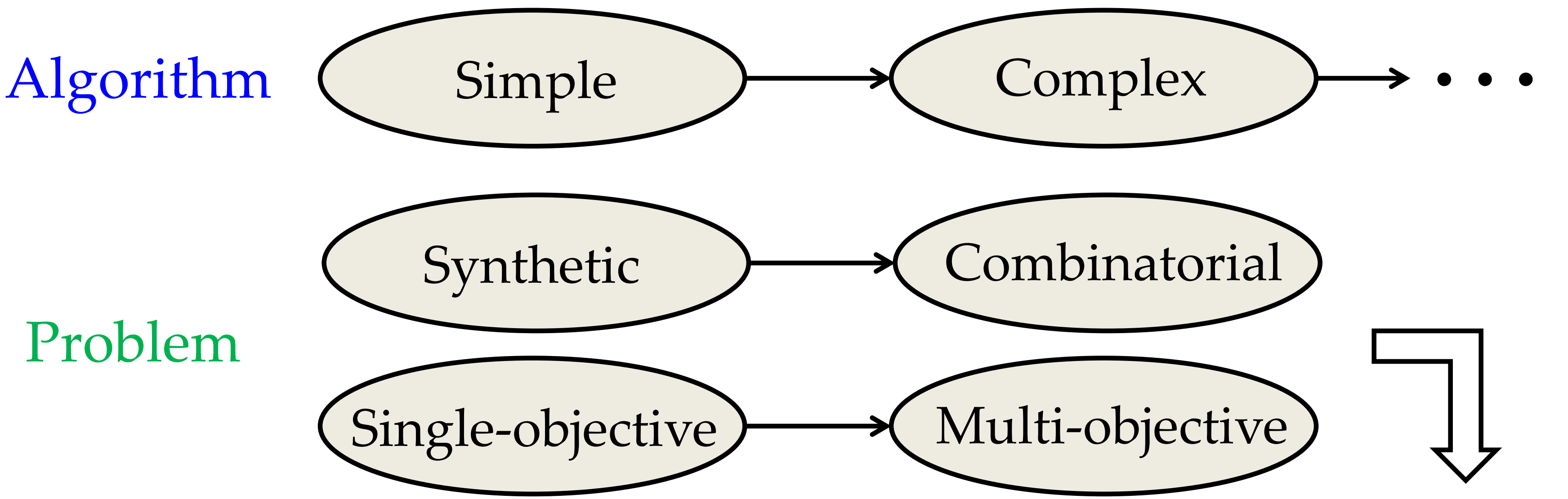
SEMO	LOTZ $\Theta(n^3)$; COCZ $O(n^2 \log n)$ [Laumanns et al., TEC'04]
GSEMO	LOTZ $O(n^3)$ [Giel, CEC'03]; $\Omega(n^2/p)$ [Doerr et al., CEC'13] bi-objective minimum spanning tree $O(m^3 w_{min}(\lvert C \rvert + \log n + \log w_{max}))$ with 2 approximation [Neumann, EJOR'07]
DEMO	multi-objective all-pairs-shortest-path $O(nP_{max}g)$ *with* $r^{3g \log n}$ approximation [Neumann & Theile, PPSN'10]
REMO	LOTZ $\Theta(n^2)$; COCZ $\Theta(n \log n)$; bi-objective minimum spanning tree $O\left(m^2 n w_{min}(\lvert C \rvert + \frac{\log n + \log w_{max}}{n w_{min}} - N_{gc}(1 - \frac{1}{m}))\right)$ with 2 approximation [Qian et al., AIJ'13]
MOEA/D	LOTZ $O(n^2 \log n)$; COCZ $\Theta(n \log n)$ [Li et al., TEC'16]

Theoretical analysis of EAs



[Holland, 1975] [Rudolph, 1998] [Droste et al., 1998]

Running time analysis



Environments with uncertainty

Theory community

Ingo Wegener (1950-2008), TU Dortmund, Germany

Stefan Droste, Thomas Jansen, Ingo Wegener:
A Rigorous Complexity Analysis of the (1 + 1) Evolutionary Algorithm for Separable Functions with Boolean Inputs. *Evolutionary Computation* 6(2): 185-196 (1998)

UK
- University of Sheffield: Pietro Oliveto
- University of Birmingham: Per Kristian Lehre
- Aberystwyth University: Thomas Jansen

Germany
- University of Potsdam: Tobias Friedrich
- University of Passau: Dirk Sudholt

France
- Ecole Polytechnique: Benjamin Doerr
- Sorbonne University: Carola Doerr

Switzerland
- ETH Zürich: Johannes Lengler

Denmark
- Technical University of Denmark: Carsten Witt

Australia
- University of Adelaide: Frank Neumann

China
- Nanjing University
- Sun Yat-sen University
- Southern University of Science and Technology

Summary

- Fitness level
  - Original fitness level
  - Refined fitness level
- Drift analysis
  - Additive drift
  - Multiplicative drift
  - Negative drift
- Switch analysis
- Results of running time analysis

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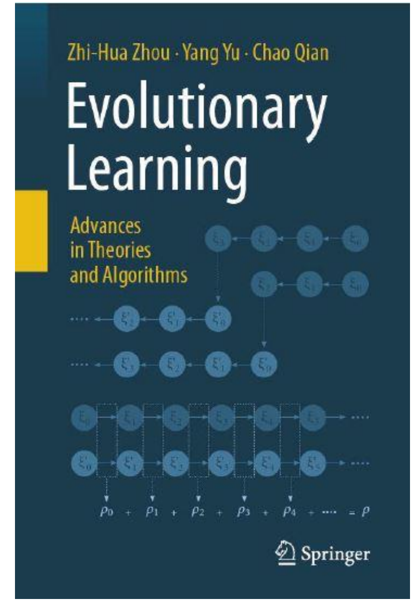
- Y. Yu, C. Qian and Z.-H. Zhou. Switch analysis for running time analysis of evolutionary algorithms. *IEEE Transactions on Evolutionary Computation*, 2015, 19(6): 777-792

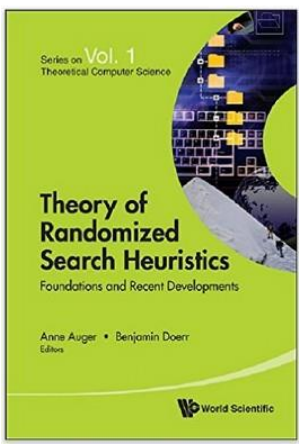
Reading books

Bioinspired Computation in Combinatorial Optimization

Algorithms and Their Computational Complexity

Authors: **Neumann**, Frank, **Witt**, Carsten





Theory of Randomized Search Heuristics

Foundations and Recent Developments

Edited by: **Anne Auger** (*INRIA, France*), **Benjamin Doerr** (*Max-Planck-Institut für Informatik, Germany*)

Assignment - 3

Task:

- Apply fitness level method to analyze the upper bound on the expected running time of (1+1)-EA solving LeadingOnes
- Apply multiplicative drift analysis to analyze the upper bound on the expected running time of (1+1)-EA solving OneMax
- Apply additive drift analysis to analyze the upper bound on the expected running time of (1+1)-EA solving OneMax

Deadline: Dec. 20