Last class

- Breadth-first search
- Uniform-cost search
- Depth-first search
- Depth-limited search
- Iterative deepening depth-first search
- Bidirectional search

Uninformed search

No additional information beyond the problem definition

Heuristic Search and Evolutionary Algorithms

Lecture 3: Informed Search

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Informed Search

Informed search uses problem-specific knowledge beyond the definition of the problem

Heuristic function:

$h(n)$ = estimated cost of the optimal path from the state at node n to a goal state

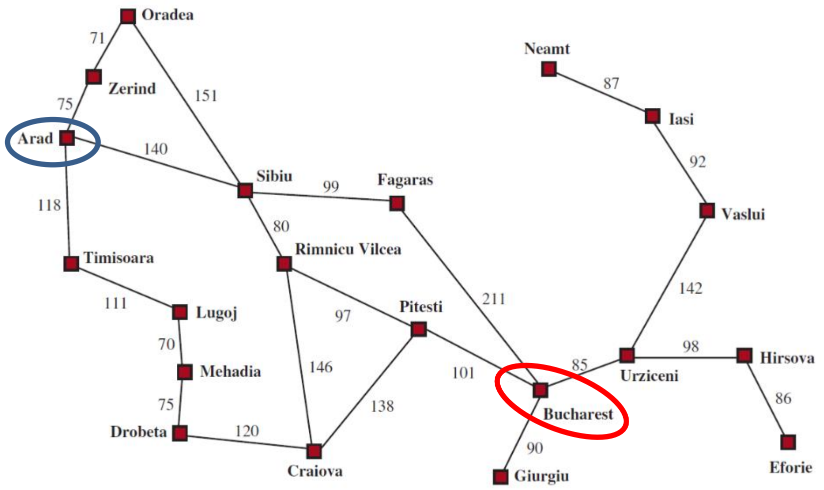
- $h(n)$ depends on the state instead of the node
- $h(n)$ measures the goodness of a state
- $h(n)$ is non-negative, and is equal to 0 if n is a goal node

How to utilize the heuristic function $h(n)$?

Greedy best-first search

Main idea: expand the node with the lowest $h(n)$ value

Problem: find the shortest path from Arad to Bucharest



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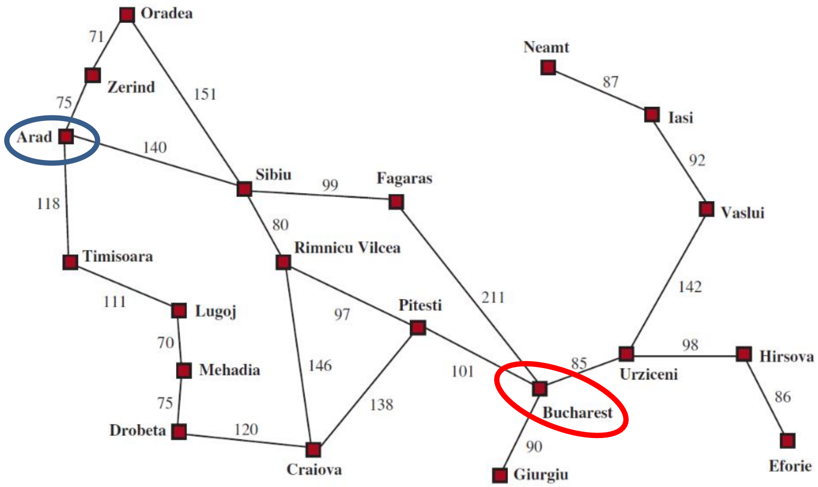
$h(n)$: straight-line distances to Bucharest

Greedy best-first search - example



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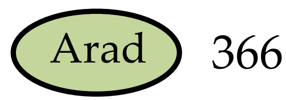
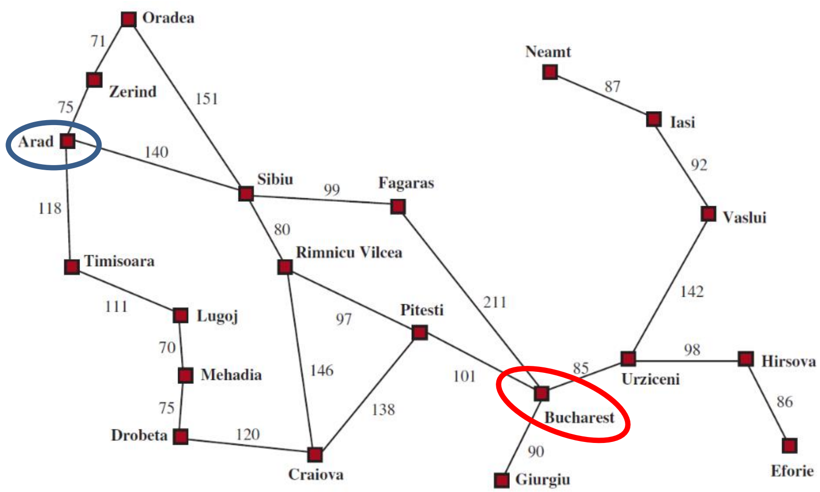


Greedy best-first search - example



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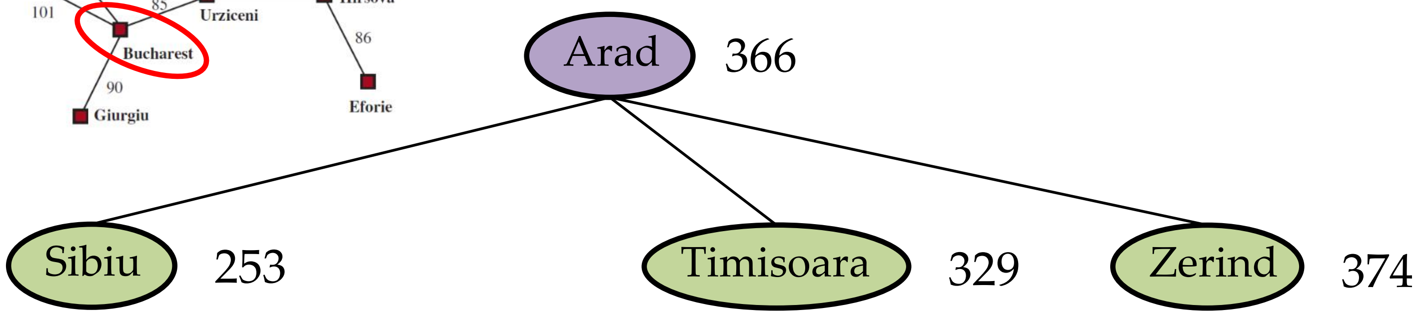
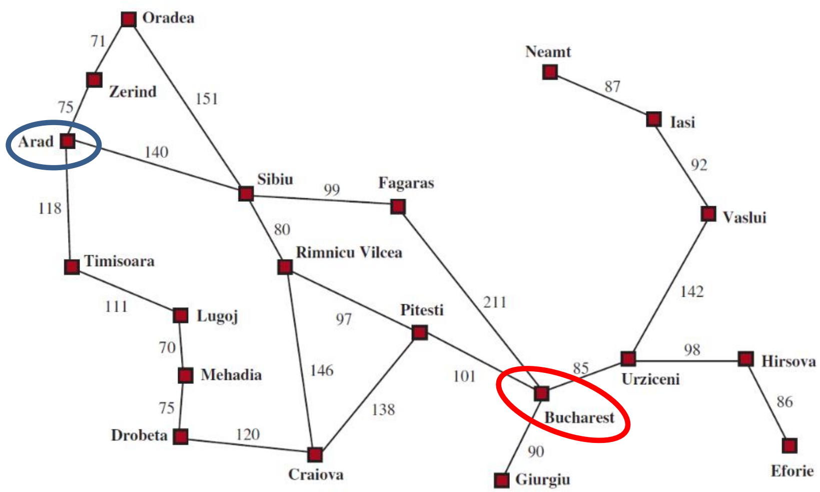


Greedy best-first search - example



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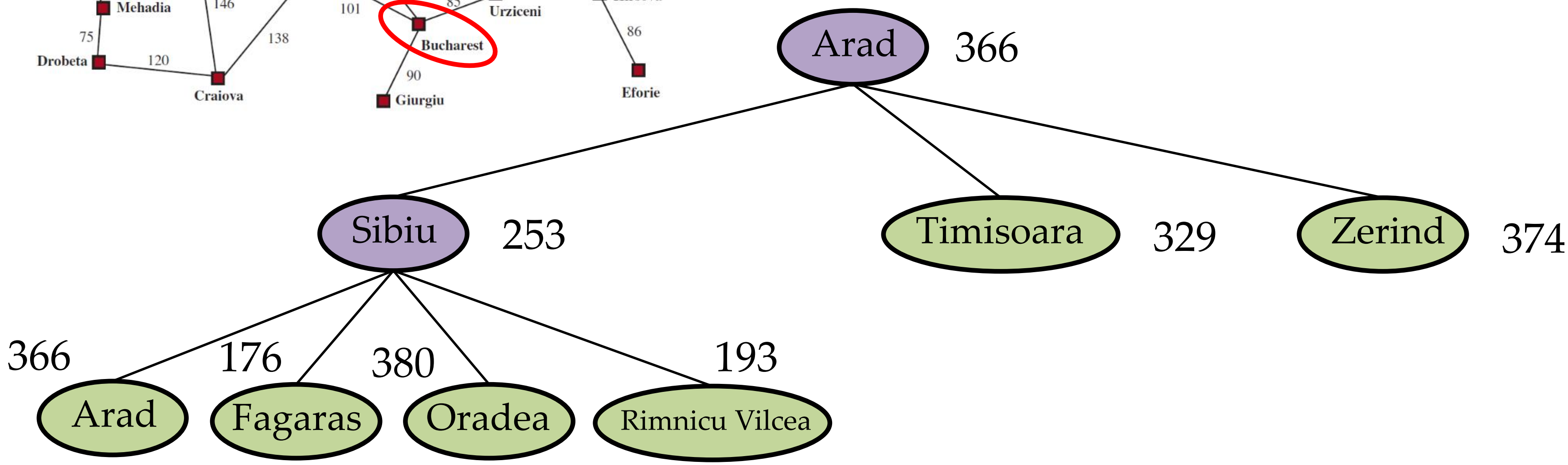
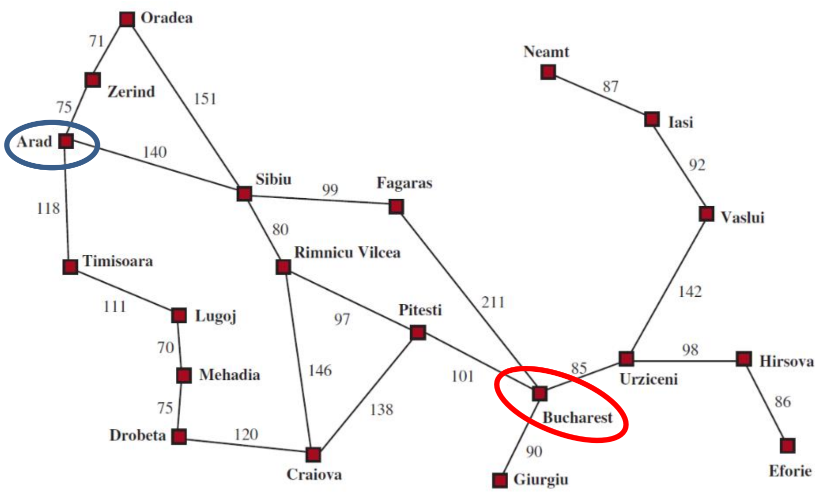


Greedy best-first search - example



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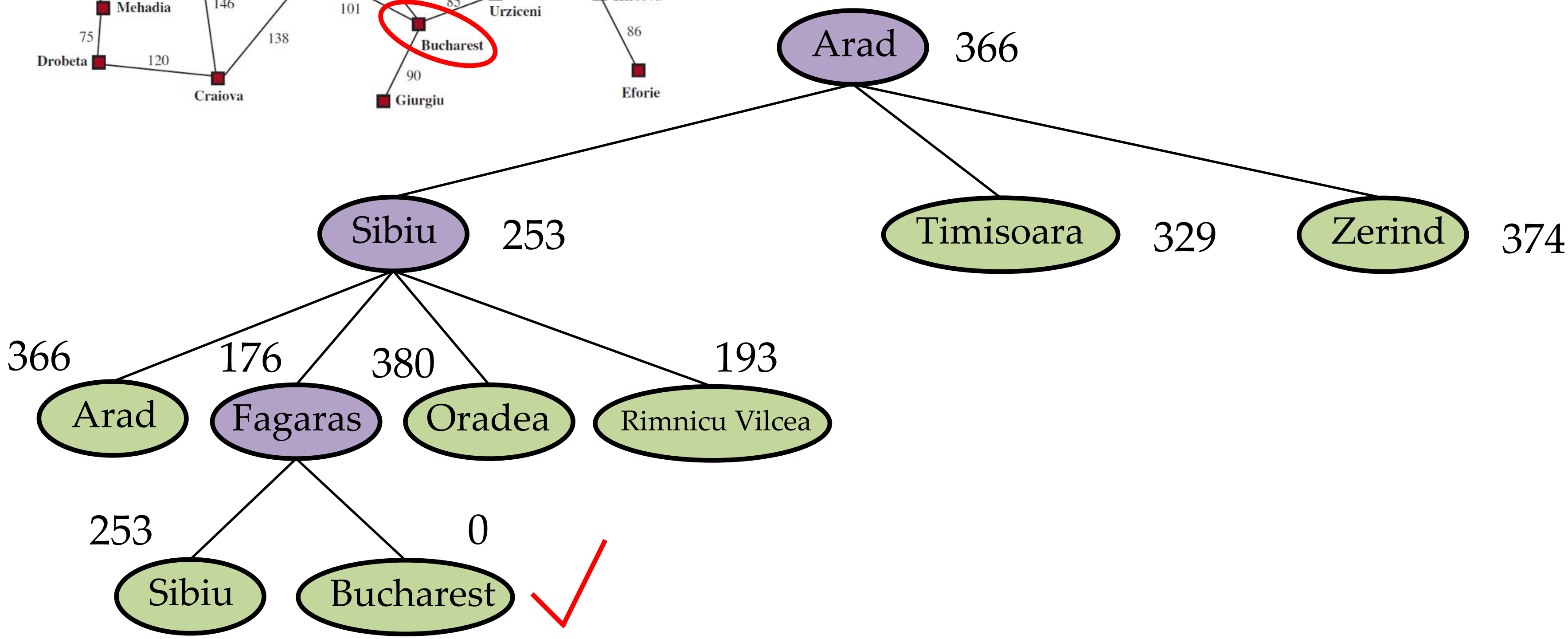
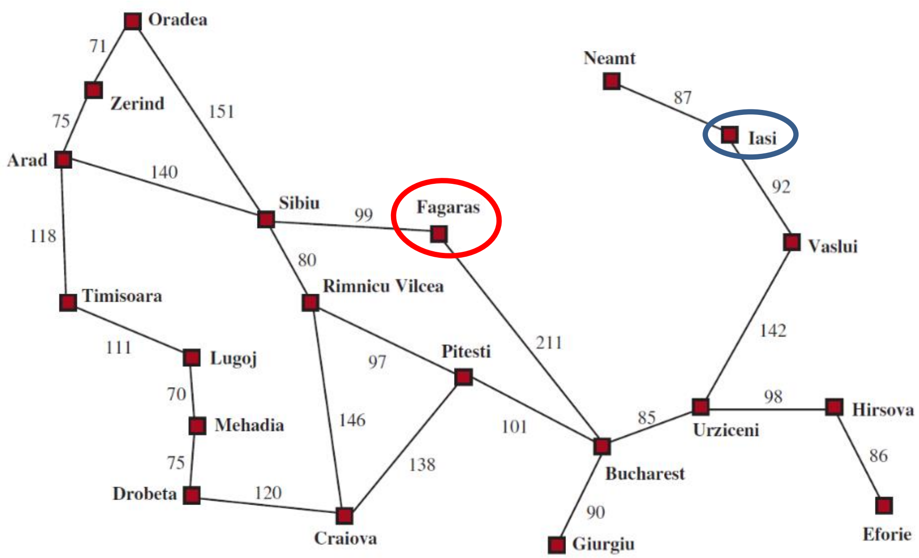


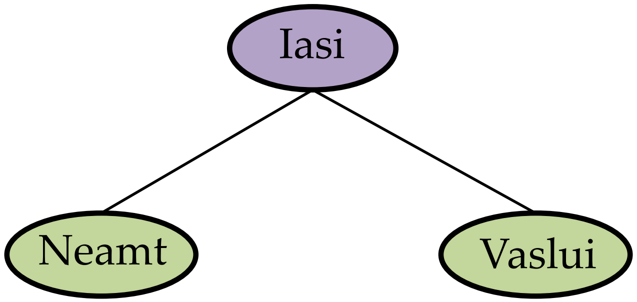
Greedy best-first search - performance

Main idea: expand the node with the lowest $h(n)$ value

- Complete

    if using tree-search, no





Greedy best-first search - performance

Main idea: expand the node with the lowest $h(n)$ value

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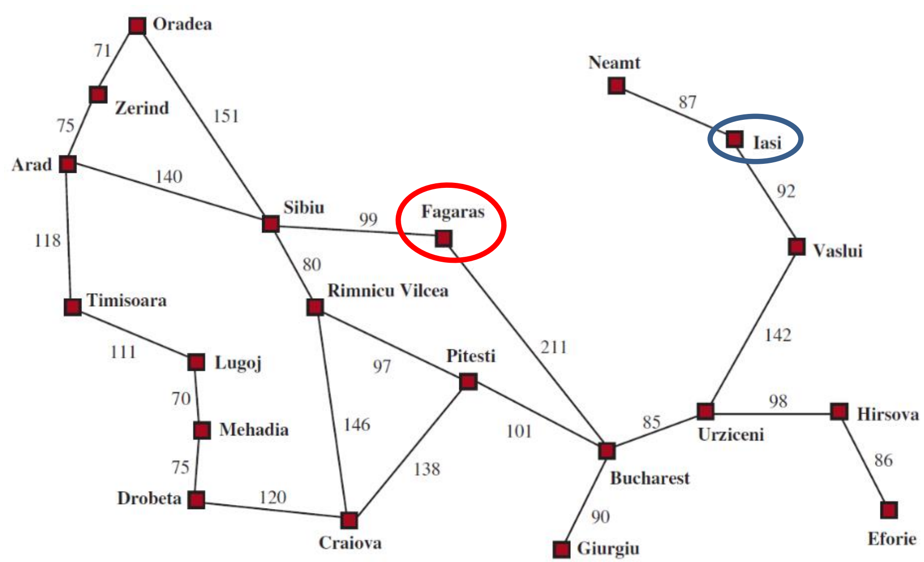
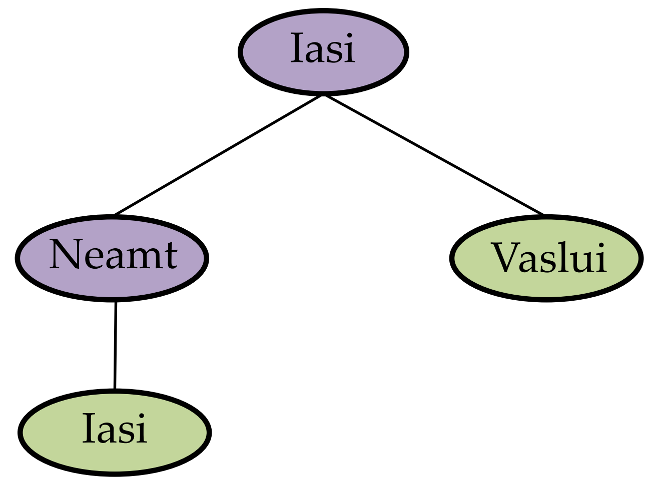




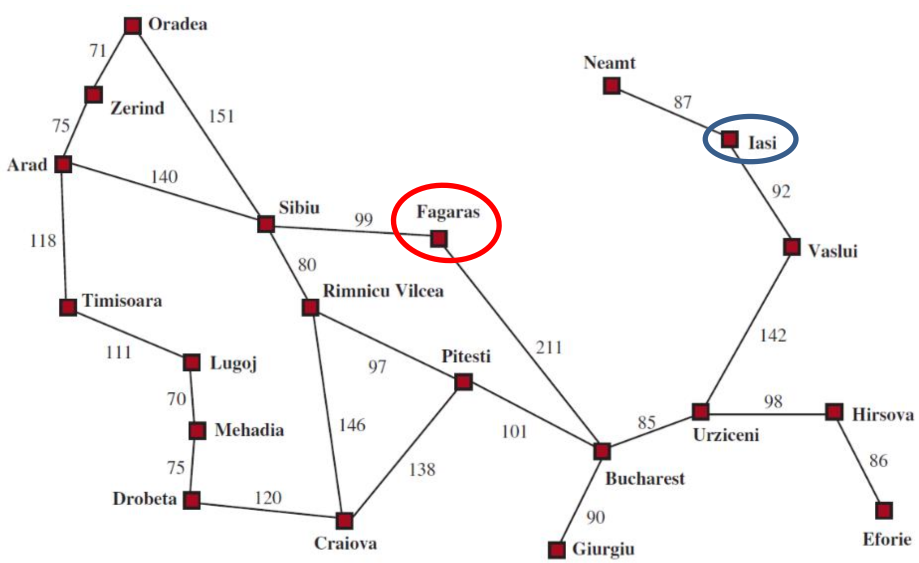
Greedy best-first search - performance

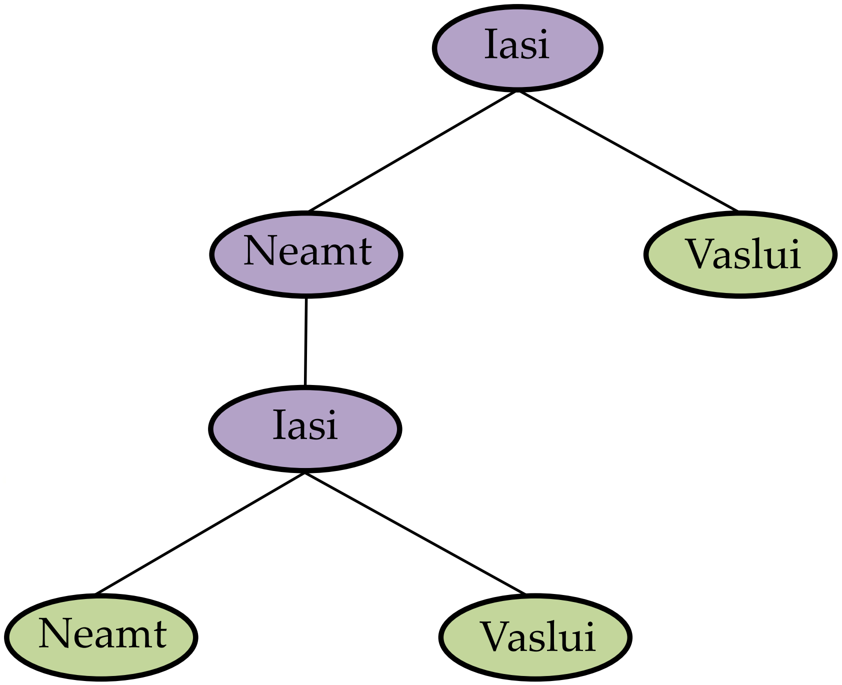
Main idea: expand the node with the lowest $h(n)$ value

- Complete

  if using tree-search, no

  if using graph-search and the state space is finite, yes





Greedy best-first search - performance

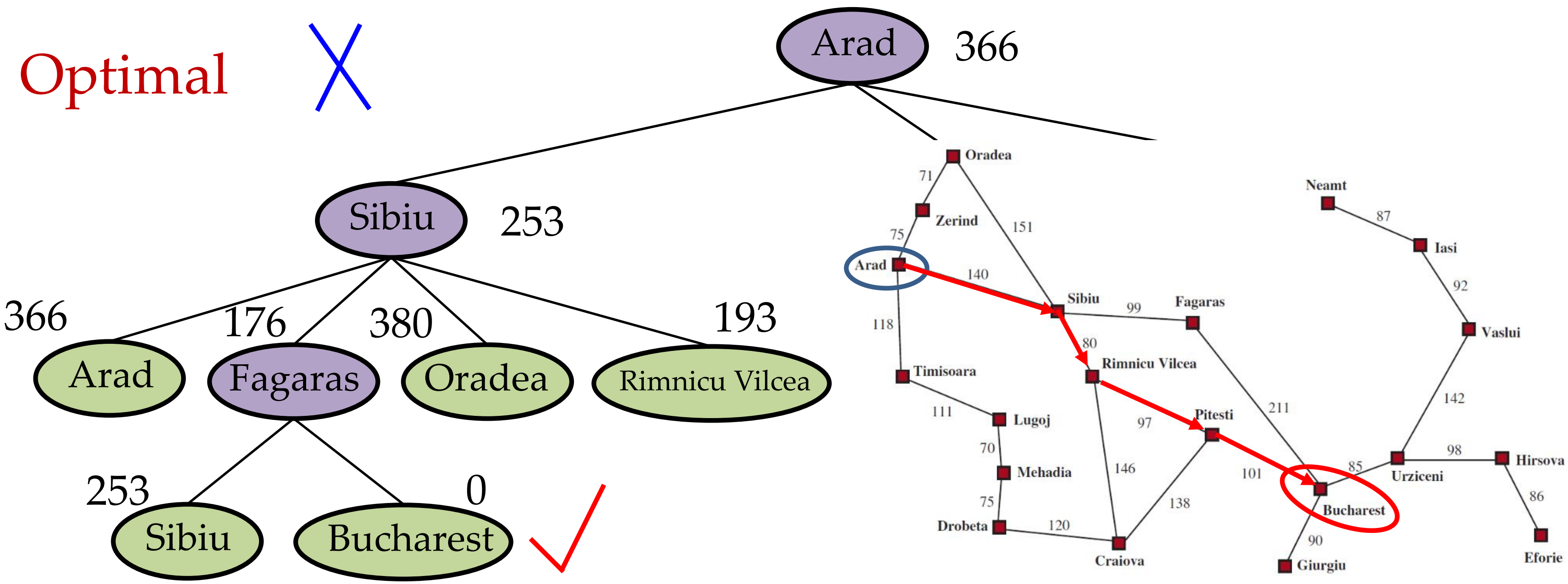
Main idea: expand the node with the lowest $h(n)$ value

- Complete

  if using tree-search, no

  if using graph-search and the state space is finite, yes

- Optimal



Greedy best-first search - performance

Main idea: expand the node with the lowest $h(n)$ value

- Complete

  if using tree-search, no

  if using graph-search and the state space is finite, yes

- Optimal ✗

- Time complexity

  $O(b^m)$, *where m is the maximum depth of any node, if using tree-search*

- Space complexity

  $O(b^m)$, *where m is the maximum depth of any node, if using tree-search*

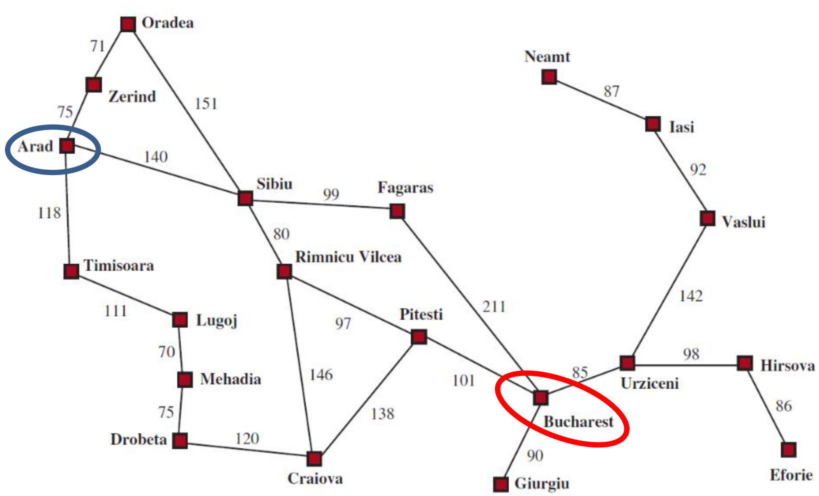
A* search

Main idea: expand the node n with the lowest value of

$$g(n) + h(n)$$

Path cost from the initial node to the node n

Estimated cost of the optimal path from the node n to a goal



$h(n)$: straight-line distances to Bucharest

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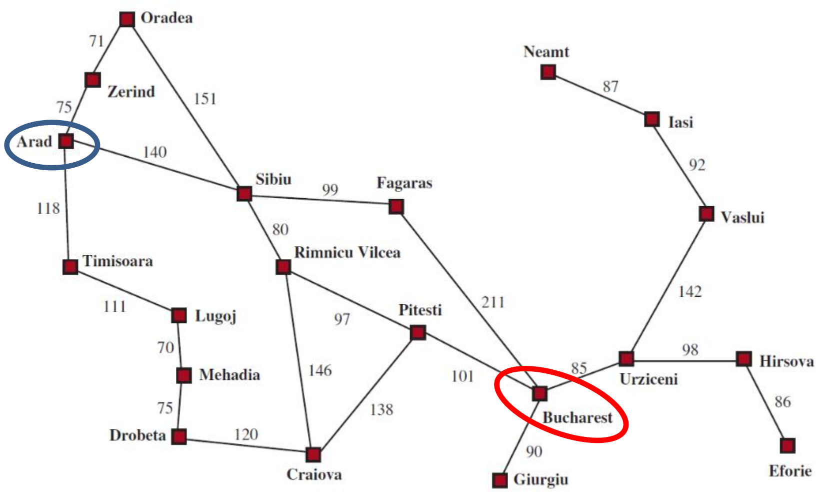
A* search - example



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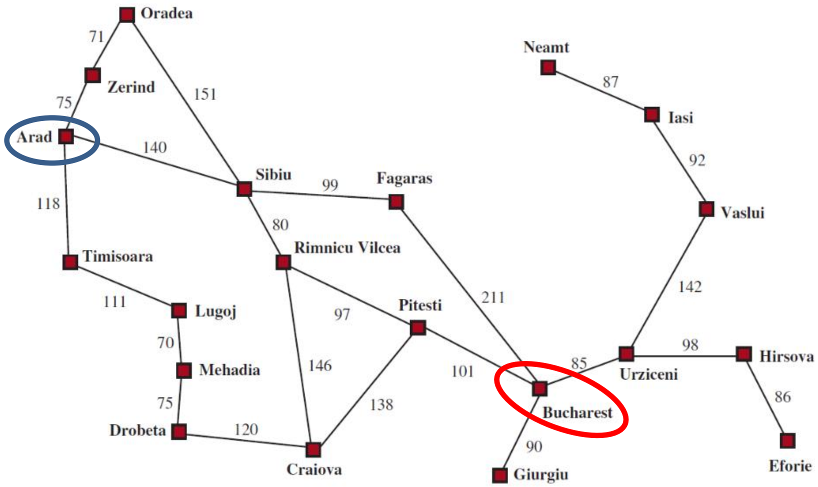
Arad 0+366=366

A* search - example



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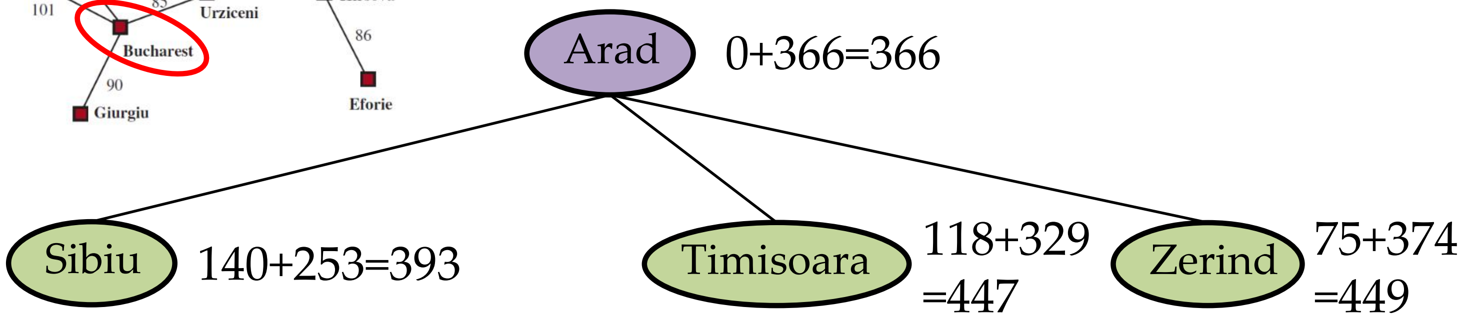
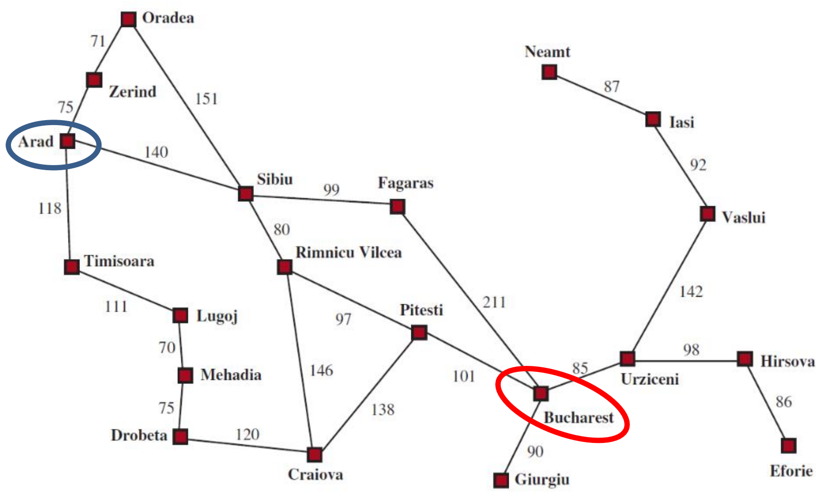


A* search - example



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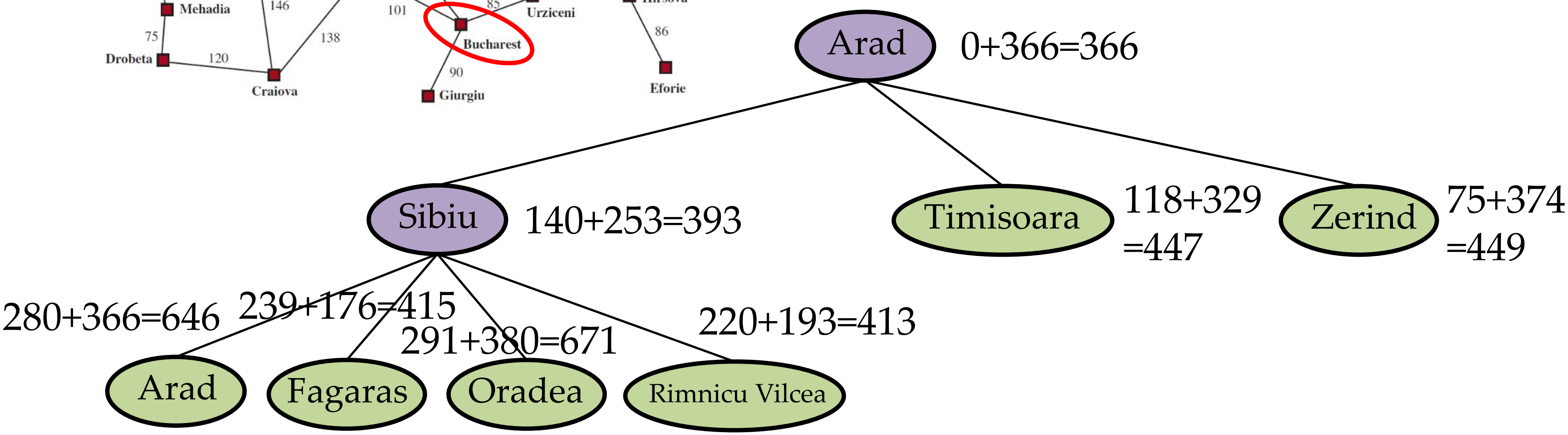
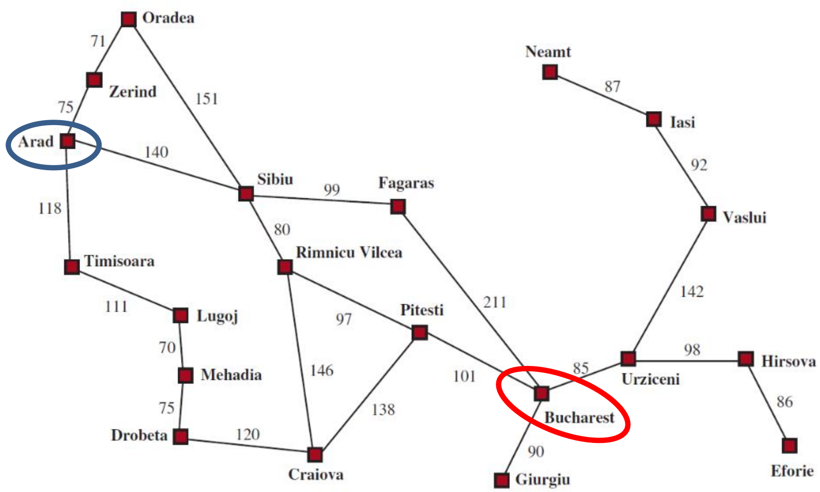


A* search - example



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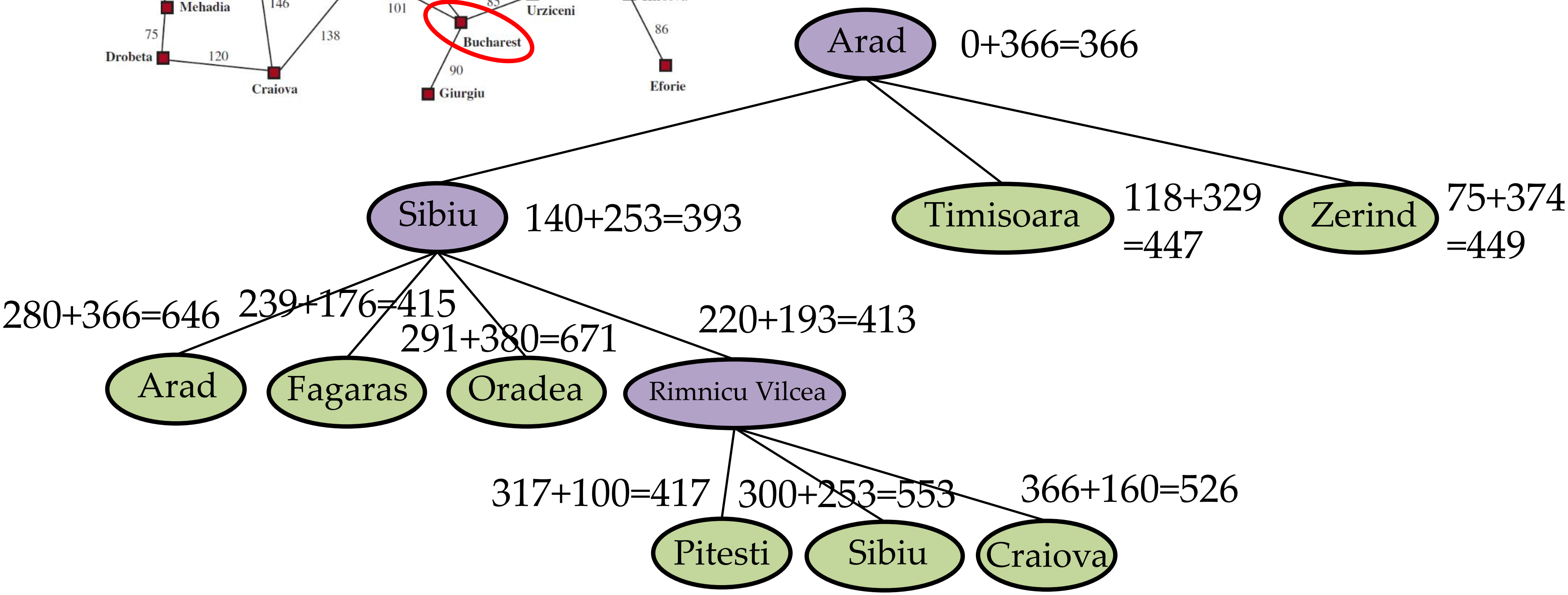
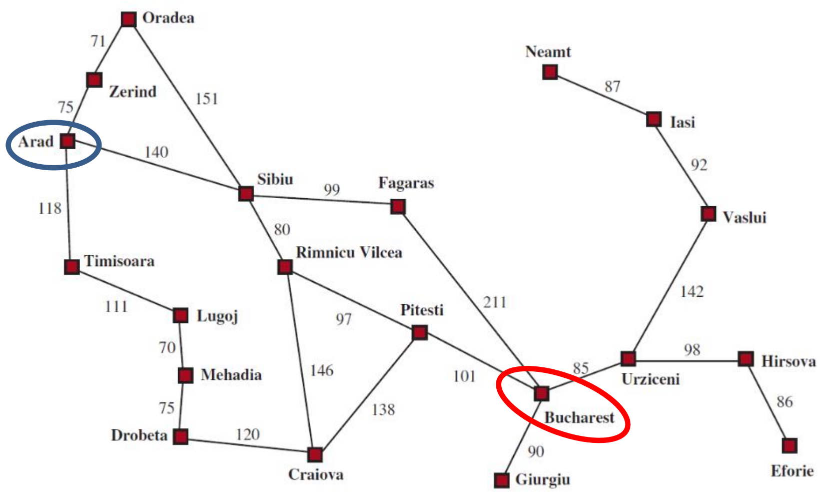


A* search - example



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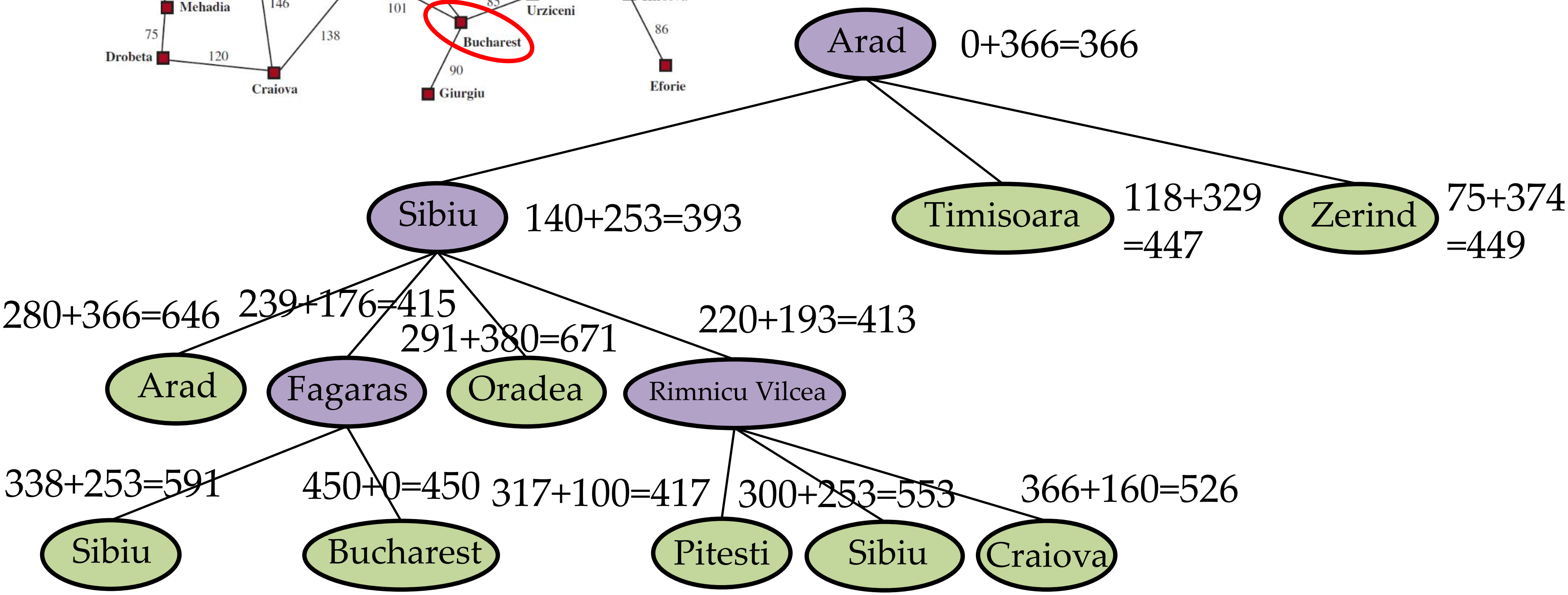
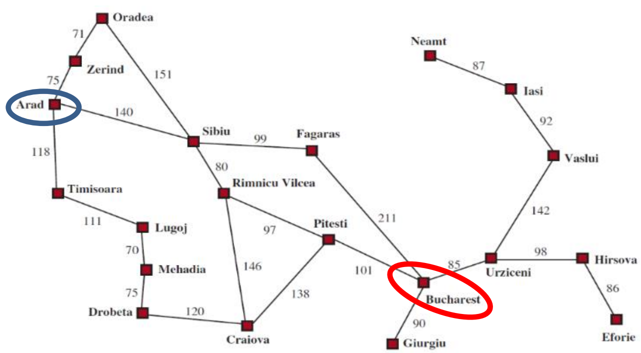


A* search - example



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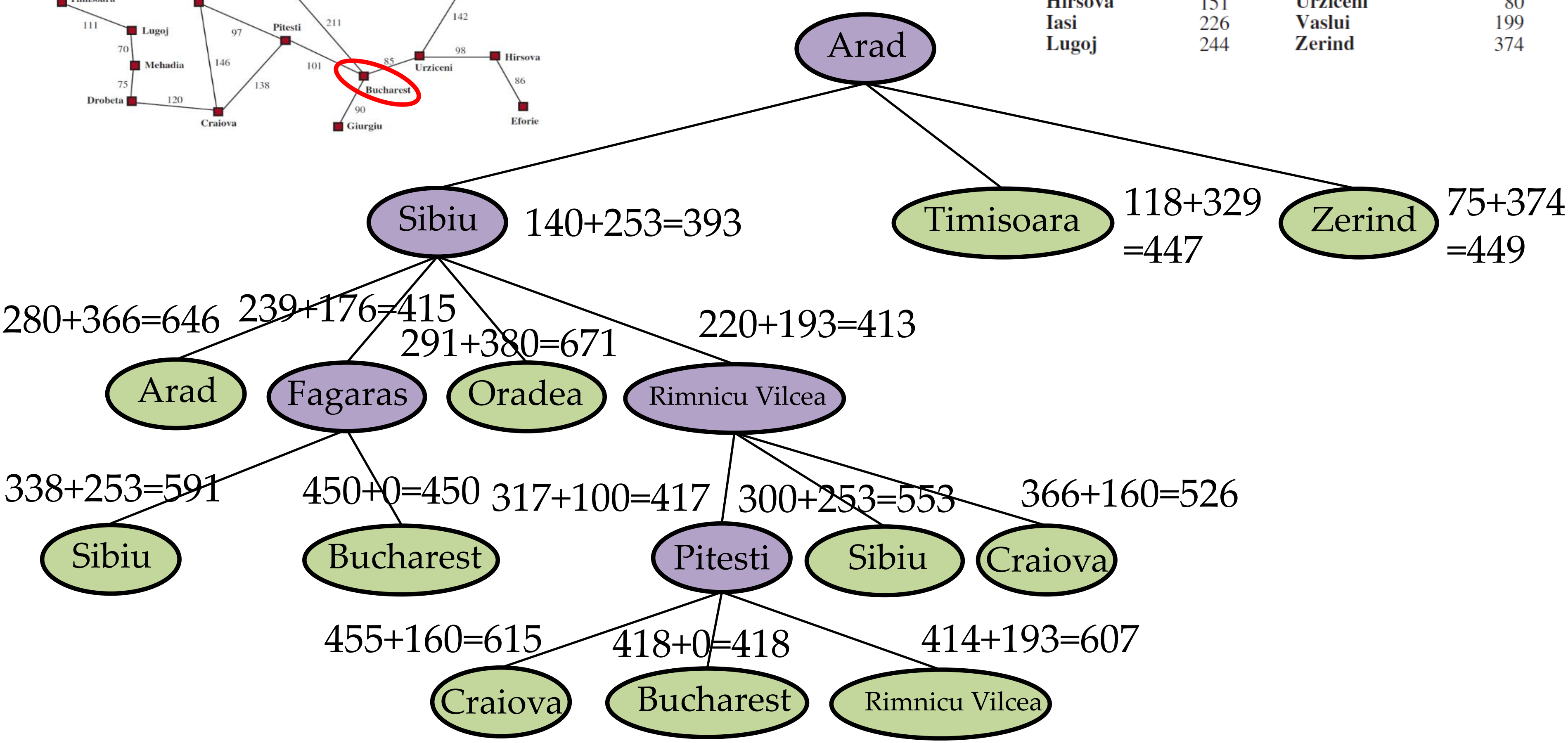
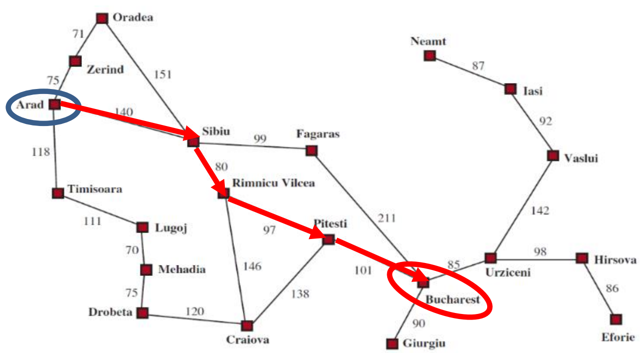


A* search - example



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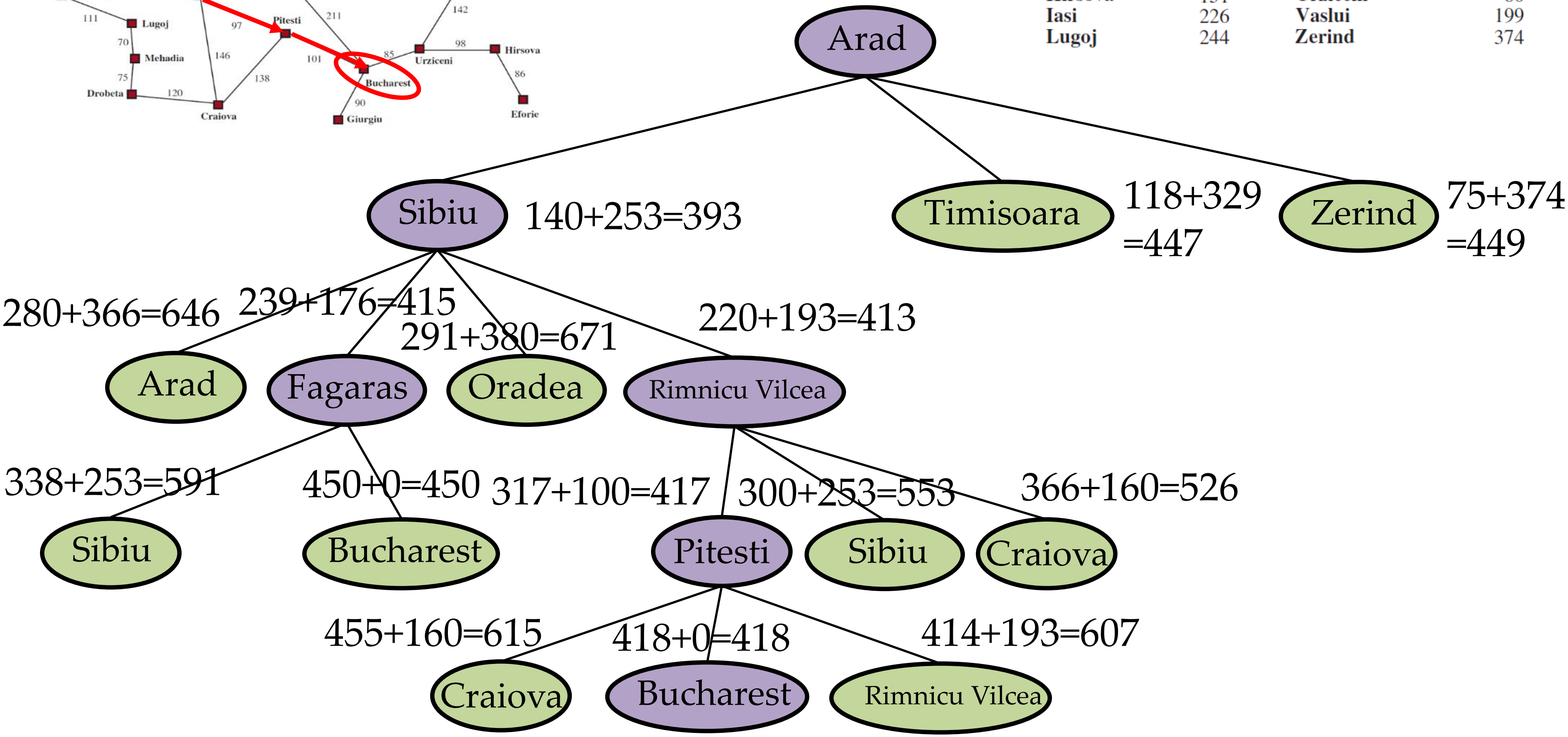


A* search - optimality

Main idea: expand the node n with the lowest value of

$$g(n) + h(n)$$

Path cost from the initial node to the node n

Estimated cost of the optimal path from the node n to a goal

- Optimal

  the tree-search version is optimal if $h(n)$ is admissible

A heuristic h is **admissible** if

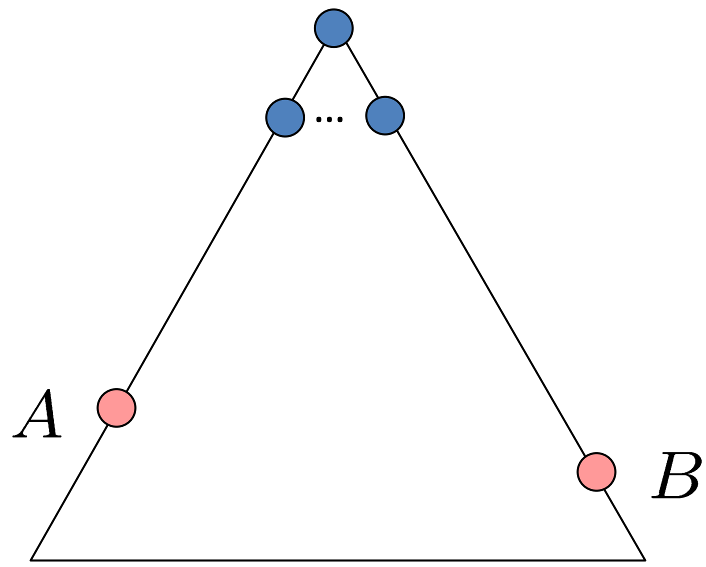
$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ is the true cost of the optimal path from n to a goal

Proof of optimality

Assume:

- A is an optimal goal node
- B is a suboptimal goal node
- h is admissible

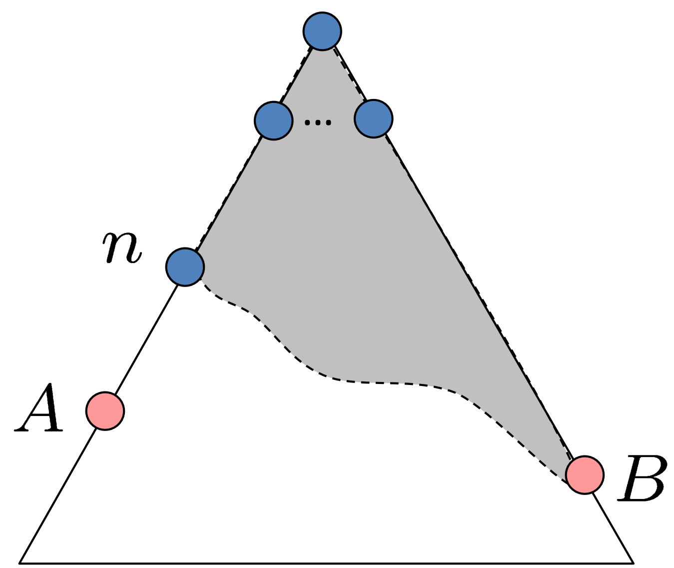


Claim:

- A will be expanded before B

Proof of optimality

- Currently, B is in the frontier
- Some ancestor n of A is in the frontier
- Claim: n will be expanded before B



To compare $g(n)+h(n)$ with $g(B)+h(B)$
and show $\boxed{g(n)+h(n) < g(B)+h(B)}$

Definition of A and B

$$g(A) < g(B)$$
$$h(A) = h(B) = 0$$

$$g(n)+h(n) \leq g(A)$$
$$= g(A)+h(A)$$

$$g(A)+h(A) < g(B)+h(B)$$

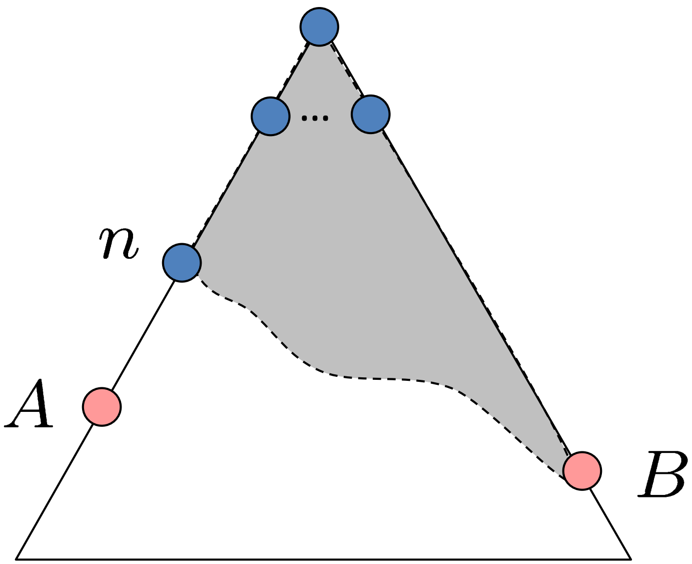
Admissibility of h $\quad h(A)=0$

Proof of optimality

- Currently, B is in the frontier
- Some ancestor n of A is in the frontier
- Claim: n will be expanded before B

$$\uparrow$$

$$g(n) + h(n) < g(B) + h(B)$$



All ancestors of A will be expanded before B

$$\Downarrow$$

A will be expanded before B

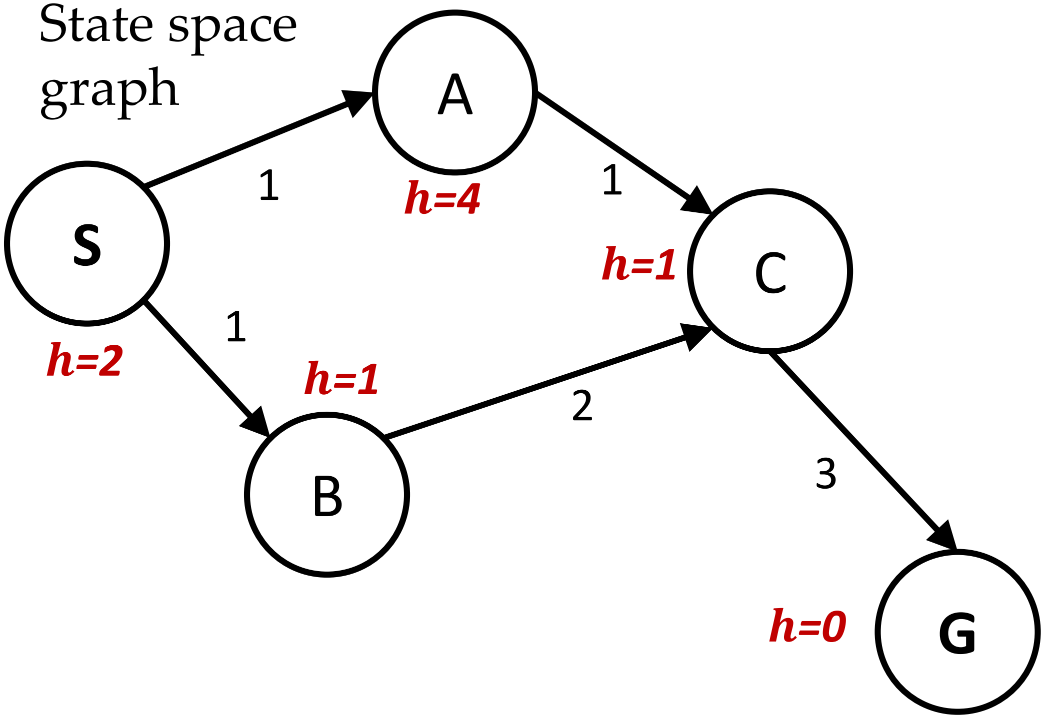
Optimal

A* search - optimality

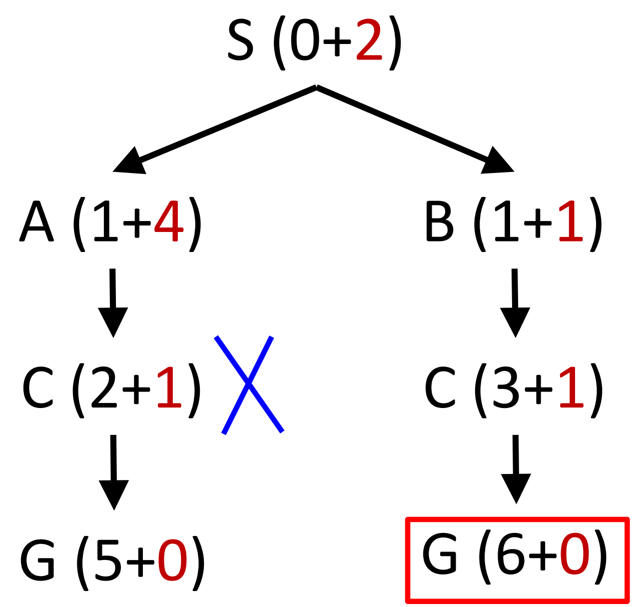
Admissible heuristic *h* $\Rightarrow$ The tree-search version of A* search is optimal

How about the graph-search version of A* search? Optimal?

State space graph



Search tree



A* search - optimality

Main idea: expand the node n with the lowest value of

$$g(n) + h(n)$$

Path cost from the initial node to the node n

Estimated cost of the optimal path from the node n to a goal

- Optimal

  the tree-search version is optimal if $h(n)$ is admissible

  the graph-search version is optimal if $h(n)$ is consistent

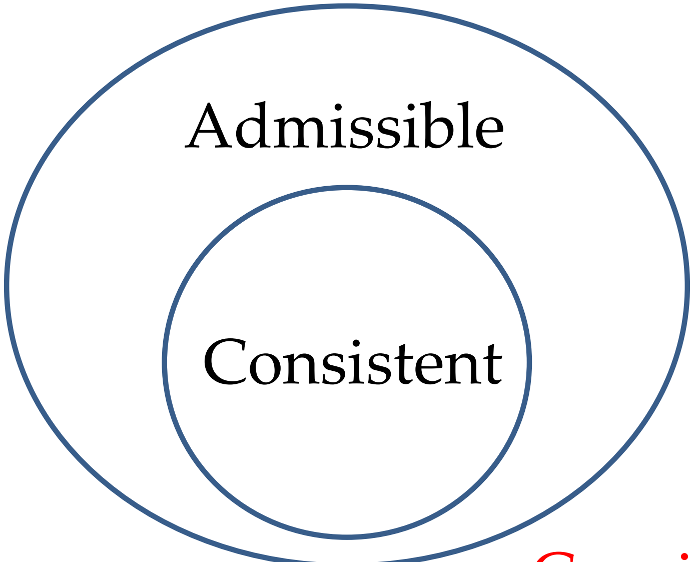
A heuristic h is **consistent** if

$$h(n) \leq c(n, a', n') + h(n')$$

where n' is the successor of n generated by action a'

Admissible and consistent heuristic

Consistent $\Rightarrow$ Admissible



- $h(n)$: estimated cost of the optimal path from the node n to a goal
- $h^*(n)$: true cost of the optimal path from the node n to a goal
- Assume that the nodes on the optimal path are $n, n', n'', \dots, n^*$

Consistent

$$h(n) \le c(n, a', n') + h(n') \le c(n, a', n') + c(n', a'', n'') + h(n'')$$

$$\le c(n, a', n') + c(n', a'', n'') + \cdots + h(n^*) = h^*(n)$$ Admissible

Consistent heuristic

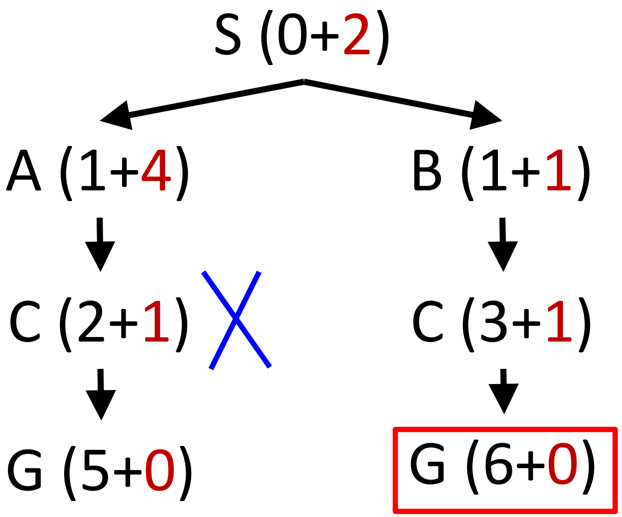
Claim: the values of $g(n) + h(n)$ along any path are non-decreasing

- Assume that n' is the successor of n generated by action a'

$$\begin{aligned} g(n') + h(n') &= g(n) + c(n, a', n') + h(n') \\ &\geq g(n) + h(n) \end{aligned}$$

Consistent

Search tree



Impossible for consistent heuristic

Proof of optimality

- The values of $g(n) + h(n)$ along any path are non-decreasing

Claim: when a goal node n is expanded, the optimal path to a goal has been found

Otherwise, there is another frontier node n' on the optimal path, whose corresponding goal node is denoted as n^*

$$g(n') + h(n') \leq g(n^*) + \overbrace{h(n^*)}^{0} < g(n) + \overbrace{h(n)}^{0}$$

$$\Downarrow$$

n' will be expanded before n, making a contradiction

A* search – performance

Main idea: expand the node n with the lowest value of

$$g(n) + h(n)$$

Path cost from the initial node to the node n

Estimated cost of the optimal path from the node n to a goal

- Optimal

  the tree-search version is optimal if $h(n)$ is admissible

  the graph-search version is optimal if $h(n)$ is consistent

Uniform-cost search: expand the node n with the lowest cost $g(n)$

Optimal

A special case of A* search with the heuristic $h(n) = 0 \longrightarrow$ Consistent

A* search – performance

Main idea: expand the node n with the lowest value of

$$g(n) + h(n)$$

- Optimal

  the tree-search version is optimal if $h(n)$ is admissible

  the graph-search version is optimal if $h(n)$ is consistent

- Complete (for consistent heuristic)

  if the branching factor b is finite and the cost (i.e., <u>the increment on $g + h$</u>) of every step exceeds some small positive constant ϵ

A* search expands nodes with non-decreasing values of $g(n) + h(n)$

A* search – performance

- Optimal

  the tree-search version is optimal if $h(n)$ is admissible

  the graph-search version is optimal if $h(n)$ is consistent

- Complete (for consistent heuristic)

  if the branching factor b is finite and "the cost" (i.e., <u>the increment on $g + h$</u>) of every step exceeds some small positive constant ϵ

- Complexity (for consistent heuristic)

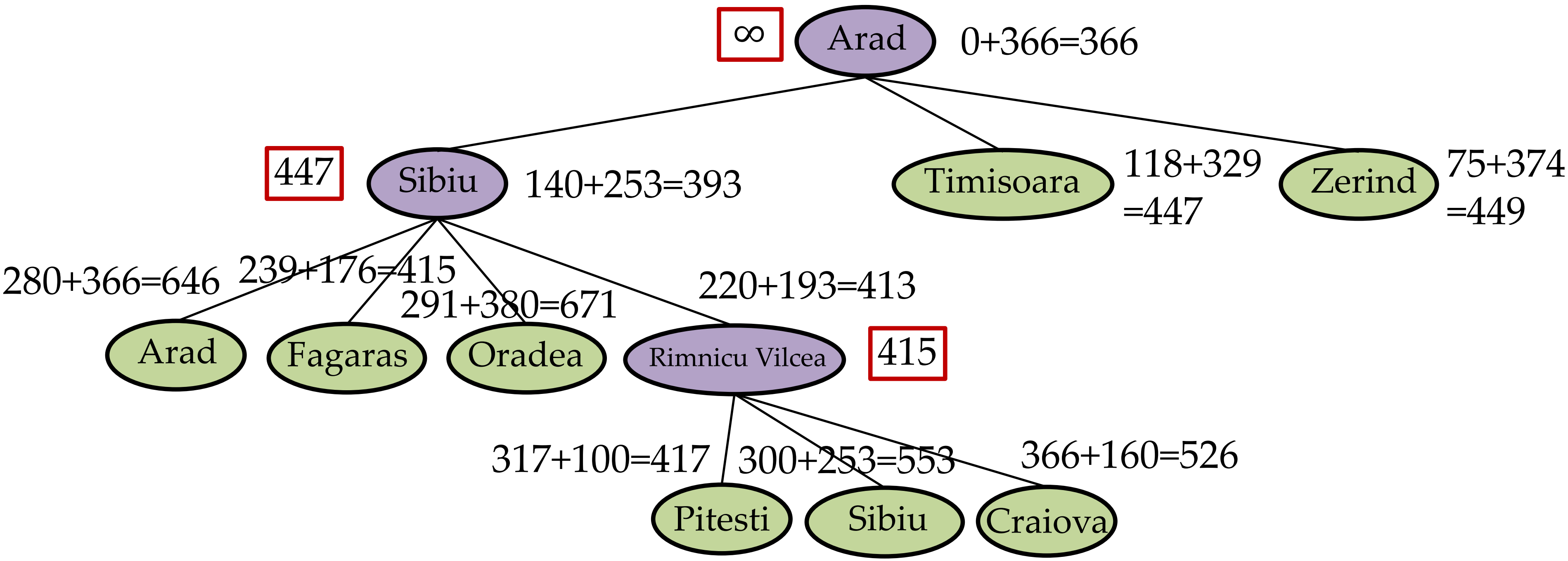
  expand all nodes with $g(n) + h(n) < C^*$ and some nodes with $g(n) + h(n) = C^*$, where C^* is the cost of the optimal path

  Time: optimally efficient for any given consistent heuristic

  Space: can still be exponential $\longrightarrow$ How to reduce?

Recursive best-first search

Main idea: for each node on the path, record the $g + h$ value of the best alternative path available from any ancestor of the node



Recursive best-first search

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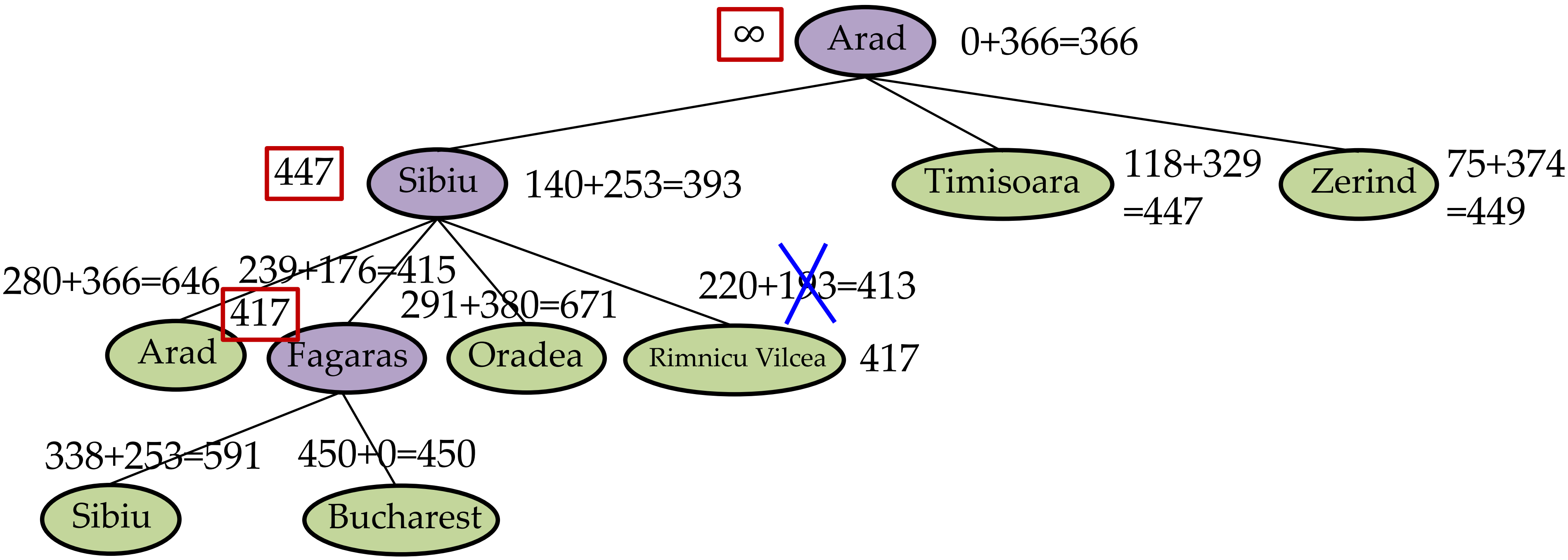
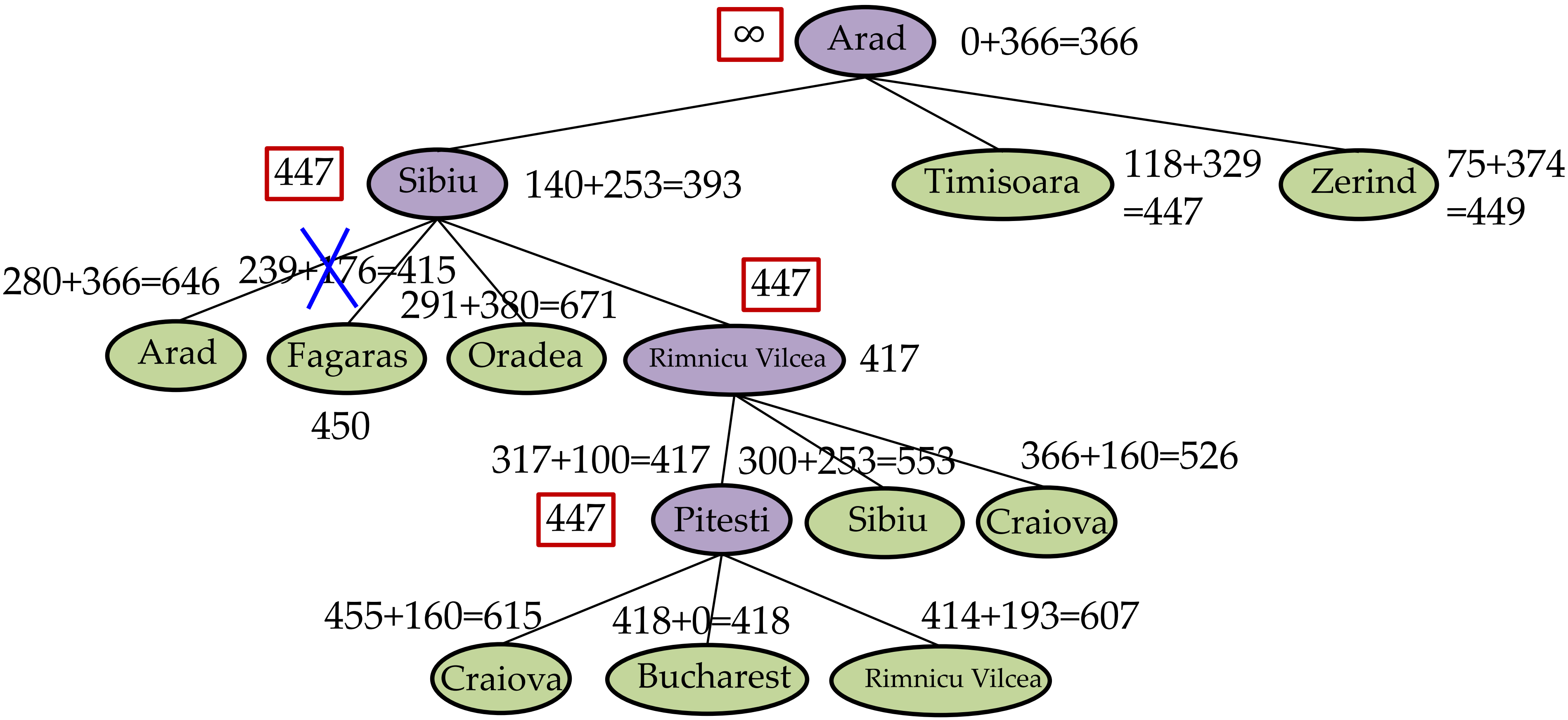


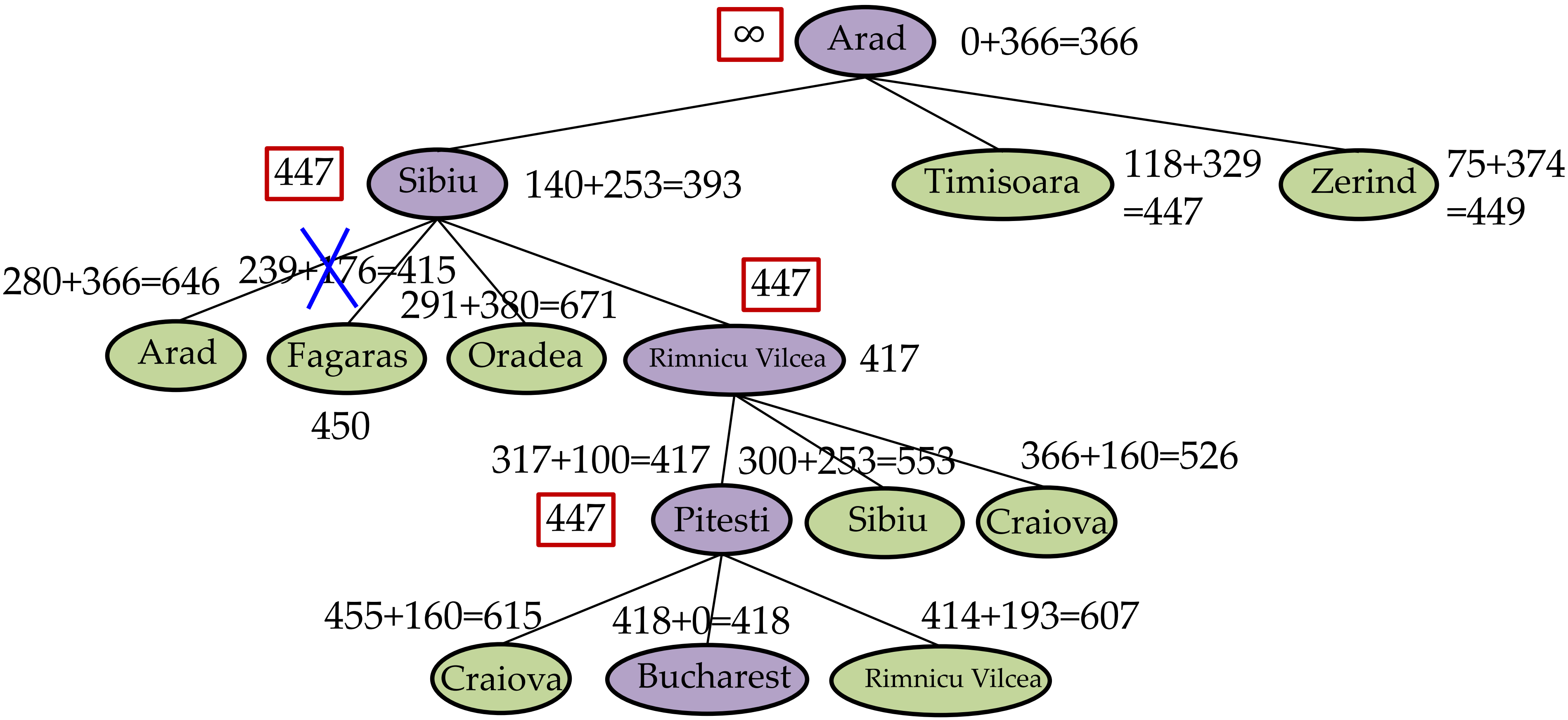
Recursive best-first search

Main idea: for each node on the path, record the $g + h$ value of the best alternative path available from any ancestor of the node



Recursive best-first search

Main idea: for each node on the path, record the $g + h$ value of the best alternative path available from any ancestor of the node



Recursive best-first search

Main idea: for each node on the path, record the $g + h$ value of the best alternative path available from any ancestor of the node;

if the best $g + h$ value of the generated nodes exceeds the recorded value of the current expanded node, unwind back to the alternative path

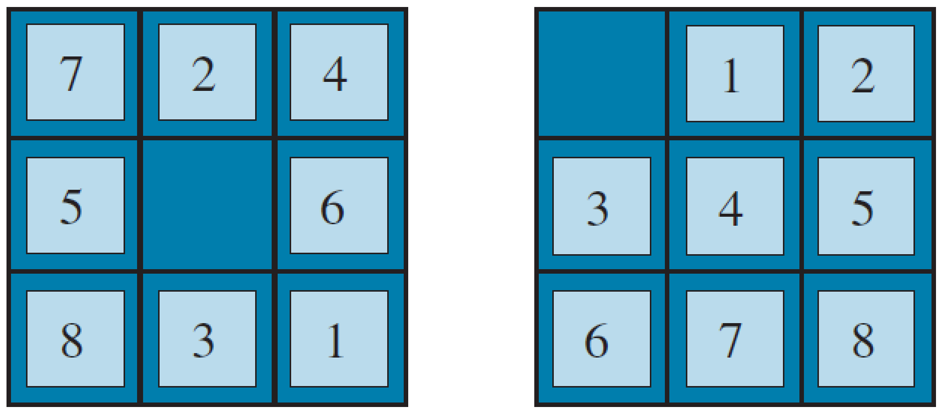
- The search process is the same as A* search
- The space complexity is linear in the depth of the deepest optimal path
- The time complexity increases because of node reexpansion

Heuristic generation from relaxed problems

Main idea: $h(n)$ = the cost of the optimal solution of a relaxed problem

Compared with the original problem, there are fewer restrictions on the actions

Example: 8-puzzle



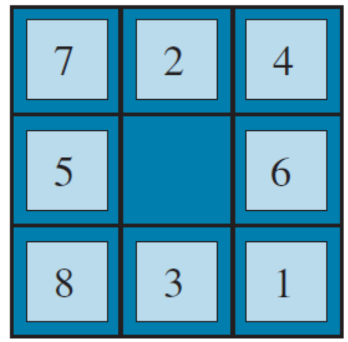
Start State Goal State

Heuristic generation from relaxed problems

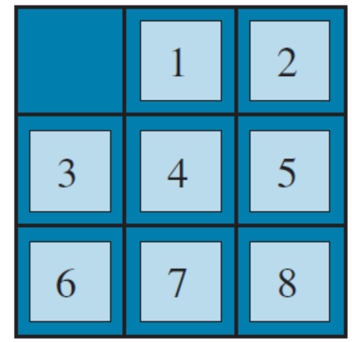
Example: 8-puzzle

$h_1 = 8$

$h_2 = 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2 = 18$



Start State



Goal State

A tile can move from square A to square B

Original: if A is adjacent to B and B is blank

Relaxed:

- if A is adjacent to B → h_2 = the sum of the $\boxed{\text{distances}}$ of the tiles from their goal positions
- if B is blank
- No restriction → h_1 = the number of misplaced tiles

Manhattan distance

Heuristic generation from relaxed problems

Main idea: $h(n)$ = the cost of the optimal solution of

$\boxed{\text{a relaxed problem}}$

Compared with the original problem, there are fewer restrictions on the actions

A supergraph of the original state space

$$h(n) \begin{cases} \text{Admissible: } 0 \leq h(n) \leq h^*(n) \\ \text{Consistent: } h(n) \leq c(n, a', n') + h(n') \end{cases}$$

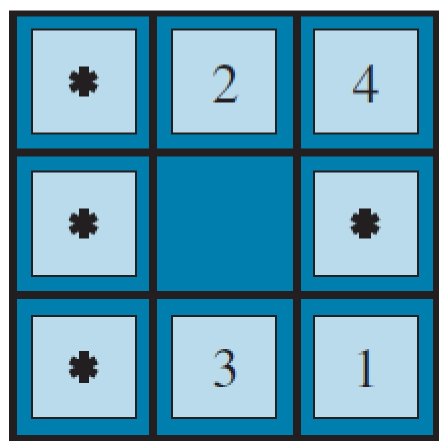
$h(n)$ should be easy to be computed

Heuristic generation from subproblems

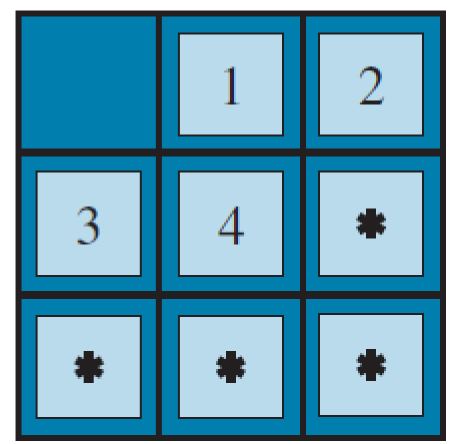
Main idea: $h(n)$ = the cost of the optimal solution of $\boxed{\text{a subproblem}}$

Compared with the original problem, there are more goal states, containing the original ones

Example: 8-puzzle



Start State



Goal State

Heuristic generation from subproblems

Main idea: $h(n)$ = the cost of the optimal solution of

a subproblem

Compared with the original problem, there are more goal states, containing the original ones

$$h(n) \begin{cases} \text{Admissible: } 0 \le h(n) \le h^*(n) \\ \text{Consistent: } h(n) \le c(n, a', n') + h(n') \end{cases}$$

Heuristic generation from new problems

Main idea: $h(n)$ = the cost of the optimal solution of a new problem

Consistent: $h(n) \le c(n, a', n') + h(n')$?

$\Uparrow$

The actions of the original problem can be performed for the new problem

Heuristic generation - example

Traveling salesman problem: to find the shortest route to visit each city exactly once, starting and ending in the same city

relax

Minimum spanning tree: to find a connected subgraph (connecting all the cites) with the minimum sum of edge weights

- Actions: *can go to non-visited cities, and return to the origin city at last*
- Goal states: *In(the start city), Visited({all the cities})*

relax

- Actions: *can go to any city, where the cost of an already performed action is 0*
- Goal states: *In(any city), Visited({all the cities})*

Heuristic generation by learning

The start state

The cost of the optimal solution

$\boldsymbol{x_1}$

7	2	4
	5	6
8	3	1

$\longrightarrow$ y_1

⋮

$\boldsymbol{x_n}$

5	2	4
7		3
8	1	6

$\longrightarrow$ y_n

Learning algorithms $\Rightarrow$ $h(n)$: not necessarily admissible

Goodness of heuristic

A* search $\begin{cases} \text{heuristic } h_1 \\ \\ \text{heuristic } h_2 \end{cases}$

Which one is good?

Performance measure: the number N of generated nodes for the same problem instance

⇕

Effective branching factor b^*

the depth of the optimal solution

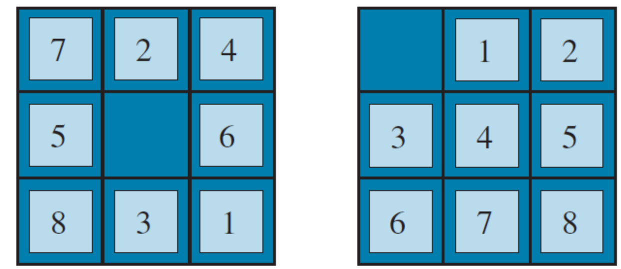
$$N = b^* + (b^*)^2 + \cdots + (b^*)^d$$

Goodness of heuristic

Example: 8-puzzle

h_1 = the number of misplaced tiles

h_2 = the sum of the distances of the tiles from their goal positions



Start State　　Goal State

	Effective branching factor b^*		
d	Iterative deepening DFS	$A^*(h_1)$	$A^*(h_2)$
2	2.45	1.79	1.79
4	2.87	1.48	1.45
6	2.73	1.34	1.30
8	2.80	1.33	1.24
10	2.79	1.38	1.22
12	2.78	1.42	1.24
14	-	1.44	1.23
16	-	1.45	1.25
18	-	1.46	1.26
20	-	-	-

For each d, the average of 100 random instances

h_2 is better than h_1

Informed search is better

Goodness of heuristic

A heuristic h_2 **dominates** another heuristic h_1 if

$$\forall n:\ h_2(n) \geq h_1(n)$$

- A* search with consistent heuristic

  expand all nodes with $g(n) + h(n) < C^*$ *and some nodes with* $g(n) + h(n) = C^*$*, where* C^* *is the cost of the optimal path*

$$g(n) + h_2(n) < C^* \Leftrightarrow h_2(n) < C^* - g(n)$$

$$\Downarrow$$

A* search with h_1 will generally expand more nodes $\Leftarrow$ $h_1(n) < C^* - g(n)$

Goodness of heuristic

A heuristic h_2 **dominates** another heuristic h_1 if

$$\forall n\colon\ h_2(n) \geq h_1(n)$$

$$\Downarrow$$

h_2 is generally better than h_1, i.e., A* search with h_2 will expand less nodes in general

Goodness of heuristic

$$\left.\begin{array}{l} h_1(n), h_2(n), \dots, h_m(n) \\ \text{Any two are non-dominated} \end{array}\right\} \begin{array}{l} \text{Composite heuristic:} \\ \max\{h_1(n), h_2(n), \dots, h_m(n)\} \end{array}$$

- The composite heuristic dominates any component heuristic $h_i(n)$

$$\forall n:\ \max\{h_1(n), h_2(n), \dots, h_m(n)\} \geq h_i(n)$$

- If $\forall i: h_i(n)$ is admissible, the composite heuristic is admissible

$$\forall n:\ \max\{h_1(n), h_2(n), \dots, h_m(n)\} \leq h^*(n)$$

- If $\forall i: h_i(n)$ is consistent, the composite heuristic is consistent

$$\max\{h_1(n), \dots, h_m(n)\} \leq c(n, a', n') + \max\{h_1(n'), \dots, h_m(n')\}$$

Summary

- Greedy best-first search
- A* search
- Recursive best-first search
- Heuristic generation
- Heuristic goodness

Informed (heuristic) search

Uses problem-specific knowledge beyond the problem definition

References

- S. J. Russell and P. Norvig. Artificial Intelligence: A Modern Approach. Chapter 3.5-3.6, Third edition.

Assignment - 1

Task: apply uninformed/informed search algorithms to play the game of Pacoban-2players

Deadline: Oct. 23