Last class

- Renewal process
- Elementary renewal theorem
- Key renewal theorem
- Alternating renewal process
- Delayed renewal process
- Renewal reward process
- Symmetric random walk

References: Chapter 3, Stochastic Processes, 2nd edition, 1995, *by Sheldon M. Ross*

Stochastic Processes

Lecture 4: Markov Chains

Chao Qian （钱超）

Associate Professor, Nanjing University, China

Email: qianc@nju.edu.cn

Homepage: http://www.lamda.nju.edu.cn/qianc/

Markov chain

A stochastic process $\{X(t), t \in T\}$ with state space S is said to be a **Markov chain** if $\forall t_1 < t_2 < \cdots < t_n < t$, $x, x_i \in S$

$$P(\, X(t) = x \mid X(t_1) = x_1, \dots, X(t_n) = x_n \,)$$
$$= P(X(t) = x \mid X(t_n) = x_n)$$

Markovian property

Here, we consider discrete-time discrete-state $\boxed{\text{homogeneous}}$ Markov chains

$\{X_n, n = 0,1,2, \dots\}$ $\quad S = \{0,1,2, \dots\}$, unless otherwise mentioned

$$\boxed{\begin{array}{c}\forall t_0 < t, x_0, x \in S\text{: } P(\, X(t) = x \mid X(t_0) = x_0 \,) \\ \text{is independent of } t_0\text{, but depends only on } t - t_0\end{array}}$$

Markov chain

Example [General Random Walk]: Let $X_i, i \geq 1$ be iid with

$$P(X_i = j) = a_j, \qquad j \in \{0, \pm 1, \pm 2, \dots\}$$

If we let

$$S_0 = 0 \qquad\qquad S_n = \sum_{i=1}^{n} X_i$$

then $\{S_n, n \geq 0\}$ is a *Markov chain* for which

$$P_{ij} = a_{j-i}$$

One-step transition probability: $P_{ij} = P(S_{n+1} = j \mid S_n = i)$

Markov chain

Example [Simple Random Walk]: The random walk $\{S_n, n \geq 0\}$, where $S_n = \sum_{i=1}^{n} X_i$, is said to be a *simple random walk* if for some p, $0 < p < 1$,

$$P(X_i = 1) = p \qquad P(X_i = -1) = q = 1 - p$$

The absolute value $\{|S_n|, n \geq 0\}$ of the simple random walk is a *Markov chain*.

$$P(|S_{n+1}| = i + 1 \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \dots, |S_1| = i_1) \; ?$$

Markov chain

Lemma: If $\{S_n, n \geq 0\}$ is a simple random walk, then $\forall i > 0$

$$\underbrace{P(\, S_n = i \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \dots, |S_1| = i_1 \,)}_{\star} = \frac{p^i}{p^i + q^i}$$

Proof: Let $i_0 = 0$, and define $j = \max\{k: i_k = 0, 0 \leq k \leq n\}$

$$\Rightarrow P(\, S_n = i \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \dots, |S_1| = i_1 \,)$$
$$= P\left(\, S_n = i \,\middle|\, |S_n| = i, |S_{n-1}| = i_{n-1}, \dots, \left|S_j\right| = 0 \,\right)$$

Now there are two possible cases for $\left|S_{j+1}\right| = i_{j+1}, \dots, |S_{n-1}| = i_{n-1}, |S_n| = i$

Case 1: $S_n = i$, then $S_{n-1} = i_{n-1}, \dots, S_{j+1} = i_{j+1}$ and has probability

$$p^{\frac{n-j}{2}+\frac{i}{2}} \cdot q^{\frac{n-j}{2}-\frac{i}{2}} \quad \left(\tfrac{n-j}{2} + \tfrac{i}{2} \text{ take the value of 1, } \tfrac{n-j}{2} - \tfrac{i}{2} \text{ take the value of -1}\right)$$

Case 2: $S_n = -i$, then $S_{n-1} = -i_{n-1}, \dots, S_{j+1} = -i_{j+1}$ and has probability

$$p^{\frac{n-j}{2}-\frac{i}{2}} \cdot q^{\frac{n-j}{2}+\frac{i}{2}} \quad \left(\tfrac{n-j}{2} - \tfrac{i}{2} \text{ take the value of 1, } \tfrac{n-j}{2} + \tfrac{i}{2} \text{ take the value of -1}\right)$$

$$\Rightarrow \star = \frac{p^{\frac{n-j}{2}+\frac{i}{2}} \cdot q^{\frac{n-j}{2}-\frac{i}{2}}}{p^{\frac{n-j}{2}+\frac{i}{2}} \cdot q^{\frac{n-j}{2}-\frac{i}{2}} + p^{\frac{n-j}{2}-\frac{i}{2}} \cdot q^{\frac{n-j}{2}+\frac{i}{2}}} = \frac{p^i}{p^i + q^i}$$

Markov chain

$$P(|S_{n+1}| = i+1 \mid |S_n| = i, |S_{n-1}| = i_{n-1}, \ldots, |S_1| = i_1)$$

$$= P(|S_{n+1}| = i+1 \mid S_n = i) \cdot \frac{p^i}{p^i+q^i}$$

Law of total probability

$$+P(|S_{n+1}| = i+1 \mid S_n = -i) \cdot \frac{q^i}{p^i+q^i}$$

$$= \frac{p^{i+1}+q^{i+1}}{p^i+q^i}$$

Hence, $\{|S_n|, n \geq 0\}$ is a Markov chain with transition probabilities

$$P_{i,i+1} = \frac{p^{i+1}+q^{i+1}}{p^i+q^i} = 1 - P_{i,i-1}, \quad i > 0$$

$$P_{01} = 1$$

Chapman-Kolmogorov equations

For a Markov chain $\{X_n, n = 0,1,2,\ldots\}$,

- one-step transition probability: $P_{ij} = P(X_{m+1} = j \mid X_m = i)$
- n-step transition probabilities:

$$P_{ij}^n = P(X_{m+n} = j \mid X_m = i)$$

How to compute it?

Chapman-Kolmogorov equations:

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \qquad \text{for all } n, m \geq 0, i, j$$

Chapman-Kolmogorov equations

Chapman-Kolmogorov equations:

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m} \qquad \text{for all } n, m \geq 0, i, j$$

Proof: $P_{ij}^{n+m} = P(X_{n+m} = j \mid X_0 = i)$ (Homogeneous by default)

Law of total probability

$$= \sum_{k=0}^{\infty} P(X_{n+m} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P(X_{n+m} = j \mid X_n = k, X_0 = i) P(X_n = k \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P_{kj}^{m} P_{ik}^{n}$$

Chapman-Kolmogorov equations

- n-step transition probabilities:

$$P_{ij}^n = P(X_{m+n} = j \mid X_m = i)$$

How to compute it?

Chapman-Kolmogorov equations:

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \qquad \text{for all } n, m \geq 0, i, j$$

Solution: Let $P^{(n)}$ denote the matrix of n-step transition probability P_{ij}^n, then by Chapman-Kolmogorov equations:

$$P^{(m+n)} = P^{(n)} \cdot P^{(m)}$$

Hence,

$$P^{(n)} = P \cdot P^{(n-1)} = P \cdot P \cdot P^{(n-2)} = \cdots = P^n$$

Communication

- State j is said to be **accessible** from state i if for some $n \geq 0$, $P_{ij}^n > 0$

- Two states i and j accessible to each other are said to **communicate**, denoted as $i \leftrightarrow j$

Proposition: Communication is an equivalence relation, i.e.,

✓ $i \leftrightarrow i$ (Follows trivially from definition)

✓ If $i \leftrightarrow j$, then $j \leftrightarrow i$ (Follows trivially from definition)

✓ If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$

$\Downarrow$ $\Downarrow$

$\exists m$, s.t. $P_{ij}^m > 0$, $\exists n$, s.t. $P_{jk}^n > 0$, $P_{ik}^{m+n} = \sum_{r=0}^{\infty} P_{ir}^m P_{rk}^n \geq P_{ij}^m P_{jk}^n > 0 \Longrightarrow i \rightarrow k$

Similarly, we can show $k \rightarrow i$

Irreducible

- Two states that communicate are said to be **in the same class**

the equivalence relation of communication ⇒ any two classes are either disjoint or identical

- A Markov chain is **irreducible** if there is only one class

⇓

All states communicate with each other

Period

- A state j has **period** d if d is the greatest common divisor of the number of transitions by which j can be reached, starting from j

$$d(j) = \gcd\{n > 0 : P_{jj}^n > 0\}$$

period of j

- If $P_{jj}^n = 0$ for all $n > 0$, then $d(j) = \infty$

- A state with period 1 is said to be **aperiodic**

Period

Proposition: If $i \leftrightarrow j$, then $d(i) = d(j)$

Proof: $i \leftrightarrow j \Rightarrow P_{ij}^{m} P_{ji}^{n} > 0$ for some m and n

Suppose $P_{ii}^{s} > 0$, then

$$\left.\begin{aligned} P_{jj}^{n+m} &\geq P_{ji}^{n} P_{ij}^{m} > 0 \\ P_{jj}^{n+s+m} &\geq P_{ji}^{n} P_{ii}^{s} P_{ij}^{m} > 0 \end{aligned}\right\} \Longrightarrow$$

$d(j)$ divides both $n+m$ and $n+s+m$

$\Downarrow$

$d(j)$ divides s

So, if $P_{ii}^{s} > 0$, then $d(j)$ divides s.

$P_{ii}^{d(i)} > 0$ is obvious, so $d(j)$ divides $d(i)$.

A similar argument yields that $d(i)$ divides $d(j)$.

$\Longrightarrow$ $d(i) = d(j)$

Recurrent

For any states i and j, define f_{ij}^n to be the probability that, starting in i, the first transition into j occurs at time n

$$f_{ij}^0 = 0$$

$$f_{ij}^n = P(X_n = j, X_k \neq j, k = 1,2,\dots,n-1 \mid X_0 = i)$$

Let f_{ij} denote the probability of ever making a transition into state j, given that the process starts in i

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$$

State j is said to be **recurrent** if $f_{jj} = 1$, and **transient** otherwise

Recurrent

Proposition: State j is recurrent if and only if $\sum_{n=1}^{\infty} P_{jj}^{n} = \infty$

Proof: j is recurrent $\Rightarrow$ with probability 1, return to j
Markov property $\Rightarrow$ once returning to j, the process restarts
So, with probability 1, the number of visits to j is ∞

$$\Rightarrow E[\textit{number of visits to } j \mid X_0 = j] = \infty$$

j is transient $\Rightarrow$ the number of visits to j is geometric with mean $\frac{1}{1-f_{jj}}$

Thus, j is recurrent if and only if

$$E[\textit{number of visits to } j \mid X_0 = j] = \infty$$

Let $I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{otherwise} \end{cases}$

$$\Longrightarrow E[\textit{number of visits to } j \mid X_0 = j] = E[\textstyle\sum_{n=0}^{\infty} I_n | X_0 = j]$$
$$= \textstyle\sum_{n=0}^{\infty} E[I_n | X_0 = j] = \sum_{n=0}^{\infty} P_{jj}^{n}$$

Recurrent

Corollary: If i is recurrent and $i \leftrightarrow j$, then j is recurrent

Proof: $i \leftrightarrow j \Rightarrow \exists m, n$ such that $P_{ij}^{n} > 0, P_{ji}^{m} > 0$

$\forall s > 0, P_{jj}^{m+n+s} \geq P_{ji}^{m} P_{ii}^{s} P_{ij}^{n}$

$$\Rightarrow \sum_{s=1}^{\infty} P_{jj}^{m+n+s} \geq P_{ji}^{m} P_{ij}^{n} \sum_{s=1}^{\infty} P_{ii}^{s} = \infty$$

$$\Rightarrow \sum_{s=1}^{\infty} P_{jj}^{s} = \infty \Rightarrow j \text{ is recurrent}$$

By Proposition on the previous page

Recurrent

Example [Simple Random Walk]: The random walk $\{S_n, n \geq 0\}$, where $S_n = \sum_{i=1}^{n} X_i$, is said to be a *simple random walk* if for some p, $0 < p < 1$,

$$P(X_i = 1) = p \qquad P(X_i = -1) = q = 1 - p$$

Which states are transient? Which are recurrent?

Solution:

All states communicates ⇒ they are either all transient or all recurrent

Only need to consider state 0 i.e., if $\sum_{n=1}^{\infty} P_{00}^{n}$ is finite or not

$P_{00}^{2n+1} = 0, n = 0,1,2, \ldots$

$P_{00}^{2n} = C_{2n}^{n} p^n (1-p)^n = \frac{(2n)!}{(n!)^2} p^n (1-p)^n, n = 1,2, \ldots$

Stirling's approximation: $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \Rightarrow P_{00}^{2n} \sim \frac{\left(4p(1-p)\right)^n}{\sqrt{\pi n}}$

Note that $a_n \sim b_n$ when $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

$$\sum_{n=1}^{\infty} \frac{\left(4p(1-p)\right)^n}{\sqrt{\pi n}} \begin{cases} p = \frac{1}{2}, 4p(1-p) = 1 \Rightarrow \sum_{n=1}^{\infty} P_{00}^{n} = \infty \Rightarrow \text{recurrent} \\ p \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right), 4p(1-p) < 1 \Rightarrow \sum_{n=1}^{\infty} P_{00}^{n} < \infty \Rightarrow \text{transient} \end{cases}$$

Recurrent

Corollary: If j is recurrent and $i \leftrightarrow j$, then $f_{ij} = 1$

Proof: $i \leftrightarrow j \Rightarrow \exists n, P_{ij}^n > 0$

$i \leftrightarrow j, j$ is recurrent $\Rightarrow i$ is recurrent

Suppose $X_0 = i$, let T_1 denote the next time we enter i (T_1 is finite by Corollary)

$X_n = j$ with probability P_{ij}^n

The number of above process needed to access state j is a geometric random variable with mean $1/P_{ij}^n$, and is thus finite with probability 1

i is recurrent $\Rightarrow$ the number of above process is infinite

$\Longrightarrow f_{ij} = 1$

Positive and Null Recurrent

Let μ_{jj} denote the expected number of transitions needed to return to state j

$$\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \displaystyle\sum_{n=1}^{\infty} n f_{jj}^{n} & \text{if } j \text{ is recurrent} \end{cases}$$

If state j is recurrent, then we say that it is **positive recurrent** if $\mu_{jj} < \infty$ and **null recurrent** if $\mu_{jj} = \infty$

Limit theorems

Let $N_j(t)$ denote the number of transitions into j by time t

By interpreting transitions into state j as being renewals,

Theorem: If $i \leftrightarrow j$, then

- ✓ $P\left(\lim_{t\to\infty} \frac{N_j(t)}{t} = \frac{1}{\mu_{jj}} \mid X_0 = i\right) = 1$ (With probability 1, $\frac{N_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$)

$$I_k = \begin{cases} 1 & \text{if } X_k = j \\ 0 & \text{otherwise} \end{cases}$$

- ✓ $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^{k}}{n} = \frac{1}{\mu_{jj}}$

($\frac{m_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty, m_D(t) = E\left[\sum_{k=1}^{t} I_k\right] = \sum_{k=1}^{t} P_{ij}^{k}$)

- ✓ If j is aperiodic, then $\lim_{n\to\infty} P_{ij}^{n} = \frac{1}{\mu_{jj}}$ (Blackwell's Theorem)
- ✓ If j has period d, then $\lim_{n\to\infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}}$ (Blackwell's Theorem)

Delayed renewal process

Properties of delayed renewal process:

$$\mu = \int_0^\infty x dF(x)$$

- With probability 1, $\frac{N_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$
- $\frac{m_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ Elementary Renewal Theorem
- If F is not lattice, then $m_D(t+a) - m_D(t) \to a/\mu$ as $t \to \infty$
- If F and G are lattice with period d, then Blackwell's Theorem

$$E[\#\text{renewals at } nd] \to d/\mu \quad \text{as } n \to \infty$$

- If F is not lattice, $\mu < \infty$ and $h(t)$ is directly Riemann integrable,

$$\int_0^\infty h(t-x) dm_D(x) = \frac{1}{\mu}\int_0^\infty h(t)dt$$ Key Renewal Theorem

Positive and Null Recurrent

If state j is recurrent, then we say that it is **positive recurrent** if $\mu_{jj} < \infty$ and **null recurrent** if $\mu_{jj} = \infty$

$$\Updownarrow \qquad \pi_j = \lim_{n\to\infty} P_{jj}^{nd(j)} = \frac{d(j)}{\mu_{jj}}$$

If state j is recurrent, then we say that it is **positive recurrent** if $\pi_j > 0$ and **null recurrent** if $\pi_j = 0$

Positive and Null Recurrent

Proposition: If i is positive (null) recurrent and $i \leftrightarrow j$, then j is positive (null) recurrent

Proof: Case 1: positive recurrent

$$i \leftrightarrow j \Rightarrow d(i) = d(j) = d \geq 1$$

$$\pi_i = \lim_{n\to\infty} P_{ii}^{nd} = \frac{d}{\mu_{ii}} > 0$$

$$i \leftrightarrow j \Rightarrow \exists s, t \geq 0, P_{ij}^{s} > 0, P_{ji}^{t} > 0$$

$$P_{jj}^{t+s+md} \geq P_{ji}^{t} P_{ii}^{md} P_{ij}^{s}$$

$$\lim_{m\to\infty} P_{jj}^{t+s+md} \geq P_{ji}^{t} P_{ij}^{s} \cdot \lim_{m\to\infty} P_{ii}^{md} = \frac{d}{\mu_{ii}} \cdot P_{ji}^{t} P_{ij}^{s} > 0$$

$$P_{jj}^{t+s} \geq P_{ji}^{t} P_{ij}^{s} > 0 \Rightarrow d \text{ divides } t+s$$

$$\pi_j = \lim_{m\to\infty} P_{jj}^{t+s+md} > 0 \Rightarrow j \text{ is positive recurrent}$$

For the null recurrent case, leave as the exercise

Stationary distribution

Definition: A probability distribution $\{\pi_j, j \geq 0\}$ is said to be **stationary** for the Markov chain if

$$\forall j:\ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

If the initial distribution, i.e., the distribution of X_0, is a stationary distribution, X_n will have the same distribution for all n.

Proof:

$$P(X_1 = j) = \sum_{i=0}^{\infty} P(X_1 = j \mid X_0 = i)\, P(X_0 = i) = \sum_{i=0}^{\infty} P_{ij}\, \pi_i = \pi_j$$

Definition of stationary

$$P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j \mid X_{n-1} = i)\, P(X_{n-1} = i) = \sum_{i=0}^{\infty} P_{ij}\, \pi_i = \pi_j$$

Stationary distribution

Theorem: An irreducible aperiodic Markov chain belongs to one of the following two classes

- Either the states are all transient or all null recurrent. In this case, $P_{ij}^n \to 0$ as $n \to \infty$ for all i, j and there exists no stationary distribution

- Or else, all states are positive recurrent, that is,

$$\pi_j = \lim_{n\to\infty} P_{ij}^n > 0$$

  In this case, $\{\pi_j, j = 0,1,2, \dots,\}$ is a stationary distribution and there exists no other stationary distribution

Proof:

Proof for the first class:

transient or null recurrent $\Rightarrow \mu_{jj} = \infty$. By Limit Theorem, $\lim_{n\to\infty} P_{ij}^n = \frac{1}{\mu_{jj}} = 0$

Stationary distribution

Suppose there exists a stationary distribution P_j, then

$$P_j = P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j | X_0 = i) P(X_0 = i) = \sum_{i=0}^{\infty} P_{ij}^n P_i$$

$$= \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_{ij}^n P_i \leq \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_i$$

Let $n \to \infty$, we have $P_j \leq \sum_{i=M+1}^{\infty} P_i$. Then, let $M \to \infty$, we have $P_j \leq 0$, which leads to a contradiction

Proof for the second class:

Note that $P_{ij}^{n+1} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj} \geq \sum_{k=0}^{M} P_{ik}^n P_{kj}$ for all M

Let $n \to \infty$, we have $\pi_j \geq \sum_{k=0}^{M} \pi_k P_{kj}$,

then let $M \to \infty$, we have $\pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj}$

Suppose $\exists j$, such that $\pi_j > \sum_{k=0}^{\infty} \pi_k P_{kj}$, then

$$\sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi_k P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = \sum_{k=0}^{\infty} \pi_k,$$

which leads to a contradiction. Thus, $\forall j$: $\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}$

Stationary distribution

Suppose P_j is a stationary distribution, then

$$P_j = P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j \mid X_0 = i) P(X_0 = i) = \sum_{i=0}^{\infty} P_{ij}^n P_i$$

- $P_j \geq \sum_{i=0}^{M} P_{ij}^n P_i$ for all M

  Let $n \to \infty$, we have $P_j \geq \sum_{i=0}^{M} \pi_j P_i$,

  then let $M \to \infty$, we have $P_j \geq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j$

- $P_j \leq \sum_{i=0}^{M} P_{ij}^n P_i + \sum_{i=M+1}^{\infty} P_i$ for all M

  Let $n \to \infty$, we have $P_j \leq \sum_{i=0}^{M} \pi_j P_i + \sum_{i=M+1}^{\infty} P_i$,

  then let $M \to \infty$, we have $P_j \leq \sum_{i=0}^{\infty} \pi_j P_i = \pi_j$

Thus, $\forall j: P_j = \pi_j$

Stationary distribution

- For an irreducible, positive recurrent and aperiodic Markov chain, $\{\pi_j, j = 0,1,2, \ldots, \}$ is the unique stationary distribution, where

$$\pi_j = \lim_{n\to\infty} P_{ij}^n = \frac{1}{\mu_{jj}}$$

- For an irreducible, positive recurrent and periodic Markov chain (where the period is d), $\{\pi_j = \frac{1}{\mu_{jj}}, j = 0,1,2, \ldots, \}$ is still the unique stationary distribution

$$\lim_{n\to\infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}} = d\pi_j$$

Transitions among classes

Proposition: Let R be a recurrent class of states. If $i \in R$, $j \notin R$, then $P_{ij} = 0$.

Proof: Suppose $P_{ij} > 0$

Then, as i and j do not communicate (since $j \notin R$)

$\Rightarrow P_{ji}^{n} = 0, \forall n$

Hence, if the process starts in state i, there is a positive

probability of at least P_{ij} that the process will never return to i

$\Rightarrow$ contradicts the fact that i is recurrent

So $P_{ij} = 0$

Transitions among classes

Proposition: If j is recurrent, then the set of probabilities $\{f_{ij}, i \in T\}$ satisfies

$$\forall i \in T: f_{ij} = \sum_{k \in T} P_{ik} f_{kj} + \sum_{k \in R} P_{ik}$$

where T denotes the set of all transient states, and R denotes the set of states communicating with j

Proof:

$$\begin{aligned} f_{ij} &= P\big(N_j(\infty) > 0 \mid X_0 = i\big) \\ &= \textstyle\sum_k P\big(N_j(\infty) > 0 \mid X_0 = i, X_1 = k\big) P(X_1 = k \mid X_0 = i) \\ &= \textstyle\sum_{k \in T} f_{kj} P_{ik} + \sum_{k \in R} f_{kj} P_{ik} + \sum_{k \notin R, k \notin T} f_{kj} P_{ik} \\ &= \textstyle\sum_{k \in T} f_{kj} P_{ik} + \sum_{k \in R} P_{ik} \end{aligned}$$

k belongs to a recurrent class that is different from R, thus $f_{kj} = 0$

Gambler's ruin problem

Gambler's ruin problem: Consider a gambler who at each play of the game has probability p of winning 1 unit and probability $q = 1 - p$ of losing 1 unit. Assuming successive plays of the game are independent.

What is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?

Solution: X_n: the player's fortune at time n

$\{X_n, n = 0,1,2, \dots\}$: a Markov chain with transition probabilities

$P_{00} = P_{NN} = 1 \qquad P_{i,i+1} = p = 1 - P_{i,i-1} \qquad i = 1,2, \dots, N-1$

$\{0\}$	$\{1,2,\dots,N-1\}$	$\{N\}$
recurrent class	transient class	recurrent class

Gambler's ruin problem

Let $f_i = f_{i,N}$ denote the probability that, starting with i, $1 \le i \le N$, the fortune will eventually reach N

just the desired probability

$$f_i = pf_{i+1} + qf_{i-1} \qquad i = 1,2,\dots,N-1 \Longrightarrow f_{i+1} - f_i = \frac{q}{p}(f_i - f_{i-1})$$

Then, $f_2 - f_1 = \frac{q}{p}(f_1 - f_0) = \frac{q}{p} f_1, f_3 - f_2 = \frac{q}{p}(f_2 - f_1) = \left(\frac{q}{p}\right)^2 f_1, \dots,$

$$f_i - f_{i-1} = \frac{q}{p}(f_{i-1} - f_{i-2}) = \left(\frac{q}{p}\right)^{i-1} f_1$$

Thus, $f_i = f_1 + f_1\left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} f_1 & \text{if } \frac{q}{p} \neq 1 \\ i f_1 & \text{if } \frac{q}{p} = 1 \end{cases}$

By $f_N = 1$, $f_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases} \xRightarrow{N \to \infty} f_i \to \begin{cases} 1-(q/p)^i & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leqslant \frac{1}{2} \end{cases}$

Gambler's ruin problem

What is the expected number of bets that the gambler, starting at i, makes before reaching either 0 or n?

Solution: X_j: the winnings on the jth bet

B: the number of bets until the fortune reaches either 0 or n

$$B = \min\left\{m: \sum_{j=1}^{m} X_j = -i \text{ or } \sum_{j=1}^{m} X_j = n - i\right\}$$

B is a stopping time for X_j, then by Wald's equation,

$E\left[\sum_{j=1}^{B} X_j\right] = E\left[X_j\right]E[B] = (2p-1)E[B]$

By $\sum_{j=1}^{B} X_j = \begin{cases} n - i & \text{with prob. } \frac{1-(q/p)^i}{1-(q/p)^N} \\ -i & \text{otherwise} \end{cases}$ $\Rightarrow$ $E[B] = \frac{1}{2p-1}\left\{\frac{n\left[1-(q/p)^i\right]}{1-(q/p)^n} - i\right\}$

(here we consider $p \neq 1/2$)

Transitions among transient states

$T = \{1, 2, \dots, t\}$: the set of transient states

How about the probability $f_{i,j}$, where both i and j are transient?
i.e., $i, j \in T$

the probability of ever making a transition into state j given that the chain starts in state i

For $i, j \in T$, $m_{i,j}$: the expected total number of time periods spent in state j given that the chain starts in state i

$$m_{i,j} = m_{j,j} \cdot f_{i,j} \quad \Rightarrow \quad f_{i,j} = m_{i,j} / m_{j,j}$$

How to compute $m_{i,j}$?

Transitions among transient states

$$m_{i,j} = \delta(i,j) + \sum_k P_{i,k}\, m_{k,j} = \delta(i,j) + \sum_{k=1}^{t} P_{i,k}\, m_{k,j}$$

$$\begin{cases} 1 & \text{if } \; i = j \\ 0 & \text{otherwise} \end{cases}$$

$$m_{k,j} = 0 \text{ for } k \notin T$$

transition probabilities among transient states

$$Q = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & \ddots & \vdots \\ P_{t1} & \cdots & P_{tt} \end{bmatrix} \qquad M = \begin{bmatrix} m_{11} & \cdots & m_{1t} \\ \vdots & \ddots & \vdots \\ m_{t1} & \cdots & m_{tt} \end{bmatrix}$$

$$M = I + QM \quad \Rightarrow \quad M = (I - Q)^{-1}$$

Transitions among transient states

Example: Consider the gambler's ruin problem with $p = 0.4$ and $N = 6$. Starting in state 3, determine

- the expected amount of time spent in state 3 $m_{3,3}$
- the expected number of visits to state 2 $m_{3,2}$
- the probability of ever visiting state 4 $f_{3,4}$

Leave as the exercise

Equivalent to f_3 under $N = 4$

Branching processes

Branching processes: Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual will, by the end of its lifetime, **have produced j new offspring with probability $P_j, j \geq 0$,** independently of the number produced by any other individual. **Let X_n denote the size of the nth generation.** The Markov chain $\{X_n, n \geq 0\}$ is called a **branching process**

Suppose that $X_0 = 1$ $\quad\quad$ $\pi_0 = \lim_{n\to\infty} P(X_n = 0)$

Let π_0 denote the probability that the population ever dies out

$$\pi_0 = P(\text{population dies out})$$

$$= \sum_{j=0}^{\infty} P(\text{population dies out} \mid X_1 = j)P_j = \sum_{j=0}^{\infty} \pi_0^j P_j$$

Branching processes

Theorem: Suppose that $P_0 > 0$ and $P_0 + P_1 < 1$. Then,

- π_0 is the smallest positive number satisfying $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$
- $\pi_0 = 1$ if and only if $\mu \leq 1$, where $\mu = \sum_{j=0}^{\infty} jP_j$ is the mean number of offspring produced by each individual

Proof: Let $\pi \geq 0$ satisfy $\pi = \sum_{j=0}^{\infty} \pi^j P_j$, prove $\pi \geq P(X_n = 0)$ for all n

$\pi = \sum_{j=0}^{\infty} \pi^j P_j \geq \pi^0 P_0 = P_0 = P(X_1 = 0)$

Assume that $\pi \geq P(X_n = 0)$, then

$$P(X_{n+1} = 0) = \sum_{j=0}^{\infty} P(X_{n+1} = 0 \mid X_1 = j) P_j$$
$$= \sum_{j=0}^{\infty} \left(P(X_n = 0)\right)^j P_j \leq \sum_{j=0}^{\infty} \pi^j P_j = \pi$$

Hence, $\pi \geq P(X_n = 0)$ for all n

Let $n \to \infty \Rightarrow \pi \geq \lim_{n\to\infty} P(X_n = 0) = \pi_0$

The proof of the second point is left as the exercise

Time-reversible Markov chains

Stationary Markov chain: An irreducible positive recurrent Markov chain is **stationary** if the initial state is chosen according to the stationary probabilities

The **reversed process** of a stationary Markov chain is also a Markov chain with transition probabilities given by

$$P_{ij}^* = \frac{\pi_j P_{ji}}{\pi_i}$$

Proof: $P(X_m = j \mid X_{m+1} = i, X_{m+2} = i_2, \ldots, X_{m+k} = i_k)$

$$= \frac{P(X_m=j, X_{m+1}=i, X_{m+2}=i_2, \ldots, X_{m+k}=i_k)}{P(X_{m+1}=i, X_{m+2}=i_2, \ldots, X_{m+k}=i_k)}$$

$$= \frac{P(X_{m+2}=i_2, \ldots, X_{m+k}=i_k \mid X_m=j, X_{m+1}=i) P(X_m=j, X_{m+1}=i)}{P(X_{m+2}=i_2, \ldots, X_{m+k}=i_k \mid X_{m+1}=i) P(X_{m+1}=i)}$$

$$= \frac{P(X_m=j, X_{m+1}=i)}{P(X_{m+1}=i)} = \frac{P(X_{m+1}=i \mid X_m=j) P(X_m=j)}{P(X_{m+1}=i)} = \frac{\pi_j P_{ji}}{\pi_i}$$

Stationary

Time-reversible Markov chains

[Definition] Time-reversible Markov chain: A stationary Markov chain is **time-reversible** if $\forall i, j$

$$P_{ij}^* = P_{ij} \quad \Longleftrightarrow \quad \pi_i P_{ij} = \pi_j P_{ji}$$

$$P_{ij}^* = \frac{\pi_j P_{ji}}{\pi_i}$$

[Necessary and Sufficient Condition]: A stationary Markov chain is *time-reversible* if and only if, starting in state i, any path back to i has the same probability as the reversed path for all i. That is, $\forall i, i_1, \ldots, i_k$:

$$P_{ii_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k i} = P_{i i_k} P_{i_k i_{k-1}} \cdots P_{i_2 i_1} P_{i_1 i}$$

Time-reversible Markov chains

Proof: [Necessary Condition]

Time-reversible: $\pi_i P_{ij} = \pi_j P_{ji}$

$$\Downarrow$$

$$P_{ii_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k i} = P_{i i_k} P_{i_k i_{k-1}} \cdots P_{i_2 i_1} P_{i_1 i}$$

$$\boxed{\pi_i} P_{ii_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k i}$$

$$= P_{i_1 i} \pi_{i_1} P_{i_1 i_2} \cdots P_{i_{k-1} i_k} P_{i_k i}$$

$$= P_{i_1 i} P_{i_2 i_1} \pi_{i_2} \cdots P_{i_{k-1} i_k} P_{i_k i}$$

$$= P_{i_1 i} P_{i_2 i_1} \cdots P_{i_k i_{k-1}} \pi_{i_k} P_{i_k i}$$

$$= P_{i_1 i} P_{i_2 i_1} \cdots P_{i_k i_{k-1}} P_{i i_k} \boxed{\pi_i}$$

Eliminate π_i on both sides, finish the proof

Time-reversible Markov chains

Proof: [Sufficient Condition]

$$P_{ii_1}P_{i_1i_2}\cdots P_{i_{k-1}i_k}P_{i_ki} = P_{ii_k}P_{i_ki_{k-1}}\cdots P_{i_2i_1}P_{i_1i}$$

$$\Downarrow$$

Time-reversible: $\pi_i P_{ij} = \pi_j P_{ji}$

$$P_{ii_1}P_{i_1i_2}\cdots P_{i_{k-1}i_k}P_{i_kj}P_{ji} = P_{ij}P_{ji_k}P_{i_ki_{k-1}}\cdots P_{i_2i_1}P_{i_1i}$$

$\Downarrow$ Summing over all states $i_1, i_2, \ldots, i_k$

$$P_{ij}^{k+1}\,P_{ji} = P_{ij}P_{ji}^{k+1}$$

$$\Downarrow$$

$$\frac{1}{n}\left(\sum_{k=1}^{n} P_{ij}^{k+1}\right)P_{ji} = \frac{1}{n}\left(\sum_{k=1}^{n} P_{ji}^{k+1}\right)P_{ij}$$

$$\Downarrow$$

Let $n \to \infty$ $\quad \pi_j P_{ji} = \pi_i P_{ij}$

Note that
$\lim_{n\to\infty} a_n = a \Rightarrow$
$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a$

Time-reversible Markov chains

Theorem: Consider an irreducible Markov chain with transition probabilities P_{ij}. If one can find nonnegative numbers π_i, $i \geq 0$, summing to unity, and a transition probability matrix $\boldsymbol{P}^* = [P_{ij}^*]$ such that

$$\pi_i P_{ij} = \pi_j P_{ji}^*$$

then π_i, $i \geq 0$ are the stationary probabilities of the original chain, and P_{ij}^* are the transition probabilities of the reverse chain

Proof: $\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji}^* = \pi_j \sum_i P_{ji}^* = \pi_j$

$\Rightarrow \pi_i$, $i \geq 0$ are the stationary probabilities of the original chain

$P_{ji}^* = \frac{\pi_i P_{ij}}{\pi_j}$ are the transition probabilities of the reverse chain

Leave as the exercise

π_i, $i \geq 0$ are also the stationary probabilities of the reverse chain

Markov chain Monte Carlo

Suppose $X \in \{x_i, i \geq 1\}$ is a discrete random variable with probability distribution $\pi_i = P(X = x_i)$, and h is a function

Problem: How to calculate $E[h(X)] = \sum_i h(x_i)\pi_i$?

Monte Carlo Method: draw samples $X_1, X_2, \ldots, X_n$ from the probability distribution of X, use $\frac{1}{n}\sum_{i=1}^{n} h(X_i)$ to estimate $E[h(X)]$

Practical situations: π_i can be calculated, but hard to be sampled

Problem: How to generate a set of independent samples of X?

Markov chain Monte Carlo

Theorem: If $\{X_n, n \geq 0\}$ is an irreducible Markov chain with stationary distribution π_i, and h is a bounded function over the state space $\{x_i, i \geq 1\}$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h(X_i) = E[h(X)] = \sum_{i} h(x_i)\pi_i$$

Now we only need to construct an irreducible Markov chain with stationary distribution being the desired probability distribution

Proof: Let $a_{i(n)}$ denote the number of transitions into x_i by time n

$\frac{1}{n}\sum_{i=1}^{n} h(X_i) = \sum_i \frac{a_{i(n)}}{n} h(x_i)$

With probability 1, $\frac{a_{i(n)}}{n} \to \frac{1}{\mu_{ii}} = \pi_i$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h(X_i) = \sum_i h(x_i)\pi_i$

Since there is already a stationary distribution, the MC must be positive recurrent

Markov chain Monte Carlo

Theorem: Suppose $\{\pi_i, i \in S\}$ is a probability distribution, there exists a time-reversible Markov chain $\{X_n, n \geq 0\}$ with state space S and stationary distribution π_i

Proof:

Target: construct P such that $\pi_i P_{ij} = \pi_j P_{ji}$ ☆

W.l.o.g., we assume $S = \{0,1,\ldots\}$, let Q be the transition probability matrix of an irreducible Markov chain such that

$$\forall i \neq j, Q_{ij} = 0 \Leftrightarrow Q_{ji} = 0$$

Now we construct P as follows:

Markov chain Monte Carlo

$$Q_{ij} = 0 \quad \Rightarrow \quad \alpha_{ij} = 1$$

$$Q_{ij} > 0 \quad \Rightarrow \quad \alpha_{ij} = \min\left\{\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right\}$$

$$P_{ij} = Q_{ij}\alpha_{ij}$$

$$P_{ii} = Q_{ii} + \sum_{j \neq i} Q_{ij}(1 - \alpha_{ij})$$

P is a transition probability matrix such that $\forall i \neq j, P_{ij} = 0 \Leftrightarrow P_{ji} = 0$, and the MC w.r.t. P is irreducible

Now we examine ☆ for $j \neq i$ (the case $j = i$ is trivial)

case 1: $\alpha_{ij} < 1$, then $\alpha_{ji} = 1$ by the definition of α_{ij}, thus

$$\pi_i P_{ij} = \pi_i Q_{ij}\alpha_{ij} = \pi_j Q_{ji} = \pi_j Q_{ji}\alpha_{ji} = \pi_j P_{ji}$$

case 2: $\alpha_{ij} = 1$, then $\pi_j Q_{ji} \geq \pi_i Q_{ij}$ and $\alpha_{ji} \leq 1$, thus

$$\pi_i P_{ij} = \pi_i Q_{ij} = \pi_j Q_{ji}\alpha_{ji} = \pi_j P_{ji}$$

Thus, ☆ holds, which implies

- π_i is the stationary distribution of the MC w.r.t. to P (sum over i)
- the MC w.r.t. to P is time-reverse

Metropolis sampling

$$Q_{ij} = 0 \Rightarrow \alpha_{ij} = 1$$

$$Q_{ij} > 0 \Rightarrow \alpha_{ij} = \min\left\{\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right\}$$

$$P_{ij} = Q_{ij}\alpha_{ij}$$

$$P_{ii} = Q_{ii} + \sum_{j \neq i} Q_{ij}(1 - \alpha_{ij})$$

Metropolis Sampling:

We need to set a transition probability matrix $\boldsymbol{Q}$

1. X_0 *is initialized with any value*
2. *Suppose the current state* $X_k = i$
3. *Sample a random number* j *from the probability distribution* $\{Q_{ij}, j \geq 0\}$
4. *If* $\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}} \geq 1$, *then* $X_{k+1} = j$ *and go to step 2*
5. *Otherwise, sample a random number* r *from the uniform distribution* $U(0,1)$. *If* $r \leq \frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}$, *then* $X_{k+1} = j$, *otherwise* $X_{k+1} = i$. *Go to step 2*

Gibbs sampling

Suppose $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ is a discrete random variable, and S is the set of all possible values of $\boldsymbol{Z}$

Assumption 1: for all $\boldsymbol{z} \in S$,

$$\pi_{\boldsymbol{z}} = P(\boldsymbol{Z} = \boldsymbol{z}) = c \cdot g(\boldsymbol{z})$$

where $c > 0$

Assumption 2: for all $1 \leq i \leq n$, *and* z_j, $1 \leq j \leq n$, $j \neq i$, the conditional probability distribution

$$P\left(Z_i = \cdot \mid Z_j = z_j \ \forall j \neq i\right)$$

exists and is known

Gibbs sampling

Set a specific transition probability matrix $\boldsymbol{Q}$

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are different on at least two dimensions, $Q_{\boldsymbol{xy}} = 0$
- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are different on only one dimension, denoted as i,

$$Q_{\boldsymbol{xy}} = \frac{1}{n} P\left(Z_i = y_i \mid Z_j = x_j \ \forall j \neq i \right) = \frac{cg(\boldsymbol{y})}{nP(Z_j = x_j \ \forall j \neq i)}$$

- If $\boldsymbol{x} = \boldsymbol{y}$, then

$$Q_{\boldsymbol{xx}} = 1 - \sum_{\boldsymbol{y} \neq \boldsymbol{x}} Q_{\boldsymbol{xy}} = 1 - \frac{1}{n} \sum_{i=1}^{n} \left(1 - P\left(Z_i = x_i \mid Z_j = x_j \ \forall j \neq i \right)\right)$$

$$= \frac{cg(\boldsymbol{x})}{n} \sum_{i=1}^{n} \frac{1}{P(Z_j = x_j \ \forall j \neq i)}$$

✓ $\forall \boldsymbol{x} \neq \boldsymbol{y}$: $Q_{\boldsymbol{xy}} = 0$ iff $Q_{\boldsymbol{yx}} = 0$

✓ The Markov chain w.r.t. $\boldsymbol{Q}$ is irreducible

Gibbs sampling

$$Q_{ij} = 0 \Rightarrow \alpha_{ij} = 1$$

$$Q_{ij} > 0 \Rightarrow \alpha_{ij} = \min\left\{\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right\}$$

$$P_{ij} = Q_{ij}\alpha_{ij}$$

$$P_{ii} = Q_{ii} + \sum_{j \neq i} Q_{ij}(1 - \alpha_{ij})$$

$$Q_{xy} > 0 \Rightarrow \alpha_{xy} = \min\left\{\frac{\pi_y Q_{yx}}{\pi_x Q_{xy}}, 1\right\} = \min\left\{\frac{cg(\boldsymbol{y}) \cdot cg(\boldsymbol{x})}{cg(\boldsymbol{x}) \cdot cg(\boldsymbol{y})}, 1\right\} = 1$$

$$\left.\begin{array}{l} \forall \boldsymbol{x} \neq \boldsymbol{y}: P_{\boldsymbol{xy}} = Q_{\boldsymbol{xy}}\alpha_{\boldsymbol{xy}} = Q_{\boldsymbol{xy}} \\ P_{\boldsymbol{xx}} = Q_{\boldsymbol{xx}} + \sum_{\boldsymbol{y} \neq \boldsymbol{x}} Q_{\boldsymbol{xy}}(1 - \alpha_{\boldsymbol{xy}}) = Q_{\boldsymbol{xx}} \end{array}\right\} \quad \boldsymbol{P} = \boldsymbol{Q}$$

Gibbs sampling

Gibbs Sampling:

1. X_0 *is initialized with any* $\boldsymbol{x}_0 \in S$
2. *Suppose the current state* $X_k = \boldsymbol{x} = (x_1, \dots, x_n) \in S$
3. *Sample a random number* i *uniformly from* $\{1,2,\dots,n\}$
4. *Sample a random value* x *from the conditional probability distribution*

$$P\left(Z_i = \cdot \mid Z_j = x_j \ \forall j \neq i\right)$$

5. $X_{k+1} = (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$. *Go to step 2*

Thus, Gibbs sampling is actually Metropolis sampling with a specific matrix Q, under some assumptions about the desired probability distribution

Semi-Markov processes

Consider a stochastic process with states 0,1, ..., which is such that, whenever it enters state $i, i \geq 0$:

- The next state it will enter is state j with probability $P_{ij}, i, j \geq 0$
- Given that the next state to be entered is state j, the time until the transition from i to j occurs has distribution F_{ij}

If we let $Z(t)$ denote the state at time t, then $\{Z(t), t \geq 0\}$ is called a **semi-Markov process**

✓ A semi-Markov process does not possess the Markovian property

✓ A Markov chain is a semi-Markov process in which

$$F_{ij}(t) = \begin{cases} 0 & t < 1 \\ 1 & t \geq 1 \end{cases}$$

Semi-Markov processes

Let X_n denote the nth state visited, then $\{X_n, n \geq 0\}$ with transition probabilities P_{ij} is called the $\boxed{\text{embedded Markov chain}}$ of the semi-Markov process

Proposition: If the semi-Markov process is irreducible and if T_{ii} has a nonlattice distribution with finite mean, then

$$P_i = \lim_{t \to \infty} P(\, Z(t) = i \mid Z(0) = j \,) = \frac{\mu_i}{\mu_{ii}}, \qquad \forall i, j$$

- τ_i: time that the process spends in state i before making a transition

$$\mu_i = E[\tau_i] \qquad\qquad P(\tau_i \leq t) = \sum_j P_{ij} F_{ij}(t)$$

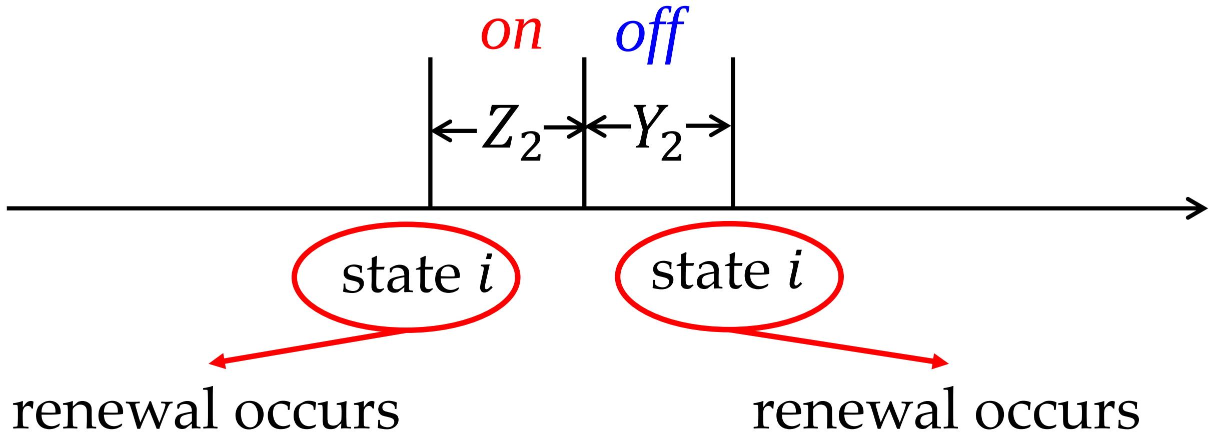
- T_{ii}: time between successive transitions into state i $\quad \mu_{ii} = E[T_{ii}]$

Semi-Markov processes

Proposition: If the semi-Markov process is irreducible and if T_{ii} has a nonlattice distribution with finite mean, then

$$P_i = \lim_{t \to \infty} P(\, Z(t) = i \mid Z(0) = j \,) = \frac{\mu_i}{\mu_{ii}}, \qquad \forall i, j$$

Proof: **A delayed alternating renewal process**



on: in state i

off: not in state i

Semi-Markov processes

Proposition: If the semi-Markov process is irreducible and if T_{ii} has a nonlattice distribution with finite mean, then

$$P_i = \lim_{t\to\infty} P(\, Z(t) = i \mid Z(0) = j\,) = \frac{\mu_i}{\mu_{ii}}, \qquad \forall i, j$$

$$\lim_{t\to\infty} P(t) = P_i$$

$$E[Z_n] = E[\tau_i] = \mu_i$$

$$E[Z_n] + E[Y_n] = E[T_{ii}] = \mu_{ii}$$

Theorem: If $E[Z_n + Y_n] < \infty$ and F is nonlattice, then

$$\lim_{t\to\infty} P(t) = P(\text{system is on at time } t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}$$

from the part of alternating renewal process in lecture 3

Semi-Markov processes

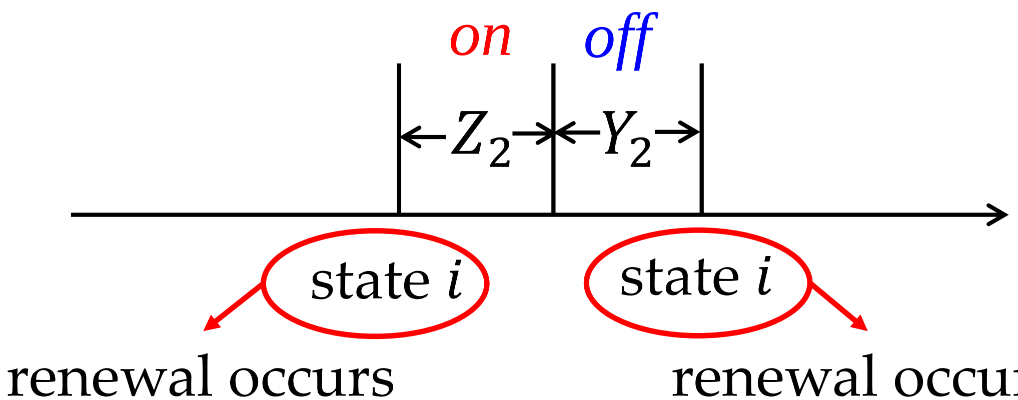
Theorem: If the semi-Markov process is irreducible and not lattice, then

$$\lim_{t\to\infty} P(\, Z(t) = i, Y(t) > x, S(t) = j \mid Z(0) = k\,) = \frac{P_{ij}\int_x^\infty \bar{F}_{ij}(y)dy}{\mu_{ii}}$$

time from t until the next transition

state entered at the first transition after t

Proof: **A delayed alternating renewal process**



on: the state is i, and will remain i for at least the next x time units; the next state is j

off: otherwise

Semi-Markov processes

Proof:

$$\lim_{t\to\infty} P(t) = P(\text{system is on at time } t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]} \quad ?$$

$$\boxed{\lim_{t\to\infty} P(\,Z(t) = i, Y(t) > x, S(t) = j \mid Z(0) = k\,)}$$

$$E[Z_n] + E[Y_n] = E[time\ of\ a\ cycle] = E[T_{ii}] = \mu_{ii}$$

$$E[Z_n] = E[\text{"on"}\ time\ in\ a\ cycle] \ = P_{ij}E[\max\{\tau_{ij} - x, 0\}]$$

time to make a transition from i to j,
i.e., a random variable having distribution F_{ij}

Semi-Markov processes

Proof:

$$\lim_{t\to\infty} P(t) = P(\text{system is on at time } t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]} \quad ?$$

$$\boxed{\lim_{t\to\infty} P(\,Z(t) = i, Y(t) > x, S(t) = j \mid Z(0) = k\,)}$$

$$E[Z_n] + E[Y_n] = E[\mathit{time\ of\ a\ cycle}] = E[T_{ii}] = \mu_{ii}$$

$$E[Z_n] = E[\text{"on"}\ \mathit{time\ in\ a\ cycle}] = P_{ij} E[\max\{\tau_{ij} - x, 0\}]$$

Semi-Markov processes

Theorem: If the semi-Markov process is irreducible and not lattice, then

$$\lim_{t\to\infty} P(\, Z(t) = i, Y(t) > x \mid Z(0) = k\,) = \frac{\int_x^\infty P(\tau_i > y)dy}{\mu_{ii}}$$

τ_i: time that the process spends in state i before making a transition

$$\sum_j \frac{P_{ij}\int_x^\infty \bar{F}_{ij}(y)dy}{\mu_{ii}} = \frac{\int_x^\infty \sum_j P_{ij}\,\bar{F}_{ij}(y)dy}{\mu_{ii}}$$

Summary

- Markov chain
- Chapman-Kolmogorov equations and classification of states
- Stationary distribution
- Transitions and gambler's ruin problem
- Branching processes
- Time-reversible Markov chains and MCMC
- Semi-Markov processes

References: Chapter 4, Markov Chains, 2nd edition, 1995, *by Sheldon M. Ross*