Last class

- Poisson process
- Properties of Poisson process
- Nonhomogeneous Poisson process
- Compound Poisson process
- Conditional Poisson process

References: Chapter 2, Stochastic Processes, 2nd edition, 1995, *by Sheldon M. Ross*

Stochastic Processes

Lecture 3: Renewal Process

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Poisson process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a **counting process** if $N(t)$ represents the total number of 'events' that have occurred up to time t.

Definition 3 [from Lecture 2]: The counting process $\{N(t), t \geq 0\}$ is said to be a **Poisson process** having rate λ, $\lambda > 0$, if

- Interarrival times $X_n, n = 1,2, \ldots$ are independent identically distributed exponential random variables having mean $1/\lambda$

Renewal process

Definition 1: The counting process $\{N(t), t \geq 0\}$ is said to be a **renewal process**, if

- Interarrival times $X_n, n = 1,2, \ldots$ are independent identically distributed non-negative random variables with a common distribution F, where $F(0) = P(X_n = 0) < 1$.

Events $\longleftrightarrow$ Renewals

$$S_n = X_1 + X_2 + \cdots + X_n\text{: the time of the } n\text{th event/renewal}$$

$S_n \sim F_n$

$$S_n \leq t \longleftrightarrow N(t) \geq n$$

$$\mu = E[X_i] = \int_0^\infty x dF(x)\text{: the expectation of } X_i$$

Renewal process

$P(N(t) = n)$?

Solution:

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n + 1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

Renewal process

$$N(\infty) = \lim_{t\to\infty} N(t)\ ?$$ $= \infty$ with prob. 1

Solution:

$$
\begin{aligned}
P(N(\infty) < \infty) &= P(X_n = \infty \text{ for some } n) \\
&= P\left(\bigcup_{n=1}^{\infty}\{X_n = \infty\}\right) \\
&\leq \sum_{n=1}^{\infty} P\left(X_n = \infty\right) \\
&= 0
\end{aligned}
$$

Renewal process

$$\lim_{t\to\infty}\frac{N(t)}{t}\ ? \qquad \to \frac{1}{\mu}\ \text{with prob. 1}$$

Solution:

$$S_{N(t)} \le t < S_{N(t)+1}$$

$$\Downarrow$$

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

$$\frac{S_{N(t)}}{N(t)} = \frac{X_1+X_2+\cdots+X_{N(t)}}{N(t)} \boxed{\to} \mu$$

$N(t) \to \infty$ as $t \to \infty$, then apply Strong Law of Large Numbers

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\cdot\frac{N(t)+1}{N(t)} \to \mu$$

Renewal process

Example: A container contains an infinite collection of coins. Each coin has its own probability of landing heads, and these probabilities are independently uniformly distributed over $(0, 1)$. Suppose we are to flip coins sequentially, at any time either flipping a new coin or one that had previously been used. If our objective is to maximize the long-run proportion of flips that lands on heads, how should we proceed?

Solution: $N(t)$: the number of tails in the first t flips

The objective: $\lim_{t\to\infty} 1 - \frac{N(t)}{t}$ 1

The strategy: chooses a coin and continues to flip it until coming up tails; discards this coin and repeats this process

Renewal process

Solution: $N(t)$: the number of tails in the first t flips

The objective: $\lim_{t\to\infty} 1 - \frac{N(t)}{t}$ 1

The strategy: chooses a coin and continues to flip it until coming up tails; discards this coin and repeats this process

The time intervals between two tails are iid random variables, thus $\{N(t), t \geq 0\}$ is a renewal process

Then we have $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$, where $\mu = \int_0^1 \boxed{\frac{1}{1-p}} dp = \infty$

mean of geometric distribution with parameter $1 - p$

Thus, $1 - \lim_{t\to\infty} \frac{N(t)}{t} = 1 - \frac{1}{\mu} = 1$

Elementary renewal theorem

Renewal function: $m(t) = E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$

Proof:

Way 1: $N(t) = \sum_{n=1}^{\infty} I_n$, $\quad I_n = \begin{cases} 1 & \text{if the } n\text{−th renewal occures in } [0,t] \\ 0 & \text{otherwise} \end{cases}$

$$E[N(t)] = E[\sum_{n=1}^{\infty} I_n] = \sum_{n=1}^{\infty} E\,[I_n] = \sum_{n=1}^{\infty} P\,(I_n = 1)$$

$$= \sum_{n=1}^{\infty} P\,(S_n \leq t) = \sum_{n=1}^{\infty} F_n\,(t)$$

Way 2: $P(N(t) \geq n) = P(S_n \leq t) = F_n(t)$

$$E[N(t)] = \sum_{n=1}^{\infty} n\,P(N(t) = n) = \sum_{n=1}^{\infty} P\,(N(t) \geq n) = \sum_{n=1}^{\infty} F_n\,(t)$$

Elementary renewal theorem

Renewal function: $m(t) = E[N(t)] < \infty$ for all $0 \leq t < \infty$

Proof:

$P(X_n = 0) < 1 \Rightarrow \exists\alpha > 0, \text{s.t. } P(X_n \geq \alpha) > 0$

Define a related process:

$$\bar{X}_n = \begin{cases} 0 & \text{if } X_n < \alpha \\ \alpha & \text{if } X_n \geq \alpha \end{cases}$$

Let $\bar{N}(t) = \sup\{n: \bar{X}_1 + \ldots + \bar{X}_n \leq t\}$, $\bar{X}_n \leq X_n \Rightarrow \bar{N}(t) \geq N(t) \Rightarrow E[\bar{N}(t)] \geq m(t)$

For the related process, renewals can only take place at time $t = n\alpha, n = 0,1,2, \ldots$

For $n \geq 1$, the number of renewals at time $n\alpha$ are independent geometric random variables with mean $1/P(X_n \geq \alpha)$; #{renewals at time 0}+1 is a geometric random variables with mean$1/P(X_n \geq \alpha)$

$$E[\bar{N}(t)] = \lfloor \frac{t}{\alpha} \rfloor \cdot \frac{1}{P(X_n \geq \alpha)} + \frac{1}{P(X_n \geq \alpha)} - 1 \leq \frac{t/\alpha + 1}{P(X_n \geq \alpha)} < \infty$$

Elementary renewal theorem

Elementary Renewal Theorem:

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty$$

Trivial?

$$\lim_{t\to\infty} \frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{with probability 1}$$

Note that the result does hold for a general random variable:

Let U be a random variable that is uniformly distributed on $(0,1)$.
Define $Y_n, n \geq 1$, by

$$Y_n = \begin{cases} 0 & \text{if } U > 1/n \\ n & \text{if } U \leq 1/n \end{cases}$$

Then, we have $Y_n \to 0$ as $n \to \infty$, but $E[Y_n] = nP(U \leq 1/n) = 1$

Elementary renewal theorem

Stopping time: An integer-valued random variable N is said to be a *stopping time* for the sequence of independent random variables $X_1, X_2, \ldots$, if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$, for all $n = 1, 2, \ldots$

Example: Let $X_n, n = 1,2, \ldots$, be independent and such that

$$P(X_n = 0) = P(X_n = 1) = \frac{1}{2}, \qquad n = 1,2, \ldots$$

Then, $N = \min\{n \mid X_1 + \cdots + X_n = 10\}$ is a stopping time

Elementary renewal theorem

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Example: Let $X_n, n = 1,2, \ldots$, be independent and such that

$$P(X_n = -1) = P(X_n = 1) = \frac{1}{2}, \qquad n = 1,2, \ldots$$

Then, $N = \min\{n \mid X_1 + \cdots + X_n = 1\}$ is a stopping time

Elementary renewal theorem

Wald's Equation: If $X_1, X_2, \ldots$ are iid random variables having finite expectations, and if N is a stopping time for $X_1, X_2, \ldots$ such that $E[N] < \infty$, then

$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X]$$

Proof: Let $I_n = \begin{cases} 1 & \text{if } N \geq n \\ 0 & \text{if } N < n \end{cases}$, then $\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$

$I_n = 1 \Leftrightarrow$ we have not stopped after $X_1, \ldots, X_{n-1}$, thus I_n is determined by $X_1, \ldots, X_{n-1}$ and independent of X_n

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} E[X_n I_n] = \sum_{n=1}^{\infty} E[X_n]E[I_n]$$

$$= E[X]\sum_{n=1}^{\infty} P(N \geq n) = E[X]E[N]$$

Next, we will give some applications of Wald's equation

Elementary renewal theorem

Example: Let $X_n, n = 1,2, \dots$, be independent and such that

$$P(X_n = 0) = P(X_n = 1) = \frac{1}{2}, \qquad n = 1,2, \dots$$

Then, $N = \min\{n \mid X_1 + \cdots + X_n = 10\}$ is a stopping time

$E[N] = ?$ 20

Wald's Equation: $E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X]$

1/2

must be 10

Elementary renewal theorem

Example: Let $X_n, n = 1,2, \dots$, be independent and such that

$$P(X_n = -1) = P(X_n = 1) = \frac{1}{2}, \qquad n = 1,2, \dots$$

Then, $N = \min\{n \mid X_1 + \cdots + X_n = 1\}$ is a stopping time

$E[N] =?$ ∞

Wald's Equation: $$E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X]$$

Note that N is finite with probability 1

Elementary renewal theorem

Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process

Corollary: If $\mu = E[X_i] < \infty$, then

$$E\left[S_{N(t)+1}\right] = \mu(m(t) + 1)$$

Proof:

To show that $N(t) + 1$ is a stopping time for the sequence of X_i

Wald's Equation: $$E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X]$$

Elementary renewal theorem

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Corollary: If $\mu = E[X_i] < \infty$, then

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Proof:

To show that $N(t) + 1$ is a stopping time for the sequence of X_i

$$N(t) + 1 = n \Leftrightarrow N(t) = n - 1$$

$$\Leftrightarrow X_1 + \cdots + X_{n-1} \leq t,\ X_1 + \cdots + X_n > t$$

$\{N(t) + 1 = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$

$\Longrightarrow$ $N(t) + 1$ is a stopping time for the sequence of X_i

Now, we can prove the elementary renewal theorem

Elementary renewal theorem

Elementary Renewal Theorem:

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty$$

Proof:

(1) $\mu < \infty$

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

$$S_{N(t)+1} > t \;\Rightarrow\; \mu(m(t)+1) > t$$

use the corollary on the previous page

$$\Longrightarrow \quad \liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

Elementary renewal theorem

(1) $\mu < \infty$ $\qquad \liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu} \qquad \limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$

Fix a constant M, define a new renewal process $\bar{X}_n = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases}$

Then, we have $\bar{X}_n \leq X_n \Rightarrow \bar{N}(t) \geq N(t) \Rightarrow \bar{m}(t) \geq m(t)$

use the corollary on the previous page

$$\bar{S}_{\bar{N}(t)+1} \leq t + M \Rightarrow (\bar{m}(t) + 1) \cdot \mu_M \leq t + M, \text{ where } \mu_M = E[\bar{X}_n]$$

$$\Rightarrow \frac{\bar{m}(t)+1}{t+M} \leq \frac{1}{\mu_M} \Rightarrow \limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M} \Rightarrow \limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu_M}$$

Let M $\to \infty \Rightarrow \mu_M \to \mu$, so $\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$

(2) $\mu = \infty$

$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu_M}$, let M $\to \infty \Rightarrow \mu_M \to \mu$, so $\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu} = 0$

Elementary renewal theorem

As $t \to \infty$, we have shown that

- ✓ $N(t) = \infty$ with prob. 1
- ✓ $\frac{m(t)}{t} = \frac{E[N(t)]}{t} \to \frac{1}{\mu}$
- ✓ $\frac{N(t)}{t} \to \frac{1}{\mu}$ with prob. 1

Theorem: Let μ and σ^2, assumed finite, represent the mean and variance of an interarrival time. Then, as $t \to \infty$,

$$P\left(\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx$$

Elementary renewal theorem

Theorem: Let μ and σ^2, assumed finite, represent the mean and variance of an interarrival time. Then, as $t \to \infty$,

$$P\left(\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx$$

Proof: Let $r_1 = t/\mu + y\sigma\sqrt{t/\mu^3}$

$$P\left(\frac{N(t)-t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right) = P(N(t) < r_1) = P\left(S_{r_1} > t\right)$$

$$= P\left(\frac{S_{r_1}-r_1\mu}{\sigma\sqrt{r_1}} > \frac{t-r_1\mu}{\sigma\sqrt{r_1}}\right) = P\left(\boxed{\frac{S_{r_1}-r_1\mu}{\sigma\sqrt{r_1}}} > \boxed{-y\left(1+\frac{y\sigma}{\sqrt{t\mu}}\right)^{-\frac{1}{2}}}\right)$$

$S_{r_1} = X_1 + \cdots + X_{r_1}, r_1 \to \infty$ as $t \to \infty$, then apply Central Limit Theorem, we have $\frac{S_{r_1}-r_1\mu}{\sigma\sqrt{r_1}} \to \mathcal{N}(0,1)$

$\to -y$ as $t \to \infty$

Key renewal theorem

Lattice: A nonnegative random variable $X \sim F$ is said to be *lattice* if there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} P(X = nd) = 1$. The largest d having this property is said to be the *period* of X.

Blackwell's Theorem:

- If F is not lattice, then for all $a \geq 0$

$$m(t+a) - m(t) \to \frac{a}{\mu} \qquad \text{as } t \to \infty$$

- If F is lattice with period d, then

$$E[\#\text{renewals at } nd] \to \frac{d}{\mu} \qquad \text{as } n \to \infty$$

Key renewal theorem

Directly Riemann Integrable: Let h be a function defined on $[0, \infty]$. For any $a > 0$, let $\overline{m}_n(a)$ and $\underline{m}_n(a)$ be the supremum and infinum of $h(t)$ over the interval $(n-1)a \leq t \leq na$. That is,

$$\overline{m}_n(a) = \sup\{h(t) \mid (n-1)a \leq t \leq na\}$$

$$\underline{m}_n(a) = \inf\{h(t) \mid (n-1)a \leq t \leq na\}$$

We say that h is *directly Riemann integrahle* if $\sum_{n=1}^{\infty} \overline{m}_n(a)$ and $\sum_{n=1}^{\infty} \underline{m}_n(a)$ are finite for all $a > 0$ and

$$\lim_{a \to 0} a \sum_{n=1}^{\infty} \overline{m}_n(a) = \lim_{a \to 0} a \sum_{n=1}^{\infty} \underline{m}_n(a)$$

Key renewal theorem

A sufficient condition for h to be *directly Riemann integrable*:

- $h(t) \geq 0$ for all $t \geq 0$
- $h(t)$ is non-increasing
- $\int_0^\infty h(t)dt < \infty$

Key Renewal Theorem:

If F is not lattice, and if $h(t)$ is directly Riemann integrable, then

$$\int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^t h(t)dt \qquad \text{as } t \to \infty$$

Key renewal theorem

Blackwell's Theorem:

- If F is not lattice, then for all $a \geq 0$

$$\Updownarrow \qquad m(t+a) - m(t) \to \frac{a}{\mu} \qquad \text{as } t \to \infty$$

Key Renewal Theorem:

If F is not lattice, and if $h(t)$ is directly Riemann integrable, then

$$\int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^t h(t)dt \qquad \text{as } t \to \infty$$

Blackwell ⇒ Key: $\lim_{t\to\infty} \frac{m(t+a)-m(t)}{a} = \frac{1}{\mu} \Rightarrow \lim_{a\to 0}\lim_{t\to\infty} \frac{m(t+a)-m(t)}{a} = \frac{1}{\mu} \Rightarrow \lim_{t\to\infty} \frac{dm(t)}{dt} = \frac{1}{\mu}$

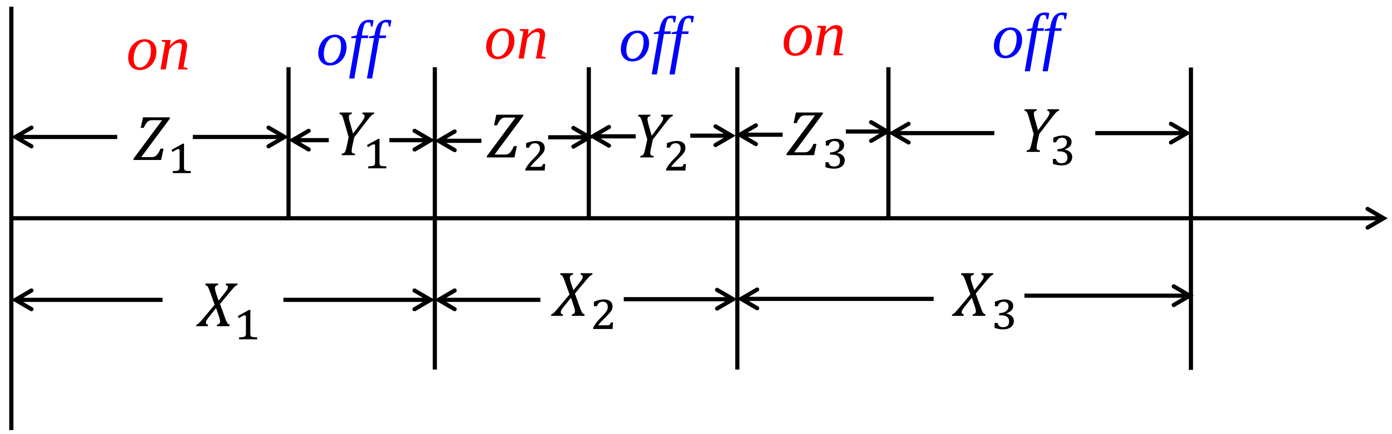
Key ⇒ Blackwell: let $h(x) = 1_{[0,a)}(x)$,

then $\int_0^t h(t-x)dm(x) = \int_0^t 1_{(t-a,t]}(x)dm(x) = m(t) - m(t-a)$

Alternating renewal process

Consider a system that can be in one of two states: *on* or *off*

$$X_n = Z_n + Y_n$$



$\{(Z_n, Y_n), n \geq 1\}$ are iid, but Z_n and Y_n can be dependent

$Z_n \sim H$ $\quad$ $Y_n \sim G$ $\quad$ $X_n = Z_n + Y_n \sim F$

Alternating renewal process

Theorem: If $E[Z_n + Y_n] < \infty$ and F is nonlattice, then

$$\lim_{t\to\infty} P(t) = P(\text{system is on at time } t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}$$

Alternating renewal process

Lemma: for $t \geq s \geq 0$

$$P\left(S_{N(t)} \leq s\right) = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y)$$

Proof:

law of total probability

$S_n \leq s, S_n \leq t, S_{n+1} > t$
$\Leftrightarrow \; S_n \leq s, N(t) = n$

$1 - F(t) = P(X_1 > t)$

$$\begin{aligned}
P\left(S_{N(t)} \leq s\right) &= \textstyle\sum_{n=0}^{\infty} P\left(S_n \leq s, S_{n+1} > t\right) \\
&= \bar{F}(t) + \textstyle\sum_{n=1}^{\infty} P\left(S_n \leq s, S_{n+1} > t\right) \\
&= \bar{F}(t) + \textstyle\sum_{n=1}^{\infty} \int_0^{\infty} P\left(S_n \leq s, S_{n+1} > t \mid S_n = y\right) dF_n(y) \\
&= \bar{F}(t) + \textstyle\sum_{n=1}^{\infty} \int_0^{s} \bar{F}\left(t-y\right) dF_n(y) \\
&= \bar{F}(t) + \int_0^{s} \bar{F}\left(t-y\right) d\left(\textstyle\sum_{n=1}^{\infty} F_n\left(y\right)\right) \\
&= \bar{F}(t) + \int_0^{s} \bar{F}\left(t-y\right) dm(y)
\end{aligned}$$

Alternating renewal process

Lemma: for $t \geq s \geq 0$

$$P\left(S_{N(t)} \leq s\right) = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y)$$

Remark:
$$\begin{cases} \bullet \quad P\left(S_{N(t)} = 0\right) = \bar{F}(t) \\ \bullet \quad dF_{S_{N(t)}}(s) = \bar{F}(t-s)dm(s) \quad \text{for } 0 < s < \infty \end{cases}$$

Intuitive explanation for remark 2:

$$dm(s) = \sum_{n=1}^{\infty} f_n(s)ds = \sum_{n=1}^{\infty} P\left(n\text{th renewal occurs in } (s, s+ds)\right)$$
$$= P(\text{renewal occurs in } (s, s+ds))$$

$$dF_{S_{N(t)}}(s) = f_{S_{N(t)}}(s)ds = P(\text{renewal in } (s, s+ds), \text{next interval} > t-s)$$
$$= \bar{F}(t-s) \cdot P\left(\text{renewal in } (s, s+ds)\right) = \bar{F}(t-s)dm(s)$$

Alternating renewal process

Theorem: If $E[Z_n + Y_n] < \infty$ and F is nonlattice, then

$$\lim_{t\to\infty} P(t) = P(\text{system is on at time } t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}$$

Proof:
$$P(t) = \underbrace{P\big(\text{system is on at time } t \mid S_{N(t)} = 0\big)}_{\triangle} P\big(S_{N(t)} = 0\big)$$
$$+ \int_0^\infty \underbrace{P\big(\text{system is on at time } t \mid S_{N(t)} = s\big)}_{\star} dF_{S_{N(t)}}(s)$$

$$\triangle = P(Z > t \mid Z + Y > t) = \frac{\overline{H}(t)}{\overline{F}(t)}$$

$$\star = P(Z > t - s \mid Z + Y > t - s) = \frac{\overline{H}(t-s)}{\overline{F}(t-s)}$$

$$\Longrightarrow \quad P(t) = \frac{\overline{H}(t)}{\overline{F}(t)} P\big(S_{N(t)} = 0\big) + \int_0^\infty \frac{\overline{H}(t-s)}{\overline{F}(t-s)} dF_{S_{N(t)}}(s)$$

$$= \overline{H}(t) + \int_0^\infty \overline{H}(t - s)\, dm(s)$$

Alternating renewal process

Key Renewal Theorem:

If F is not lattice, and if $h(t)$ is directly Riemann integrable, then

$$\int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^t h(t)dt \qquad \text{as } t \to \infty$$

$$P(t) = \bar{H}(t) + \int_0^\infty \bar{H}(t-s)dm(s)$$

- $\bar{H}(t) \geq 0$ for all $t \geq 0$
- $\bar{H}(t)$ is non-increasing
- $\int_0^\infty \bar{H}(t)dt = E[Z_n] < \infty$

$\bar{H}(t)$ is directly Riemann integrable

$\bar{H}(t) \to 0$ as $t \to 0$

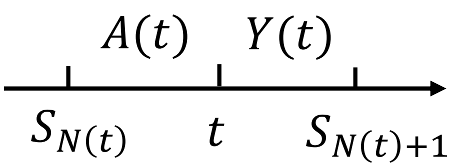
$$\lim_{t\to\infty} P(t) = \frac{1}{\mu}\int_0^\infty \bar{H}(t)dt = \frac{E[Z_n]}{E[Z_n]+E[Y_n]}$$

Alternating renewal process

Example: Consider a renewal process,

$A(t) = t - S_{N(t)}$: time from t since the last renewal, called "age" at t

$Y(t) = S_{N(t)+1} - t$: time from t until the next renewal, called "residual life" at t



$$\lim_{t\to\infty} P(A(t) \le x)?$$

Consider an alternating renewal process: $\begin{cases} \text{on} & \text{if } A(t) \le x \\ \text{off} & \text{otherwise} \end{cases}$

$$\lim_{t\to\infty} P(A(t) \le x) \dashrightarrow \frac{E[Z]}{E[X]}$$

$$= E[min(X, x)]/E[X]$$

$$= \int_0^\infty P\,(min(X, x) > y)dy/E[X]$$

$$= \int_0^x P\,(X > y)dy/E[X]$$

$$= \int_0^x \bar{F}\,(y)dy/\mu$$

$$\lim_{t\to\infty} P(Y(t) \le x)?$$

Leave as the exercise

Alternating renewal process

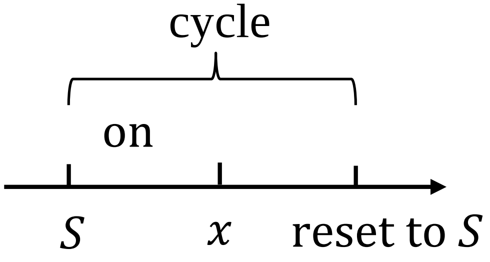
Example: Suppose that customers arrive at a store, which sells a single type of commodity, in accordance with a renewal process having nonlattice interarrival distribution F. The amounts desired by the customers are assumed to be independent with a common distribution G.

The store uses the following (s, S) ordering policy: if the inventory level after serving a customer is below s, then an order is instantaneously placed to bring it up to S; otherwise no order is placed.

Let $X(t)$ denote the inventory level at time t, and suppose $X(0) = S$

$\lim_{t\to\infty} P(X(t) \geq x)$?

Consider an alternating renewal process: $\begin{cases} \text{on} & \text{if } X(t) \geq x \\ \text{off} & \text{otherwise} \end{cases}$



Alternating renewal process

Let $X(t)$ denote the inventory level at time t, and suppose $X(0) = S$

$\lim_{t\to\infty} P(X(t) \geq x)$? $\dashrightarrow$ $\dfrac{E[\text{ amount of time the inventory } \geq x \text{ in a cycle }]}{E[\text{ time of a cycle }]}$

Let $Y_i, X_i, i \geq 1$, denote the demand and interarrival time of the i-th customer, and $N_x = \min\{n: Y_1 + \cdots + Y_n > S - x\}$

Then, amount of "on" time in cycle $= \sum_{i=1}^{N_x} X_i$, time of a cycle $= \sum_{i=1}^{N_s} X_i$, thus

$m_G(t) = \sum_{n=1}^{\infty} G_n(t)$

$$\lim_{t\to\infty} P(X(t) \geq x) = \frac{E\left[\sum_{i=1}^{N_x} X_i\right]}{E\left[\sum_{i=1}^{N_s} X_i\right]} = \frac{E[N_x]}{E[N_s]} = \frac{m_G(S-x)+1}{m_G(S-s)+1}$$

X_i is independent of N_x, N_s

$N_x - 1$ can be interpreted as the number of renewals by time $S - x$

Finally, we have $\lim_{t\to\infty} P(X(t) \geq x) = \frac{m_G(S-x)+1}{m_G(S-s)+1}, \quad s \leq x \leq S$

Delayed renewal process

Definition: Let $\{X_n, n = 1,2, \ldots\}$ be a sequence of independent nonnegative random variables with X_1 having distribution G, and X_n having distribution $F, n > 1$. Let $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_n$, $n \geq 1$, and define

$$N_D(t) = sup\{n: S_n \leq t\}$$

$\{N_D(t), t \geq 0\}$ is called a *delayed renewal process*

When $G = F$, $\{N_D(t), t \geq 0\}$ is an ordinary renewal process

$$\begin{aligned} P(N_D(t) = n) &= P(N_D(t) \geq n) - P(N_D(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) = G * F_{n-1}(t) - G * F_n(t) \end{aligned}$$

$$\begin{aligned} m_D(t) = E[N_D(t)] &= \sum_{n=1}^{\infty} P(N_D(t) \geq n) \\ &= \sum_{n=1}^{\infty} P(S_n \leq t) = \sum_{n=1}^{\infty} G * F_{n-1}(t) \end{aligned}$$

Delayed renewal process

Properties of delayed renewal process:

$$\mu = \int_0^\infty x dF(x)$$

- With probability 1, $\frac{N_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$
- $\frac{m_D(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ Elementary Renewal Theorem
- If F is not lattice, then $m_D(t+a) - m_D(t) \to a/\mu$ as $t \to \infty$
- If F and G are lattice with period d, then Blackwell's Theorem

$$E[\#\text{renewals at } nd] \to d/\mu \quad \text{as } n \to \infty$$

- If F is not lattice, $\mu < \infty$ and $h(t)$ is directly Riemann integrable,

$$\int_0^\infty h(t-x) dm_D(x) = \frac{1}{\mu} \int_0^\infty h(t) dt$$

Key Renewal Theorem

Delayed renewal process

Example: Suppose that a sequence of iid discrete random variables $X_1, X_2, \ldots$ is observed, and suppose that we keep track of the number of times that a given subsequence of outcomes, or pattern, occurs.

Suppose the pattern is $x_1, x_2, \ldots, x_k$ and say that it occurs at time n if $X_n = x_k, X_{n-1} = x_{k-1}, \ldots, X_{n-k+1} = x_1$.

Sequence: (1,0,1,0,1,0,1,1,1,0,1,0,1, ...)

Pattern 0,1,0,1 appears at time 5, 7, 13

If we let $N(t)$ denote the number of times the pattern occurs by time t, then $\{N(t), t \geq 1\}$ is a delayed renewal process

Delayed renewal process

N_A: time until pattern A occurs for the first time

$N_{A|B}$: additional time needed for pattern A to appear starting with pattern B

$P(A \text{ before } B)$: probability that pattern A occurs before pattern B

How to analyze them?

As G and F are lattice with period $d = 1$, by Blackwell's theorem

$$\frac{1}{\mu} = \lim_{n\to\infty} E[\#\text{patterns at } n]$$

$$= \lim_{n\to\infty} P(\text{pattern occurs at time } n) = \prod_{i=1}^{k} P(X = x_i)$$

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

Case: each random variable is 1 with prob. p and 0 with prob. q

How to calculate N_A?

- *If the occurrence of a pattern does not influence the next occurrence of the pattern, then*

$$\mathrm{E}[N_A] = 1/\prod_{i=1}^{k} P(X = x_i)$$

$$\text{e.g., } \mathrm{E}[N_{01}] = 1/pq$$

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

Case: each random variable is 1 with prob. p and 0 with prob. q

How to calculate N_A?

- *If the occurrence of a pattern will influence the next occurrence of the pattern, then*

$$\mathrm{E}[N_{0101}] = \mathrm{E}\left[N_{0101\,|\,01}\right] + \mathrm{E}[N_{01}]$$
$$= \frac{1}{p^2q^2} + \frac{1}{pq}$$

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

Case: each random variable is 1 with prob. p and 0 with prob. q

How to calculate N_A?

- *If the occurrence of a pattern will influence the next occurrence of the pattern, then*

$$
\begin{aligned}
\mathrm{E}[N_{1011011}] &= \mathrm{E}\left[N_{1011011\,|\,1011}\right] + \mathrm{E}[N_{1011}] \\
&= \frac{1}{p^5q^2} + \mathrm{E}\left[N_{1011\,|\,1}\right] + \mathrm{E}[N_1] \\
&= \frac{1}{p^5q^2} + \frac{1}{p^3q} + \frac{1}{p}
\end{aligned}
$$

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

Case: each random variable is 1 with prob. p and 0 with prob. q

How to calculate N_A?

- *If the occurrence of a pattern will influence the next occurrence of the pattern, then*

$$\mathrm{E}[N_{\underbrace{11\cdots1}_{k \text{ consecutive 1s}}}]$$

Leave as the exercise

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

How to calculate $P(A \text{ before } B)$?

Use the relationship between $\mathrm{E}[N_A]$, $\mathrm{E}[N_B]$ and $P(A \text{ before } B)$

Let $M = \min\{N_A, N_B\}$

$$
\begin{aligned}
E[N_A] &= E[M] + E[N_A - M] \\
&= E[M] + E[\underbrace{N_A - M \mid B \text{ before } A}_{N_A - N_B}](1 - P(A \text{ before } B)) \\
&= E[M] + E\left[N_{A|B}\right](1 - P(A \text{ before } B))
\end{aligned}
$$

$$
\mathrm{E}[N_B] = E[M] + E\left[N_{B|A}\right]P(A \text{ before } B)
$$

Delayed renewal process

Expected time between patterns: $\mu = 1/\prod_{i=1}^{k} P(X = x_i)$

How to calculate $P(A \text{ before } B)$?

$$P(A \text{ before } B) = \frac{E[N_B] + E\left[N_{A|B}\right] - E[N_A]}{E\left[N_{B|A}\right] + E[N_{A|B}]}$$

Case: each random variable is 1 with prob. p and 0 with prob. q

$A = 1010 \quad B = 0100$

Delayed renewal process

$$P(A \text{ before } B) = \frac{E[N_B] + E\left[N_{A|B}\right] - E[N_A]}{E\left[N_{B|A}\right] + E[N_{A|B}]}$$

$A = 1010 \quad B = 0100$

Leave as the exercise

Renewal reward process

Definition: Consider a renewal process $\{N(t), t \geq 0\}$ having interarrival times $X_n, n \geq 1$ with distribution F, and suppose that each time a renewal occurs we receive a reward. We denote by R_n the reward earned at the time of the nth renewal. Assume that the $R_n, n \geq 1$ are iid.

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

$\{R(t), t \geq 0\}$ is called a *renewal reward process*

Note that $\{(X_n, R_n), n \geq 1\}$ are iid, but R_n may depend on X_n

Renewal reward process

Theorem: If $E[R] < \infty$ and $E[X] < \infty$, then

- With probability 1, $\frac{R(t)}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$
- $\frac{E[R(t)]}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$

Proof:

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \cdot \frac{N(t)}{t}$$

$\lim_{t\to\infty} N(t) = \infty$ with prob. 1

$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[X]}$ with prob. 1

Renewal reward process

Theorem: If $E[R] < \infty$ and $E[X] < \infty$, then

- With probability 1, $\dfrac{R(t)}{t} \to \dfrac{E[R]}{E[X]}$ as $t \to \infty$
- $\boxed{\dfrac{E[R(t)]}{t} \to \dfrac{E[R]}{E[X]} \quad \text{as } t \to \infty}$

Proof:

$$E[R(t)] = E\left[\sum_{n=1}^{N(t)} R_n\right] = E\left[\sum_{n=1}^{N(t)+1} R_n\right] - E[R_{N(t)+1}]$$

$N(t) + 1$: a stopping time for the sequence of X_i, and thus a stopping time for the sequence of R_i

By Wald's equation $= (m(t) + 1)E[R] - E[R_{N(t)+1}]$

Renewal reward process

$$\frac{E[R(t)]}{t} = \boxed{\frac{(m(t)+1)E[R]}{t}} - \frac{E\left[R_{N(t)+1}\right]}{t}$$

$t \to \infty$ By elementary renewal theorem

$E[R]/E[X]$

We only need to show that $\lim_{t\to\infty} \frac{E\left[R_{N(t)+1}\right]}{t} \to 0$

$$g(t) = E\left[R_{N(t)+1}\right]$$

$$= E\left[R_{N(t)+1} | S_{N(t)} = 0\right]\bar{F}(t) + \int_0^t E\left[R_{N(t)+1} | S_{N(t)} = s\right]\bar{F}(t-s)dm(s)$$

Remark:

- $P\left(S_{N(t)} = 0\right) = \bar{F}(t)$
- $dF_{S_{N(t)}}(s) = \bar{F}(t-s)dm(s)$ for $0 < s < \infty$

Renewal reward process

$$g(t) = E\left[R_{N(t)+1}\right]$$
$$= E\left[R_{N(t)+1} | S_{N(t)} = 0\right]\bar{F}(t) + \int_0^t E\left[R_{N(t)+1} | S_{N(t)} = s\right] \bar{F}(t-s)dm(s)$$
$$= E[R|X > t]\bar{F}(t) + \int_0^t E[R|X > t-s]\,\bar{F}(t-s)dm(s)$$
$$= h(t) + \int_0^t h(t-s)\,dm(s)$$

Let $h(t) = E[R|X > t]\bar{F}(t)$

$$h(t) = \int_t^\infty E[R|X = x]\,dF(x) \le E[|R|] = \int_0^\infty E[|R| \mid X = x]\,dF(x) < \infty$$

$h(t) \to 0$ as $t \to \infty$, thus, $\forall \epsilon > 0, \exists T > 0$, such that $\forall t \ge T$, $|h(t)| < \epsilon$

Renewal reward process

$$g(t) = h(t) \ + \int_0^t h(t-s)\, dm(s)$$

$$\frac{|g(t)|}{t} \leq \frac{|h(t)|}{t} + \int_0^{t-T} \frac{|h(t-s)|}{t} dm(s) + \int_{t-T}^{t} \frac{|h(t-s)|}{t} dm(\mathrm{s})$$
$$\leq \frac{\epsilon}{t} + \frac{\epsilon m(t-T)}{t} + E[|R\,|] \frac{m(t)-m(t-T)}{t}$$
$$\to 0 + \frac{\varepsilon}{\mathrm{E}[X]} + \frac{E[|R\,|]}{t} \frac{T}{\mathrm{E}[X]} \quad (\text{as } t \to \infty)$$
$$= \frac{\varepsilon}{\mathrm{E}[X]}$$

Elementary renewal theorem: $\frac{m(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$

Blackwell's Theorem: $m(t+a) - m(t) \to \frac{a}{\mu}$ as $t \to \infty$

Renewal reward process

We have assumed that the reward R_n is earned all at once at the end of the renewal cycle

Theorem: If $E[R] < \infty$ and $E[X] < \infty$, then

- With probability 1, $\frac{R(t)}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$
- $\frac{E[R(t)]}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$

If the reward R_n is earned gradually during the renewal cycle, the above theorem still holds

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)} R_n + R_{N(t)+1}}{t}$$

Renewal reward process

Example: Suppose that travelers arrive at a train depot in accordance with a renewal process having a mean interarrival time μ. Whenever there are N travelers waiting in the depot, a train leaves.

If the depot incurs a cost at the rate of nc dollars per unit time whenever there are n travelers waiting and an additional cost of K each time a train is dispatched, what is the average cost per unit time incurred by the depot?

Solution: A cycle is completed whenever a train leaves $\Rightarrow$ A renewal reward process

$$\text{Average cost per unit time} = \frac{E[\text{cost of a cycle}]}{E[\text{length of a cycle}]}$$

Renewal reward process

$$\text{Average cost per unit time} = \frac{E[\text{cost of a cycle}]}{E[\text{length of a cycle}]}$$

$$E[\text{length of cycle}] = N\mu$$

Let X_n denote the time between the n-th and $(n+1)$-th arrival in a cycle, then

$$\begin{aligned} E[\text{ cost of a cycle }] &= E[cX_1 + 2cX_2 + \cdots + (N-1)cX_{N-1}] + K \\ &= \frac{c\mu N(N-1)}{2} + K \end{aligned}$$

Thus, the average cost is

$$\frac{c(N-1)}{2} + \frac{K}{N\mu}$$

Renewal reward process

Example: Suppose that customers arrive at a single-server service station in accordance with a nonlattice renewal process. Upon arrival, a customer is immediately served if the server is idle, and he or she waits in line if the server is busy. The service times of customers are assumed to be iid, and are also assumed independent of the arrival stream.

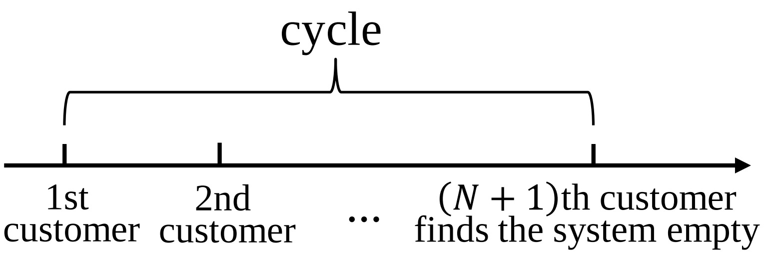
X_i: Interarrival time Y_i: Service time Assume $E[Y_i] < E[X_i] < \infty$

Suppose that the first customer arrives at time 0 and let $n(t)$ denote the number of customers in the system at time t

$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t n(s)\,ds$: long-run average #customers in the system

$n(s)$: the rate of a reward earned at time s

$$L = \frac{E[\text{ reward during a cycle }]}{E[\text{ time of a cycle }]} = \frac{E\left[\int_0^T n(s)ds\right]}{E[T]}$$



Renewal reward process

Example: Suppose that customers arrive at a single-server service station in accordance with a nonlattice renewal process. Upon arrival, a customer is immediately served if the server is idle, and he or she waits in line if the server is busy. The service times of customers are assumed to be iid, and are also assumed independent of the arrival stream.

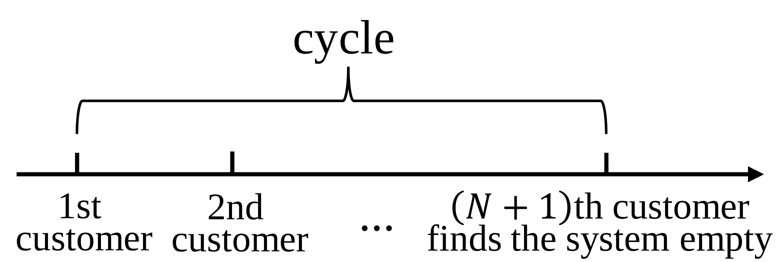
X_i: Interarrival time $\quad$ Y_i: Service time $\quad$ Assume $E[Y_i] < E[X_i] < \infty$

let W_i denote the amount of time the ith customer spends in the system

$W = \lim_{n\to\infty} \frac{W_1+\cdots+W_n}{n}$: long-run average time a customer spends in the system

W_i: the reward earned at day i

$$W = \frac{E[\text{ reward during a cycle }]}{E[\text{ time of a cycle }]} = \frac{E\left[\sum_{i=1}^{N} W_i\right]}{E[N]}$$



Renewal reward process

$$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t n(s)\,ds = \frac{E[\int_0^T n(s)\,ds]}{E[T]}$$ long-run average #customers in the system

$$W = \lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{n} = \frac{E[\sum_{i=1}^{N} W_i]}{E[N]}$$ long-run average time a customer spends in the system

T is the length of a cycle, N is the number of customers served in that cycle $\Rightarrow T = \sum_{i=1}^{N} X_i$

N is a stopping time for $X_1, X_2, \ldots$, because

$$N = n \Leftrightarrow X_1 + \cdots + X_k < Y_1 + \cdots + Y_k,\ 1 \le k \le n-1,$$
$$\text{and } X_1 + \cdots + X_n > Y_1 + \cdots + Y_n$$

By Wald's equation: $E[T] = E[N]E[X] = E[N]/\lambda$, thus $L = \lambda W \frac{E\left[\int_0^T n(s)ds\right]}{E\left[\sum_{i=1}^{N} W_t\right]}$

$\int_0^T n(s)ds = \sum_{i=1}^{N} W_t$
= total paid during a cycle

$L = \lambda \cdot W$

$\lambda = 1/E[X_i]$

Symmetric random walk

Definition: Let $Y_1, Y_2, \ldots$ be iid with

$$P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}$$

and define
$$Z_0 = 0, Z_n = \sum_{i=1}^{n} Y_i$$

$\{Z_n, n \geq 0\}$ is called the *symmetric random walk* process

We want to know

$$\lim_{n\to\infty} \frac{\text{amount of time in } [0,2n] \text{ that } Z_n \text{ is positive}}{2n}$$

Symmetric random walk

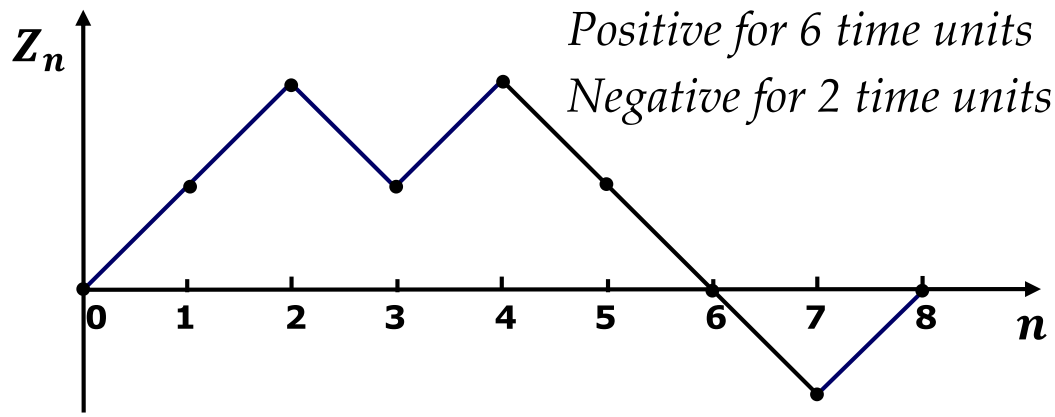
$$u_n = P(Z_{2n} = 0) = \binom{2n}{n} \frac{1}{2^{2n}}$$

Lemma: $P(Z_1 \neq 0, Z_2 \neq 0, \ldots, Z_{2n} \neq 0) = u_n$

Leave as the exercise

Theorem: Let $E_{k,n}$ denote the event that by time $2n$ the symmetric random walk will be positive for $2k$ time units and negative for $2n - 2k$ time units, and let $b_{k,n} = P(E_{k,n})$ Then

$$\boxed{b_{k,n} = u_k u_{n-k}}$$



Positive for 6 time units

Negative for 2 time units

Symmetric random walk

Theorem: Let $E_{k,n}$ denote the event that by time $2n$ the symmetric random walk will be positive for $2k$ time units and negative for $2n - 2k$ time units, and let $b_{k,n} = P(E_{k,n})$ Then

$$b_{k,n} = u_k u_{n-k}$$

Proof:

For $n = 1$: $b_{0,1} = b_{1,1} = \frac{1}{2}, u_0 = 1, u_1 = \frac{1}{2}$

Assume $b_{k,m} = u_k u_{m-k}$ for all values of m such that $m < n$, we need to show that the equation holds for n

Symmetric random walk

the time of the first return to 0

$$b_{n,n} = \sum_{r=1}^{n} P\big(E_{n,n} \mid \boxed{T} = 2r\big)P(T = 2r) + P\big(E_{n,n} \mid T > 2n\big)P(T > 2n)$$

$$P\big(E_{n,n} \mid T = 2r\big) = b_{n-r,\,n-r}/2\,,\, P(E_{n,\,n} \mid T > 2n) = \frac{1}{2}$$

Then, $b_{n,n} = \frac{1}{2}\sum_{r=1}^{n} b_{n-r,\,n-r}P(T = 2r) + \frac{1}{2}P(T > 2n)$

inductive hypothesis

$= \frac{1}{2}\sum_{r=1}^{n} u_{n-r}\underbrace{u_0}_{1}P(T = 2r) + \frac{1}{2}P(T > 2n)$

Note that $\sum_{r=1}^{n} u_{n-r}P(T = 2r) = \sum_{r=1}^{n} P(Z_{2n-2r} = 0)P(T = 2r)$

$= \sum_{r=1}^{n} P\big(Z_{2n} = 0 \mid T = 2r\big)P(T = 2r)$

$= P(Z_{2n} = 0) = u_n$

Thus, $b_{n,n} = \frac{1}{2}u_n + \frac{1}{2}P(T > 2n) = \frac{1}{2}u_n + \frac{1}{2}u_n = u_n$

Symmetric random walk

$$\text{for } 0 < k < n: \qquad b_{k,n} = \sum_{r=1}^{n} \underbrace{P\big(E_{k,n} \mid T = 2r\big)}_{\frac{1}{2} b_{k-r,n-r} + \frac{1}{2} b_{k,n-r}} P(T = 2r)$$

inductive hypothesis

$$\begin{aligned}
b_{k,n} &= \tfrac{1}{2}\textstyle\sum_{r=1}^{k} \ b_{k-r,n-r} P(T = 2r) + \tfrac{1}{2}\textstyle\sum_{r=1}^{n-k} \ b_{k,n-r} P(T = 2r) \\
&= \tfrac{1}{2} u_{n-k} \textstyle\sum_{r=1}^{k} \ u_{k-r} P(T = 2r) + \tfrac{1}{2} u_k \textstyle\sum_{r=1}^{n-k} \ u_{n-r-k} P(T = 2r) \\
&= \tfrac{1}{2} u_{n-k} u_k + \tfrac{1}{2} u_k u_{n-k} \\
&= u_{n-k} u_k
\end{aligned}$$

Symmetric random walk

Theorem: Let $E_{k,n}$ denote the event that by time $2n$ the symmetric random walk will be positive for $2k$ time units and negative for $2n-2k$ time units, and let $b_{k,n} = P(E_{k,n})$ Then

$$b_{k,n} = u_k u_{n-k} \sim 1/\left(\pi\sqrt{k(n-k)}\right)$$

$$u_n = P(Z_{2n} = 0) = \binom{2n}{n}\frac{1}{2^{2n}} \sim 1/\sqrt{\pi n}$$

Stirling's approximation: $n! \sim \sqrt{2\pi n}\cdot(n/e)^n$

Now we can analyze

$$\lim_{n\to\infty}\frac{\text{amount of time in } [0,2n] \text{ that } Z_n \text{ is positive}}{2n} = \xi$$

Symmetric random walk

$$b_{k,n} = u_k u_{n-k} \sim 1/\left(\pi\sqrt{k(n-k)}\right)$$

$$\boxed{\lim_{n\to\infty} \frac{\text{amount of time in } [0,2n] \text{ that } Z_n \text{ is positive}}{2n} = \xi}$$

For $x \in [0,1]$,

Random variable

$P(\xi \leq x)$

Arc Sine Laws

$$= \sum_{k=0}^{nx} b_{k,n} \approx \frac{1}{\pi}\int_0^{nx} \frac{1}{\sqrt{y(n-y)}}\,dy = \frac{1}{\pi}\int_0^{x} \frac{1}{\sqrt{w(1-w)}}\,dw = \frac{2}{\pi}\arcsin\sqrt{x}$$

$w = \frac{y}{n}$

$(arc\sin x)' = \frac{1}{\sqrt{1-x^2}}$

Different from $\lim_{n\to\infty} P(Z_n > 0) \to \frac{1}{2}$

Summary

- Renewal process
- Elementary renewal theorem
- Key renewal theorem
- Alternating renewal process
- Delayed renewal process
- Renewal reward process
- Symmetric random walk

References: Chapter 3, Stochastic Processes, 2nd edition, 1995, *by Sheldon M. Ross*