Last class

- What is stochastic process
- Stochastic process in AI
- Preliminaries

References: Chapter 1 & 10, Stochastic Processes, 2nd edition, 1995, *by Sheldon M. Ross*

Stochastic Processes

Lecture 2: Poisson Process

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Counting process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a **counting process** if $N(t)$ represents the total number of 'events' that have occurred up to time t.

A stochastic process $\{N(t), t \geq 0\}$ is a **counting process**, if

- $N(t) \geq 0$
- $N(t)$ is integer valued
- if $s < t$, then $N(s) \leq N(t)$
- for $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$

Poisson process

Definition 1: The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

- $N(0) = 0$
- The process has independent increments
- The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all s, $t \geq 0$,

$$P(N(s+t) - N(s) = n) = \boxed{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}, \qquad n = 0,1,2,\ldots$$

Implies that the process also has stationary increments

Properties of stochastic process

A stochastic process $\{X(t) \mid t \in T\}$ is said to have independent increments if $\forall t_0 < t_1 < t_2 < \cdots < t_n$, the random variables

$$X(t_1) - X(t_0), X(t_2) - \mathrm{X}(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent

A stochastic process $\{X(t) \mid t \in T\}$ is said to have stationary increments if $\forall s > 0$,

$$X(t+s) - X(t)$$

has the same distribution for all t

Poisson process

Definition 2: The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

- $N(0) = 0$
- The process has stationary and independent increments
- $P(N(h) = 1) = \lambda h + o(h)$
- $P(N(h) \geq 2) = o(h)$

$$P(N(h) = 0) = 1 - \lambda h + o(h)$$

Why are these two definitions equivalent?

Equivalence between Definitions 1 and 2

Definition 2 $\Rightarrow$ **Definition 1**

We only need to show $P_n(t) = P(N(t) = n) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

Proof: To analyze $P_n'(t)$

For $n = 0$:

$$\begin{aligned} P_0(t+h) &= P(N(t+h) = 0) = P(N(t) = 0, N(t+h) - N(t) = 0) \\ &= P(N(t) = 0) \cdot P(N(t+h) - N(t) = 0) = P_0(t) \cdot (1 - \lambda h + o(h)) \end{aligned}$$

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h} \Longrightarrow \frac{P'_0(t)}{P_0(t)} = -\lambda$$

$$\log P_0(t)|_0^t = -\lambda t \xRightarrow{P_0(0) = 1} \log P_0(t) = -\lambda t \Longrightarrow P_0(t) = e^{-\lambda t}$$

Equivalence between Definitions 1 and 2

For $n \geq 1$:
$$\begin{aligned} P_n(t+h) &= P(N(t+h)=n) \\ &= P(N(t+h)-N(t)=0, N(t)=n) \\ &\quad +P(N(t+h)-N(t)=1, N(t)=n-1) \\ &\quad +P(N(t+h)-N(t)\geq 2, N(t+h)=n) \\ &= P_n(t)(1-\lambda h)+P_{n-1}(t)\lambda h+o(h) \end{aligned}$$

$$\Downarrow$$

$$\frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t)+\lambda P_{n-1}(t)+\frac{o(h)}{h}$$

$$\Downarrow$$

$$P_n'(t) = -\lambda P_n(t)+\lambda P_{n-1}(t)$$

$$\Downarrow$$

$$e^{\lambda t}[P_n'(t)+\lambda P_n(t)] = \lambda e^{\lambda t}P_{n-1}(t)$$

$$\Downarrow$$

$$\frac{de^{\lambda t}P_n(t)}{dt} = \lambda e^{\lambda t}P_{n-1}(t)$$

Equivalence between Definitions 1 and 2

Mathematical induction: $n = 0$ holds. Suppose the equation holds for $n \leq k-1$, we need to proof it holds for $n = k$, $k \geq 1$

$$\frac{de^{\lambda t}P_k(t)}{dt} = \lambda e^{\lambda t}P_{k-1}(t) = \lambda \frac{(\lambda t)^{k-1}}{(k-1)!}$$

$$\Downarrow$$

$$e^{\lambda t}P_k(t) - P_k(0) = \frac{(\lambda t)^k}{k!}$$

$$\Downarrow$$

$$P_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

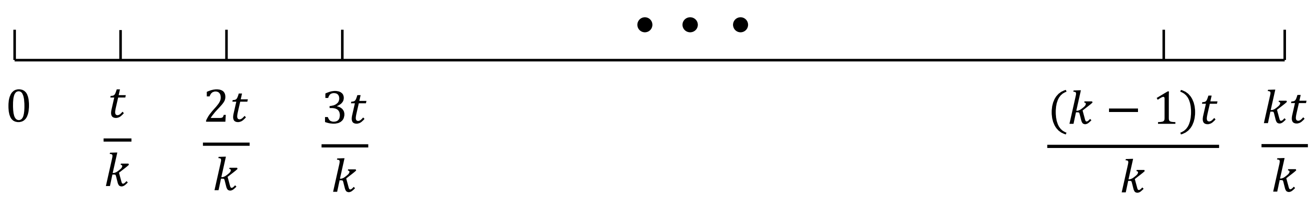
Definition 2 $\overset{?}{\Longleftarrow}$ **Definition 1** Leave as the exercise

Equivalence between Definitions 1 and 2

Definition 2 $\Rightarrow$ Definition 1

We only need to show $P_n(t) = P(N(t) = n) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

Intuitive analysis:



$k \to \infty$

$N(t)$ can be viewed as #subintervals in which an event occurs

Need to show $\underline{P(\text{2 or more everts in any subinterval})} \to 0$

$\leq \sum_{i=1}^{k} P(\text{2 or more everts in the } i\text{–th subinterval})$

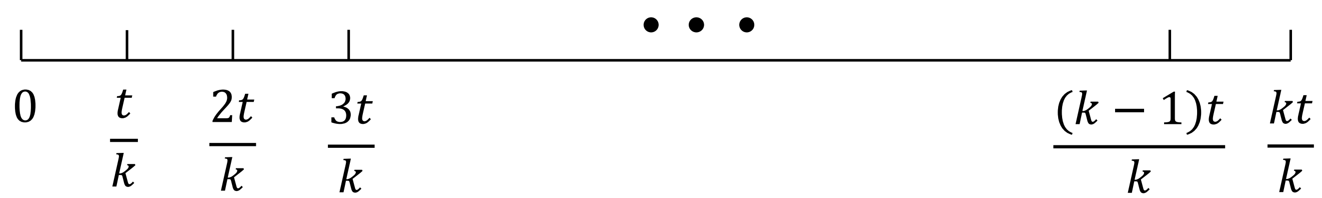
$= k \cdot o(\frac{t}{k}) = t\,\dfrac{o(t/k)}{t/k} \to 0$

Equivalence between Definitions 1 and 2

Definition 2 $\Rightarrow$ Definition 1

We only need to show $P_n(t) = P(N(t) = n) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

Intuitive analysis:



$k \to \infty$

$N(t)$ can be viewed as #subintervals in which an event occurs

Binomial distribution with parameters k and $\lambda \cdot \frac{t}{k} + o\left(\frac{t}{k}\right)$

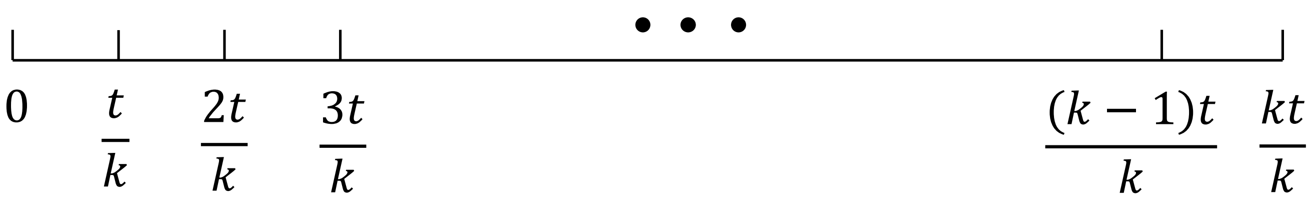
Because: Independent and stationary increments

Equivalence between Definitions 1 and 2

Definition 2 $\Rightarrow$ Definition 1

We only need to show $P_n(t) = P(N(t) = n) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

Intuitive analysis:



$k \to \infty$

$N(t)$ can be viewed as #subintervals in which an event occurs

Binomial distribution with parameters k and $\lambda \cdot \frac{t}{k} + o\left(\frac{t}{k}\right)$

Poisson distribution with $\lambda t + \lim_{k\to\infty} k \cdot o\left(\frac{t}{k}\right) = \lambda t$

Poisson process

Interarrival times $\boldsymbol{X_n}$: the time between the $(n-1)$st and the nth event

Proposition. $X_n, n = 1,2, \ldots$ are independent identically distributed exponential random variables having mean $1/\lambda$

Proof:

$$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

$$\begin{aligned} P(X_2 > t \mid X_1 = s) &= P(0 \text{ events occur in } (s, s+t] \mid X_1 = s) \\ &= P(0 \text{ evets occur in } (s, s+t]) \\ &= P(0 \text{ events occur in } (0, t]) \\ &= P(N(t) = 0) = e^{-\lambda t} \end{aligned}$$

law of total probability

$$P(X_2 > t) = e^{-\lambda t}$$

Poisson process

Definition 3: The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

- Interarrival times $X_n, n = 1,2, \ldots$ are independent identically distributed exponential random variables having mean $1/\lambda$

Definition 1 $\Rightarrow$ **Definition 3** ✓

Definition 3 $\Rightarrow$ **Definition 1** ?

Equivalence between Definitions 1 and 3

Definition 3 $\Rightarrow$ **Definition 1**

Proof:

Memoryless property of exponentially distributed random variables X_n $\Rightarrow$ The process has stationary and independent increments

Then, we only need to show $P(N(t)=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$

Define the *arrival time* S_n of the nth event, also called the *waiting time* until the nth event

$$S_n = X_1 + X_2 + \cdots + X_n$$

Equivalence between Definitions 1 and 3

$$P(N(t) \geq n) = P(S_n \leq t)$$

X_n are iid exponential random variables having mean $1/\lambda$

$S_n = X_1 + X_2 + \cdots + X_n$ $\Downarrow$ **?** Leave as the exercise

S_n has a gamma distribution with parameters n and λ, i.e.,

$$P(S_n \leq t) = \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

Properties of Poisson process

Define the *arrival time* S_n of the nth event, also called the *waiting time* until the nth event

$$S_n = X_1 + X_2 + \cdots + X_n$$

Theorem: Given $N(t) = n$, the arrival times $\boxed{S_1, S_2, \dots, S_n}$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

$Y_1, Y_2, \dots, Y_n$ probability density $f(y_1, y_2, \dots, y_n) = \frac{1}{t^n}$

$Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ probability density $f(y_1, y_2, \dots, y_n) = \frac{n!}{t^n}$

Properties of Poisson process

Theorem: Given $N(t) = n$, the arrival times $S_1, S_2, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Proof: Assume $0 < t_1 < t_2 < \cdots < t_{n+1} = t,\quad t_i + h_i < t_{i+1}$

$$P(t_i \le S_i \le t_i + h_i, i = 1,2,\cdots,n \mid N(t) = n)$$

$$= \frac{P(\text{exactly 1 event in } [t_i, t_i + h], i = 1,2,\cdots,n, \text{no events else alone})}{P(N(t) = n)}$$

$$= \frac{\left(\prod_{i=1}^{n} e^{-\lambda h_i} \lambda h_i\right) \cdot e^{-\lambda(t - \sum_{i=1}^{n} h_i)}}{e^{-\lambda t}(\lambda t)^n / n!} = \frac{n!}{t^n} \prod_{i=1}^{n} h_i$$

Divide both sides by $\prod_{i=1}^{n} h_i$ and let $h_i \to 0$

$$\Longrightarrow \quad f(t_1, t_2, \cdots, t_n \mid N(t) = n) = \frac{n!}{t^n}$$

Next, we will show how to apply this theorem

Properties of Poisson process

Example: Suppose that travelers arrive at a train depot in accordance with a Poisson process with rate λ. If the train departs at time t, the expected sum of the waiting times of travelers arriving in $(0, t)$?

Solution:

$$E\left[\sum_{i=1}^{N(t)}(t - S_i) \mid N(t) = n\right] = nt - E[\sum_{i=1}^{n} S_i \mid N(t) = n]$$

Using the theorem on the previous page $\Longrightarrow$

$$= nt - E\left[\sum_{i=1}^{n} Y_{(i)}\right] = nt - E[\sum_{i=1}^{n} Y_i]$$

$$= nt - nE[Y_i] = \frac{tn}{2}$$

$$E\left[\sum_{i=1}^{N(t)}(t - S_i)\right] = E\left[E\left[\sum_{i=1}^{N(t)}(t - S_i) \mid N(t)\right]\right]$$

$$= E\left[\frac{t}{2}N(t)\right] = \frac{t}{2}E[N(t)]$$

$$= \frac{t}{2} \cdot \lambda t = \frac{\lambda t^2}{2}$$

Properties of Poisson process

Example: Suppose that a device is subject to shocks that occur in accordance with a Poisson process having rate λ. The ith shock gives rise to a damage D_i. The D_i, $i \geq 1$, are assumed to be iid and also to be independent of $\{N(t) \mid t \geq 0\}$, where $N(t)$ denotes the number of shocks in $[0, t]$. The damage due to a shock is assumed to decrease exponentially in time. That is, if a shock has an initial damage D, its damage after time t is $De^{-\alpha t}$. The expected total damage at time t?

Solution:

$$E\left[\sum_{i=1}^{N(t)} D_i e^{-\alpha(t-S_i)}\right]$$

$$= E\left[E\left[\sum_{i=1}^{N(t)} D_i e^{-\alpha(t-S_i)} \mid N(t)\right]\right]$$

Properties of Poisson process

$$E\left[\sum_{i=1}^{n} D_i e^{-\alpha(t-S_i)} \mid N(t)=n\right]$$

$$= \sum_{i=1}^{n} E\left[D_i e^{-\alpha(t-s_i)} \mid N_t = n\right]$$

$$= \sum_{i=1}^{n} E\left[D_i\right] \cdot e^{-\alpha t} \cdot E\left[e^{\alpha S_i} \mid N(t)=n\right]$$

$$= E[D] \cdot e^{-\alpha t} \cdot E\left[\sum_{i=1}^{n} e^{\alpha S_i} \mid N(t)=n\right]$$

$$= E[D] \cdot e^{-\alpha t} \cdot E\left[\sum_{i=1}^{n} e^{\alpha Y_{(i)}}\right]$$

$$= E[D] \cdot e^{-\alpha t} \cdot E\left[\sum_{i=1}^{n} e^{\alpha Y_i}\right]$$

$$= E[D] \cdot e^{-\alpha t} \cdot n \cdot \frac{1}{t}\int_0^t e^{\alpha x}\, dx$$

$$= \frac{n}{\alpha t} E[D] \cdot \left(1 - e^{-\alpha t}\right)$$

$\Longrightarrow$

$$E\left[\sum_{i=1}^{N(t)} D_i e^{-\alpha(t-S_i)}\right]$$

$$= \frac{E[D]}{\alpha t} \cdot \left(1 - e^{-\alpha t}\right) \cdot E[N(t)]$$

$$= \frac{\lambda E[D]}{\alpha} \cdot \left(1 - e^{-\alpha t}\right)$$

Properties of Poisson process

Suppose that each event of a Poisson process with rate λ is classified as being either a type-I or type-II event, and the probability of an event being classified as type-I depends on the time when it occurs. If an event occurs at time s, then, independently of all else, it is classified as type-I with probability $P(s)$ and type-II with probability $1 - P(s)$.

Proposition: If $N_i(t)$ represents the number of type-i events that occur by time t, $i = 1, 2$, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables having respective means $\lambda t p$ and $\lambda t(1-p)$, where

$$p = \frac{1}{t}\int_0^t P(s)\, ds$$

Properties of Poisson process

Proof: $\forall m, n \geq 0, P(N_1(t) = m, N_2(t) = n)$

$$= P(N_1(t) = m, N_2(t) = n, N(t) = m + n)$$

$$= \underline{P(N_1(t) = m, N_2(t) = n \mid N(t) = m + n)}P(N(t) = m + n)$$

$P(\text{\#events that happen at time } S_1, S_2, \ldots S_{m+n} \text{ are type}-\text{I, type}-\text{II are } m, n \mid N(t) = m + n)$

$$= P\left(\cdots Y_{(1)}, Y_{(2)}, \ldots, Y_{(m+n)} \cdots\right) = P(\cdots Y_1, Y_2, \ldots Y_{m+n}, \cdots) = C_{m+n}^{m} \boxed{p}^{m}(1-p)^{n}$$

$$\frac{1}{t}\int_0^t p\,(s)ds$$

$$P(N_1(t) = m, N_2(t) = n) = C_{m+n}^{m} p^{m}(1-p)^{n} e^{-\lambda t} \frac{(\lambda t)^{m+n}}{(m+n)!}$$

$$= e^{-\lambda p t} \frac{(\lambda p t)^{m}}{m!} \cdot e^{-\lambda(1-p)t} \frac{(\lambda(1-p)t)^{n}}{n!}$$

$$P(N_1(t) = m) = \sum_n P\,(N_1(t) = m, N_2(t) = n) = e^{-\lambda p t} \frac{(\lambda p t)^{m}}{m!} \cdot 1$$

$$P(N_2(t) = n) = e^{-\lambda(1-p)t} \frac{(\lambda(1-p)t)^{n}}{n!}$$

Properties of Poisson process

Example: Suppose that customers arrive at a service station with a Poisson process of rate λ. Upon arrival the customer is immediately served by one of an infinite number of possible servers, and the service times are assumed to be independent with a common distribution G.

The number $N_1(t)$ of customers that have completed service by t

The number $N_2(t)$ of customers that are in service at t

How about their distribution?

Solution: For a customer arriving at time s

Type-I customer: service completed by t

service time $\leq t - s$

$P(s) = G(t - s)$

Type-II customer: in service at t

Properties of Poisson process

Example: Suppose that customers arrive at a service station with a Poisson process of rate λ. Upon arrival the customer is immediately served by one of an infinite number of possible servers, and the service times are assumed to be independent with a common distribution G.

The number $\boxed{N_1(t)}$ of customers that have completed service by t

type-I customers by time t

Poisson distribution with mean

$$\lambda pt = \lambda t \cdot \frac{1}{t}\int_0^t P(s)ds = \lambda\int_0^t G(t-\mathrm{s})ds = \lambda\int_0^t G(s)ds$$

The number $N_2(t)$ of customers that are in service at t

Poisson distribution with mean

$$\lambda(1-p)t = \lambda\int_0^t \big(1-G(t-\mathrm{s})\big)ds = \lambda\int_0^t (1-G(s))ds$$

Properties of Poisson process

Theorem: Let $\{N_i(t), t \geq 0\}$ be a Poisson process having rate λ_i, where $i \in \{1,2, \ldots, n\}$. Suppose they are independent. Let

$$N(t) = N_1(t) + N_2(t) + \cdots + N_n(t)$$

Then, $\{N(t), t \geq 0\}$ is a Poisson process having rate $\sum_{i=1}^{n} \lambda_i$

Proof: Leave as the exercise

Next, we will show how to apply this theorem

Properties of Poisson process

Example: $X_1, X_2, \dots, X_n$ are iid exponential random variables having mean $1/\lambda$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote their order statistics.

To prove:

$nX_{(1)}, (n-1)(X_{(2)} - X_{(1)}), \dots, (n-i+1)(X_{(i)} - X_{(i-1)}), \dots, X_{(n)} - X_{(n-1)}$ are iid exponential random variables having mean $1/\lambda$

Solution: Let $X_i \sim Exp(\lambda)$ denote the life of component i.

Once it fails, replace it with one of the same type. Let $N_i(t)$ represents the total number of fails of i up to time t. Then, $\{N_i(t), t \geq 0\}$ is a Poisson process with rate λ. Let $N(t) = \sum_{i=1}^{n} N_i(t)$, apply the Theorem on the previous page, we have $\{N(t), t \geq 0\}$ is a Poisson process with rate $n\lambda$.

Then $X_{(1)}$ (time of the first event)$\sim Exp(n\lambda)$, so $nX_{(1)} \sim Exp(\lambda)$.

After the first component fails, we ignore this component, and assume time starts from the beginning (using the memoryless property of exponentially distributed random variables), so $(n-1)(X_{(2)} - X_{(1)}) \sim Exp(\lambda)$ ……

Nonhomogeneous Poisson process

Definition N1: The counting process $\{N(t), t \geq 0\}$ is said to be a **nonhomogeneous or nonstationary Poisson process** with intensity function $\lambda(t)$, $t > 0$, if

- $N(0) = 0$
- The process has independent increments
- $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$
- $P(N(t+h) - N(t) \geq 2) = o(h)$

$$P(N(t+h) - N(t) = 0) = 1 - \lambda(t)h + o(h)$$

Nonhomogeneous Poisson process

Proposition. For a nonhomogeneous Poisson process $\{N(t), t \geq 0\}$ with intensity function $\lambda(t)$, the number of events in interval $(t, t+s]$ is Poisson distributed with mean $m(t+s) - m(t)$. That is, for all $s, t \geq 0$,

$$P(N(t+s) - N(t) = n) = e^{-(m(t+s)-m(t))} \frac{(m(t+s) - m(t))^n}{n!}$$

where $m(t) = \int_0^t \lambda(x)\, dx$

Proof: Leave as the exercise

Nonhomogeneous Poisson process

Definition N2: The counting process $\{N(t), t \geq 0\}$ is said to be a **nonhomogeneous or nonstationary Poisson process** with intensity function $\lambda(t)$, $t > 0$, if

- $N(0) = 0$
- The process has independent increments
- The number of events in $(t, t+s]$ is Poisson distributed with mean $m(t+s) - m(t)$. That is, for all $s, t \geq 0$,

$$P(N(t+s) - N(t) = n) = e^{-(m(t+s)-m(t))} \frac{(m(t+s) - m(t))^n}{n!}$$

where $m(t) = \int_0^t \lambda(x)\, dx$

Nonhomogeneous Poisson process

Relation-1 between homogeneous and nonhomogeneous Poisson processes

A nonhomogeneous Poisson process can be viewed as a random sample from a homogeneous Poisson process

A nonhomogeneous Poisson process with intensity $\lambda(t)$

Why?

A homogeneous Poisson process with rate λ, where $\lambda \geq \lambda(t)$

If an event occurring at time t is counted with probability $\frac{\lambda(t)}{\lambda}$
then the process of counted events is

Nonhomogeneous Poisson process

Proof:

To prove the condition of Definition N1

- $N'(0) = 0$
- The process has independent increments
- $P(N'(t+h) - N'(t) \geq 2) = o(h)$
- $$\begin{aligned} P(N'(t+h) - N'(t) = 1) &= P(N(t+h) - N(t) = 1) \cdot \frac{\lambda(t)}{\lambda} \\ &\quad + P(N(t+h) - N(t) \geq 2) \cdot p \\ &= (\lambda h + o(h)) \cdot \frac{\lambda(t)}{\lambda} + o(h) \\ &= \lambda(t)h + o(h) \end{aligned}$$

To prove the condition of Definition N2

Leave as the exercise

Nonhomogeneous Poisson process

Relation-2 between homogeneous and nonhomogeneous Poisson processes

Let $m(t) = \int_0^t \lambda(x)\, dx$

Let $\{N^*(t), t \geq 0\}$ be a homogeneous Poisson process with rate 1

$N(t) = N^*(m(t))$ $\Downarrow$

$\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity $\lambda(t)$

Why? Leave as the exercise

Nonhomogeneous Poisson process

Relation-2 between homogeneous and nonhomogeneous Poisson processes

Let $m(t) = \int_0^t \lambda(x)\, dx$

Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson process with intensity $\lambda(t)$

$$N^*(t) = N(m^{-1}(t)) \Downarrow$$

$\{N^*(t), t \geq 0\}$ is a homogeneous Poisson process with rate 1

Why? Leave as the exercise

Nonhomogeneous Poisson process

Proposition: For a counting process $\{N(t), t \geq 0\}$, let S_i denote the occurring time of the ith event. Suppose that $m(0) = 0$ and $m'(t) = \lambda(t) > 0$. If

$m(S_1), m(S_2) - m(S_1), \ldots, m(S_n) - m(S_{n-1}), \ldots$ are iid exponential random variables having mean 1

Then $\{N(t), t \geq 0\}$ is a **nonhomogeneous Poisson process** with intensity function $\lambda(t)$

Proof: Let $N^*(t) = N(m^{-1}(t))$, then the occurring time of $\{N^*(t), t \geq 0\}$ is

$$0 < m(s_1) < m(s_2) < \cdots < m(s_n) < \cdots$$

By **Definition 3**, $\{N^*(t), t \geq 0\}$ is a homogeneous Poisson process with rate 1. Then, by **Relation-2.1,** $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)$.

Nonhomogeneous Poisson process

Definition N3: For a counting process $\{N(t), t \geq 0\}$, let S_i denote the occurring time of the ith event. Suppose that $m(0) = 0$ and $m'(t) = \lambda(t) > 0$. If

$m(S_1), m(S_2) - m(S_1), \ldots, m(S_n) - m(S_{n-1}), \ldots$ are iid exponential random variables having mean 1

Then $\{N(t), t \geq 0\}$ is a **nonhomogeneous Poisson process** with intensity function $\lambda(t)$

This is a generalization of Definition 3 of Poisson process

Next we will give one concrete nonhomogeneous Poisson process

Nonhomogeneous Poisson process

Example: Suppose that customers arrive at a service station with a Poisson process $\{N(t), t \geq 0\}$ of rate λ. Upon arrival the customer is immediately served, and the service times are assumed to be independent with a common distribution G.

$N'(t)$: the number of customers that have completed service by t

$$\Downarrow$$

$\{N'(t), t \geq 0\}$: a nonhomogeneous Poisson process with intensity $\lambda G(t)$

To verify the condition of Definition N2

- $N'(0) = 0$ ✓
- The process has independent increments **?**
- $N'(t+s) - N'(t)$ is Poisson distributed with mean $m(t+s) - m(t)$ **?**

$$m(t) = \lambda \int_0^t G(x)\, dx$$

Nonhomogeneous Poisson process

Proof:

$N'(t+s) - N'(t)$ is Poisson distributed with mean $m(t+s) - m(t)$, where $m(t) = \lambda \int_0^t G(x)\,dx$

Type-I customer: service completed $(t, t+s]$

$$P(\boxed{y}) = \begin{cases} G(t+s-y) - G(t-y) & if\ y \le t \\ G(t+s-y) & if\ t < y \le t+s \\ 0 & if\ y > t+s \end{cases}$$

the arrival time
of customer i

$N'(t+s) - N'(t) =$ #Type-I customers by time ∞

Properties of Poisson process

Suppose that each event of a Poisson process with rate λ is classified as being either a type-I or type-II event, and the probability of an event being classified as type-I depends on the time when it occurs. If an event occurs at time s, then, independently of all else, it is classified as type-I with probability $P(s)$ and type-II with probability $1 - P(s)$.

Proposition: If $N_1(t)$ represents the number of type-i events that occur by time t, $i = 1, 2$, then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables having respective means $\boxed{\lambda t p}$ and $\lambda t(1-p)$, where

$\lambda \int_0^\infty P(y)dy$

$$p = \frac{1}{t}\int_0^t P(s)\, ds$$

Nonhomogeneous Poisson process

Proof:

$N'(t+s) - N'(t)$ is Poisson distributed with mean $m(t+s) - m(t)$, where $m(t) = \lambda \int_0^t G(x)\,dx$

Type-I customer: service completed $(t, t+s]$

$$P(y) = \begin{cases} G(t+s-y) - G(t-y) & if\ y \le t \\ G(t+s-y) & if\ t < y \le t+s \\ 0 & if\ y > t+s \end{cases}$$

$N'(t+s) - N'(t) =$ #Type-I customers by time ∞

$$\begin{aligned} \lambda \int_0^\infty P(y)dy &= \lambda \int_0^t G(t+s-y) - G(t-y)dy + \lambda \int_t^{t+s} G(t+s-y)dy \\ &= \lambda \int_0^{t+s} G(t+s-y) - \lambda \int_0^t G(t-y)dy = m(t+s) - m(t) \end{aligned}$$

Nonhomogeneous Poisson process

Proof:

The process has independent increments

$\forall t_0 < t_1 < t_2 < \cdots < t_n$, the random variables

$$\underline{N'(t_1) - N'(t_0)}, \underline{N'(t_2) - N'(t_1)}, \ldots, N'(t_n) - N'(t_{n-1})$$

are independent

Type-I customer: service completed $(t_0, t_1]$

Type-II customer: service completed $(t_1, t_2]$

$\vdots$

note that the generalization of the proposition in page 39 is used

Compound Poisson process

Definition: Let $X_1, X_2, \ldots$ be a sequence of iid random variables having distribution function F, and suppose that this sequence is independent of $\{N(t), t \geq 0\}$, a Poisson process with rate λ. If

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

then $\{S(t), t \geq 0\}$ is said to be a **compound Poisson process**

Example: Suppose that customers arrive at a store at a Poisson rate λ, and the amounts of money spent by each customer are iid random variables, independent of the arrival process.

$S(t)$: the total amount spent by all customers arriving by time t

Compound Poisson process

Example: Suppose that events are occurring with a Poisson process of rate α, and that whenever an event occurs a certain contribution results. Specifically, an event occurring at time s will, independent of the past, result in a contribution whose value is a random variable with distribution F_s.

Sum of the contributions by time t

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

$N(t)$: number of events occurring by time t

X_i: contribution made when the i-th event occurs

X_i are neither independent nor identically distributed

But $\{S(t), t \geq 0\}$ is a compound Poisson process   Why?

Compound Poisson process

Proof:

Let $Y(S)$ denote the contribution made when an event happens at time S, then $S(t) = \sum_{i=1}^{N(t)} Y(S_i)$,

$$[S(t)|N(t) = n] = \left[\sum_{i=1}^{N(t)} Y(S_i) \middle| N(t) = n\right]$$

[·] denotes the distribution

$$= [\sum_{i=1}^{n} Y(U_{(i)})]$$

$$= [\sum_{i=1}^{n} Y(U_i)],$$

where $U_i \;\; i = 1, \ldots, n \;\; iid \sim U(0, t)$.

Then, $S(t) = \sum_{i=1}^{N(t)} Y(U_i)$,

$Y(U_i) \;\; i = 1, \ldots, n \;\; iid \sim F(x)$, $\qquad F(x) = \frac{1}{t}\int_0^t F_s(x)ds$

and is independent of $\{N(t), t \geq 0\}$

$\Longrightarrow$ $\{S(t), t \geq 0\}$ is a compound Poisson process

Compound Poisson process

For a compound Poisson process $\{S(t), t \geq 0\}$, How to compute $E[S(t)]$?

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

Way 1: $E[S(t)] = E\big[E[S(t)|N(t)]\big]$

X_i is independent of $N(t)$

$$E[S(t)|N(t) = n] = E[\textstyle\sum_{i=1}^{n} X_i] = nE[X]$$

$$E[S(t)] = E[N(t)]E[X] = \lambda t E[X]$$

Way 2: Only applicable when X takes discrete values.

Suppose X takes values in $\{x_1, x_2, \dots, x_m\}$,
and $p_j = P\big(X = x_j\big), j = 1, \dots, m$.
Denote $N_j(t) = \#\{k: X_k = x_j, 1 \leq k \leq N(t)\}$, then $S(t) = \sum_{j=1}^{m} x_j N_j(t)$.
Denote $X = x_j$ as a type-j event, then $N_j(t)$ is the number of type-j events until time t. Thus, $N_1(t), \dots, N_m(t)$ are independent, $\forall j: N_j(t) \sim \mathrm{Poi}\big(\lambda p_j t\big)$

$\Longrightarrow$ $E[S(t)] = \sum_{j=1}^{m} x_j E[N_j(t)] = \sum_{j=1}^{m} x_j \lambda p_j t = \lambda t \sum_{j=1}^{m} x_j p_j = \lambda t E[X]$

Compound Poisson process

For a compound Poisson process $\{S(t), t \geq 0\}$, $S(t) = \sum_{i=1}^{N(t)} X_i$

How to compute $E[S(t)^n]$?

$N(t) \sim \text{Poi}(\lambda t)$

How to compute the probability distribution of $S(t)$?

Compound Poisson random variable: Let $X_1, X_2, \ldots$ be a sequence of iid random variables having distribution function F, and suppose that this sequence is independent of N, a Poisson random variable with mean λ.

$$W = \sum_{i=1}^{N} X_i$$

Compound Poisson process

$$W = \sum_{i=1}^{N} X_i$$

Proposition. Let X be a random variable having distribution F that is independent of W. Then, for any function $h(x)$,

$$E[Wh(W)] = \lambda E[Xh(W+X)]$$

Proof:

$$\begin{aligned}
E[Wh(W)] &= \sum_{n=0}^{\infty} E[Wh(W)|N=n] \cdot e^{-\lambda} \lambda^n / n! \\
&= \sum_{n=0}^{\infty} (e^{-\lambda}\lambda^n/n!) \cdot E[\textstyle\sum_{j=1}^{n} X_j h(\sum_{i=1}^{n} X_i)] \\
&= \sum_{n=0}^{\infty} (e^{-\lambda}\lambda^n/n!) \cdot \textstyle\sum_{j=1}^{n} E[X_j h(\sum_{i=1}^{n} X_i)] \\
&\overset{X_i \text{ iid}}{=} \sum_{n=0}^{\infty} (e^{-\lambda}\lambda^n/n!) \cdot n \cdot E[X_n h(\textstyle\sum_{i=1}^{n} X_i)] \\
&= \sum_{n=1}^{\infty} (e^{-\lambda}\lambda^n/(n-1)!) \cdot \int E[X_n h(\textstyle\sum_{i=1}^{n} X_i)|X_n = x] dF(x) \\
&= \lambda \sum_{n=1}^{\infty} (e^{-\lambda}\lambda^{n-1}/(n-1)!) \int x E[h\left(\textstyle\sum_{i=1}^{n-1} X_i + x\right)] dF(x) \\
&= \lambda \int x \sum_{m=0}^{\infty} \underbrace{(e^{-\lambda}\lambda^m/m!)}_{P(N=m)} \underbrace{E[h(\textstyle\sum_{i=1}^{m} X_i + x)]}_{E[h\left(\sum_{i=1}^{N} X_i + x\right)|N=m]} dF(x) \\
&= \lambda \int x\, E[h(W+x)] dF(x) \\
&= \lambda \int E[Xh(W+X)|X=x] dF(x) \\
&= \lambda E[Xh(W+X)]
\end{aligned}$$

Compound Poisson process

$$W = \sum_{i=1}^{N} X_i$$

Corollary. If X has distribution F, then for any positive integer n,

$$E[W^n] = \lambda \sum_{j=0}^{n-1} \binom{n-1}{j} E[W^j] E[X^{n-j}]$$

Proof:

Let $h(x) = x^{n-1}$, then

$$\begin{aligned} E[W^n] &= E[Wh(W)] \\ &= \lambda E[X(W+X)^{n-1}] \\ &= \lambda E[\textstyle\sum_{j=0}^{n-1} \binom{n-1}{j} W^j X^{n-j}] \end{aligned}$$

W is independent of X

Compound Poisson process

$$W = \sum_{i=1}^{N} X_i$$

Corollary. When X_i are positive integer valued random variables, suppose

$$\alpha_j = P(X_i = j), \qquad j \geq 1 \qquad\qquad P_j = P(W = j), \qquad j \geq 0$$

Then, $\qquad P_0 = e^{-\lambda} \qquad P_n = \frac{\lambda}{n} \sum_{j=1}^{n} j \alpha_j P_{n-j}$

Proof: $P_0 = P(W = 0) = P(N = 0) = e^{-\lambda}$

For $n \geq 1$, let $h(W) = \begin{cases} \frac{1}{n} & W = n \\ 0 & W \neq n \end{cases}$, then

$$P_n = E[I(W = n)] = E[Wh(W)] = \lambda E[Xh(W + X)] = \lambda \sum_j j\, E[h(W + j)]\alpha_j$$

$$= \lambda \sum_j j \cdot \frac{1}{n} P(W + j = n)\alpha_j = \lambda \sum_{j=1}^{n} \frac{j}{n} \alpha_j P(W = n - j)$$

Conditional Poisson process

Definition: Let Λ be a positive random variable having distribution G and let $\{N(t), t \geq 0\}$ be a counting process such that, given that $\Lambda = \lambda$, $\{N(t), t \geq 0\}$ is a Poisson process with rate λ. Then, $\{N(t), t \geq 0\}$ is called a **conditional Poisson process**

Stationary but not independent $\Rightarrow$ not a Poisson process

$$P(N(t+s) - N(s) = n)$$

$$= \int_0^\infty P\left(N(t+s) - N(s) = n \mid \Lambda = \lambda\right) dG(\lambda)$$

$$= \int_0^\infty \frac{e^{-\lambda t}}{n!} (\lambda t)^n dG(\lambda)$$

is independent of s, but depends on G

Conditional Poisson process

How to compute the conditional distribution of Λ given that $N(t) = n$?

$P(\Lambda \leq x \mid N(t) = n)$?

Solution:

$$P(\Lambda \in (\lambda, \lambda + d\lambda) \mid N(t) = n)$$
$$= \frac{P(N(t) = n \mid \Lambda \in (\lambda, \lambda + d\lambda))P(\Lambda \in (\lambda, \lambda + d\lambda))}{P(N(t) = n)}$$
$$= \frac{\frac{e^{-\lambda t}(\lambda t)^n}{n!} \cdot dG(\lambda)}{\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} dG(\lambda)}$$

Thus, $P(\Lambda \leq x | N(t) = n) = \dfrac{\int_0^x e^{-\lambda t} (\lambda t)^n dG(\lambda)}{\int_0^\infty e^{-\lambda t} (\lambda t)^n dG(\lambda)}$

Summary

- Poisson process
- Properties of Poisson process
- Nonhomogeneous Poisson process
- Compound Poisson process
- Conditional Poisson process

References: Chapter 2, Stochastic Processes, 2nd edition, 1995, *by Sheldon M. Ross*