

# Convex Functions (I)

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# Outline

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## □ Basic Properties

- Definition
- First-order Conditions, Second-order Conditions
- Jensen's inequality and extensions
- Epigraph

## □ Operations That Preserve Convexity

- Nonnegative Weighted Sums
- Composition with an affine mapping
- Pointwise maximum and supremum
- Composition
- Minimization
- Perspective of a function

## □ Summary



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# Convex Function

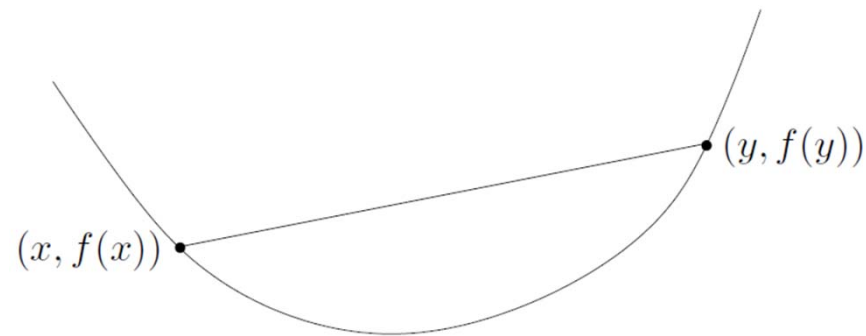
□  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if

■  $\text{dom } f$  is convex

$\theta x + (1 - \theta)y \in \text{dom } f, \forall \theta \in [0,1], x, y \in \text{dom } f$

■  $\forall \theta \in [0,1], x, y \in \text{dom } f$

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$





# Convex Function

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$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

□  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is strictly convex if

■  $\forall \theta \in (0,1), x \neq y$

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$$



# Convex Function

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□  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if

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$$\theta x + (1 - \theta)y \in \text{dom } f, \forall \theta \in [0,1], x, y \in \text{dom } f$$

■  $\forall \theta \in [0,1], x, y \in \text{dom } f$

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

□  $f$  is concave if  $-f$  is convex

■  $\text{dom } f$  is convex

□ Affine functions are both convex and concave, and vice versa.



# Extended-value Extensions

□ The extended-value extension of  $f$  is

$$\blacksquare \tilde{f}(x) = \begin{cases} f(x) & x \in \text{dom } f \\ \infty & x \notin \text{dom } f \end{cases}$$

$$\blacksquare \tilde{f}: \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\infty\}$$

$$\tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta)\tilde{f}(y)$$

$$\blacksquare \text{dom } f = \{x \mid \tilde{f}(x) < \infty\}$$

□ Example

$$\blacksquare f(x) = f_1(x) + f_2(x), \text{dom } f = \text{dom } f_1 \cap \text{dom } f_2$$

$$\blacksquare \tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$$

$$\tilde{f}(x) = \infty, \text{ if } x \notin \text{dom } f_1 \text{ or } x \notin \text{dom } f_2$$



# Extended-value Extensions

□ The extended-value extension of  $f$  is

$$\blacksquare \tilde{f}(x) = \begin{cases} f(x) & x \in \text{dom } f \\ \infty & x \notin \text{dom } f \end{cases}$$

$$\blacksquare \tilde{f}: \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\infty\}$$

$$\tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta)\tilde{f}(y)$$

□ Example

■ Indicator Function of a Set  $C$

$$\tilde{I}_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$





# Zeroth-order Condition

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## □ Definition

- High-dimensional space

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

- A function is convex if and only if it is convex when restricted to any line that intersects its domain.

- $x \in \text{dom } f, v \in \mathbf{R}^n, t \in \mathbf{R}, x + tv \in \text{dom } f$
- $f$  is convex  $\Leftrightarrow g(t) = f(x + tv)$  is convex
- One-dimensional space

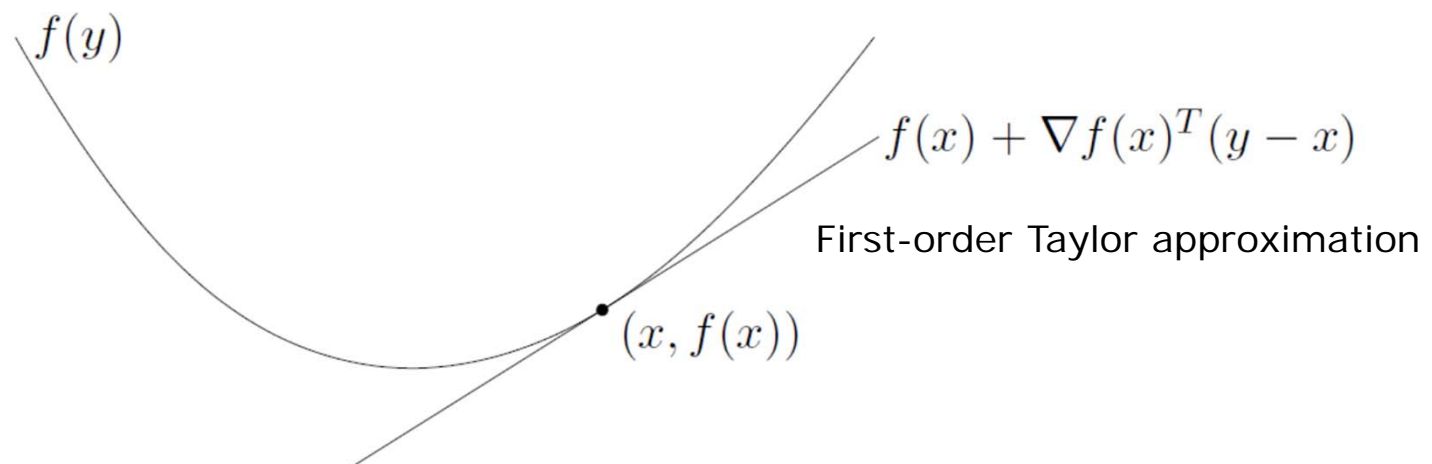


# First-order Conditions

□  $f$  is differentiable. Then  $f$  is convex if and only if

- $\text{dom } f$  is convex
- For all  $x, y \in \text{dom } f$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$





# First-order Conditions

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□  $f$  is differentiable. Then  $f$  is convex if and only if

■  $\text{dom } f$  is convex

■ For all  $x, y \in \text{dom } f$

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

■ Local Information  $\Rightarrow$  Global Information

■  $\nabla f(x) = 0 \Rightarrow f(y) \geq f(x), \forall y \in \text{dom } f$

□  $f$  is strictly convex if and only if

$$f(y) > f(x) + \nabla f(x)^\top (y - x)$$



# Proof

□  $f$  is convex  $\Leftrightarrow f: \mathbf{R} \rightarrow \mathbf{R}, f(y) \geq f(x) + f'(x)(y - x), x, y \in \text{dom } f$

■ Necessary condition:

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y), 0 \leq t \leq 1$$

$$\Rightarrow f(y) \geq f(x) + \frac{f(x+t(y-x))-f(x)}{t}$$

$$\xrightarrow{t \rightarrow 0} \Rightarrow f(y) \geq f(x) + f'(x)(y - x)$$

■ Sufficient condition:

$$\left. \begin{array}{l} z = \theta x + (1 - \theta)y \\ f(x) \geq f(z) + f'(z)(x - z) \\ f(y) \geq f(z) + f'(z)(y - z) \end{array} \right\} \Rightarrow \left. \begin{array}{l} f(x) \geq f(z) + (1 - \theta)f'(z)(x - y) \\ f(y) \geq f(z) - \theta f'(z)(x - y) \end{array} \right\}$$

$$\Rightarrow \theta f(x) + (1 - \theta)f(y) \geq f(z) \Rightarrow f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

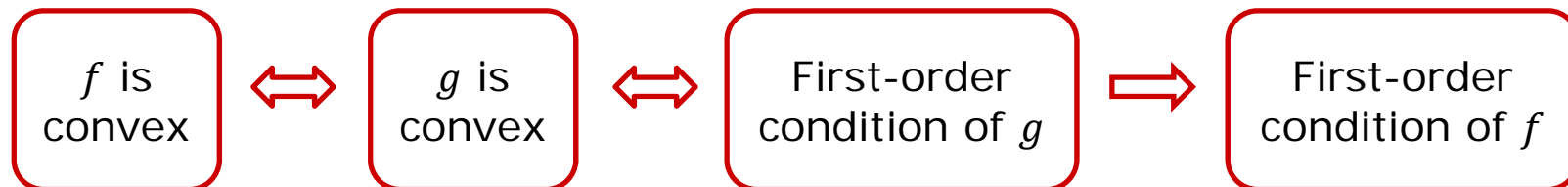


# Proof

- $f$  is convex  $\Leftrightarrow f: \mathbf{R} \rightarrow \mathbf{R}, f(y) \geq f(x) + f'(x)(y - x), x, y \in \text{dom } f$
- $f$  is convex  $\Leftrightarrow f: \mathbf{R}^n \rightarrow \mathbf{R}, f(y) \geq f(x) + \nabla f(x)^\top (y - x), x, y \in \text{dom } f$

$$g(t) = f(ty + (1 - t)x), \quad g'(t) = \nabla f(ty + (1 - t)x)^\top (y - x)$$

- $f$  is convex  $\Rightarrow g(t)$  is convex  $\Rightarrow g(1) \geq g(0) + g'(0) \Rightarrow f(y) \geq f(x) + \nabla f(x)^\top (y - x)$





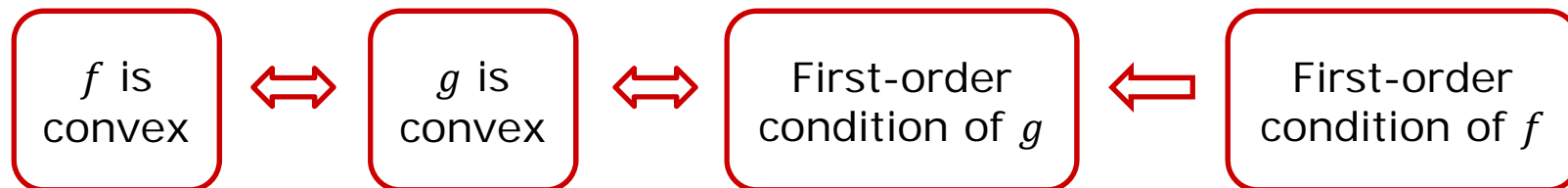
# Proof

- $f$  is convex  $\Leftrightarrow f: \mathbf{R} \rightarrow \mathbf{R}, f(y) \geq f(x) + f'(x)(y - x), x, y \in \text{dom } f$
- $f$  is convex  $\Leftrightarrow f: \mathbf{R}^n \rightarrow \mathbf{R}, f(y) \geq f(x) + \nabla f(x)^\top (y - x), x, y \in \text{dom } f$

$$g(t) = f(ty + (1 - t)x), \quad g'(t) = \nabla f(ty + (1 - t)x)^\top (y - x)$$

$$\blacksquare \quad f(ty + (1 - t)x) \geq f(\tilde{t}y + (1 - \tilde{t})x) + \nabla f(\tilde{t}y + (1 - \tilde{t})x)^\top (y - x)(t - \tilde{t})$$

$$\Rightarrow g(t) \geq g(\tilde{t}) + g'(\tilde{t})(t - \tilde{t}) \Rightarrow g(t) \text{ is convex} \Rightarrow f \text{ is convex}$$





# Second-order Conditions

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□  $f$  is twice differentiable. Then  $f$  is convex if and only if

- $\text{dom } f$  is convex
- For all  $x \in \text{dom } f$ ,  $\nabla^2 f(x) \succeq 0$

□ Attention

- $\nabla^2 f(x) \succ 0 \Rightarrow f$  is strictly convex
- $f$  is strict convex  $\not\Rightarrow \nabla^2 f(x) \succ 0$   
 $f(x) = x^4$  is strict convex but  $f''(0) = 0$
- $\text{dom } f$  is convex is necessary,  $f(x) = 1/x^2$



# Examples

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## □ Functions on $\mathbf{R}$

- $e^{ax}$  is convex on  $\mathbf{R}$ ,  $\forall a \in \mathbf{R}$
- $x^a$  is convex on  $\mathbf{R}_{++}$  when  $a \geq 1$  or  $a \leq 0$ , and concave for  $0 \leq a \leq 1$
- $|x|^p$ , for  $p \geq 1$ , is convex on  $\mathbf{R}$
- $\log x$  is concave on  $\mathbf{R}_{++}$
- Negative entropy  $x \log x$  is convex on  $\mathbf{R}_{++}$





# Examples

## □ Functions on $\mathbf{R}^n$

■ Every norm on  $\mathbf{R}^n$  is convex

■  $f(x) = \max\{x_1, \dots, x_n\}$

■ Quadratic-over-linear:  $f(x, y) = \frac{x^2}{y}$

✓  $\text{dom } f = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$

■  $f(x) = \log(e^{x_1} + \dots + e^{x_n})$

$\max\{x_1, \dots, x_n\} \leq f(x) \leq \max\{x_1, \dots, x_n\} + \log n$

■  $f(x) = (\prod_{i=1}^n x_i)^{1/n}$  is concave on  $\mathbf{R}_{++}^n$

■  $f(X) = \log \det X$  is concave on  $\mathbf{S}_{++}^n$



# Examples

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## □ Functions on $\mathbf{R}^n$

■ Every norm on  $\mathbf{R}^n$  is convex

✓  $f(x)$  is a norm on  $\mathbf{R}^n$

$$\begin{aligned} \checkmark \quad f(\theta x + (1 - \theta)y) &\leq f(\theta x) + f((1 - \theta)y) \\ &= \theta f(x) + (1 - \theta)f(y) \end{aligned}$$

■  $f(x) = \max\{x_1, \dots, x_n\} = \max_i x_i$

$$\begin{aligned} \checkmark \quad f(\theta x + (1 - \theta)y) &= \max_i \{\theta x_i + (1 - \theta)y_i\} \\ &\leq \theta \max_i \{x_i\} + (1 - \theta) \max_i \{y_i\} \end{aligned}$$

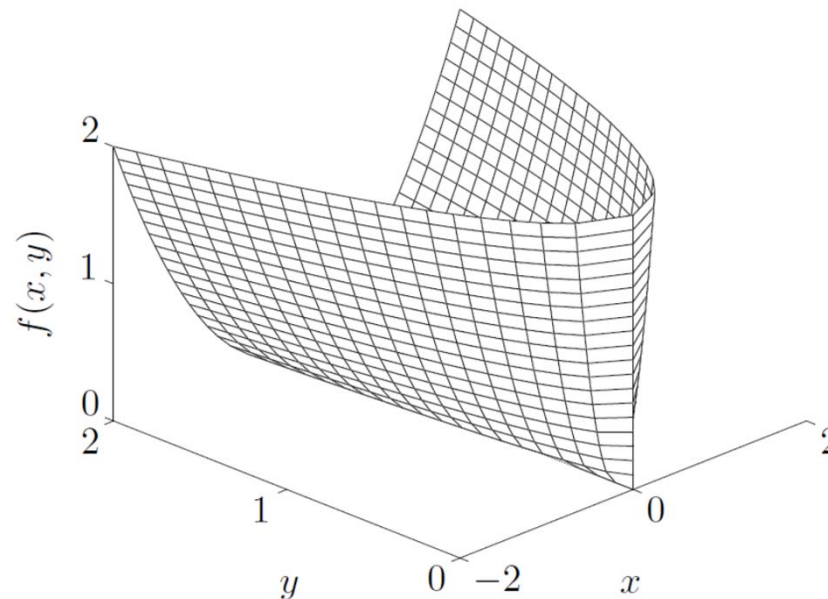


# Examples

## □ Functions on $\mathbf{R}^n$

■  $f(x, y) = \frac{x^2}{y}, \text{ dom } f = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$

✓  $\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^\top \succcurlyeq 0$

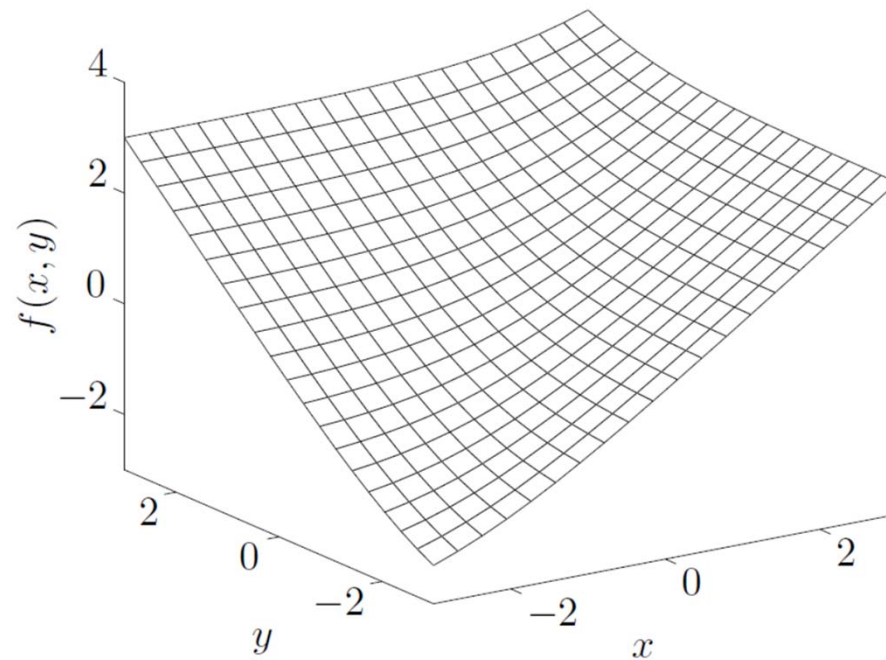




# Examples

## □ Functions on $\mathbf{R}^n$

■  $f(x) = \log(e^{x_1} + \dots + e^{x_n})$





# Examples

## □ Functions on $\mathbf{R}^n$

■  $f(x) = \log(e^{x_1} + \dots + e^{x_n})$

✓  $\nabla^2 f(x) = \frac{1}{(\mathbf{1}^\top z)^2} \left( (\mathbf{1}^\top z) \text{diag}(z) - z z^\top \right)$

✓  $z = (e^{x_1}, \dots, e^{x_n})$

✓  $v^\top \nabla^2 f(x) v = \frac{1}{(\mathbf{1}^\top z)^2} \left( \left( \sum_{i=1}^n z_i \right) \left( \sum_{i=1}^n v_i^2 z_i \right) - \left( \sum_{i=1}^n v_i z_i \right)^2 \right) \geq 0$

✓ Cauchy-Schwarz inequality:  $(a^\top a)(b^\top b) \geq (a^\top b)^2$



# Examples

## □ Functions on $\mathbf{R}^n$

■  $f(X) = \log \det X$  is concave on  $\mathbf{S}_{++}^n$

✓  $g(t) = f(Z + tV), Z + tV \succ 0, Z \succ 0$

✓  $g(t) = \log \det(Z + tV)$

$$= \log \det \left( Z^{\frac{1}{2}} \left( I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}} \right) Z^{\frac{1}{2}} \right)$$

$$= \sum_{i=1}^n \log(1 + t\lambda_i) + \log \det Z$$

✓  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$

✓  $g'(t) = \sum_{i=1}^n \frac{\lambda_i}{1+t\lambda_i}, g''(t) = -\sum_{i=1}^n \frac{\lambda_i^2}{(1+t\lambda_i)^2}$

$\det(AB) = \det(A) \det(B)$  <https://en.wikipedia.org/wiki/Determinant>



# Sublevel Sets

## □ $\alpha$ -sublevel set

$$C_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\}$$

- $f(x)$  is convex  $\Rightarrow C_\alpha$  is convex,  $\forall \alpha \in \mathbf{R}$
- $C_\alpha$  is convex,  $\forall \alpha \in \mathbf{R} \not\Rightarrow f(x)$  is convex  
e.g.,  $f(x) = -e^x$

## □ $\alpha$ -superlevel set

$$C_\alpha = \{x \in \text{dom } f \mid f(x) \geq \alpha\}$$

- $f(x)$  is concave  $\Rightarrow C_\alpha$  is convex,  $\forall \alpha \in \mathbf{R}$
- $G(x) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ ,  $A(x) = \frac{1}{n} \sum_{i=1}^n x_i$
- $\{x \in \mathbf{R}_+^n \mid G(x) \geq \alpha A(x)\}$  is convex,  $\alpha \in [0,1]$



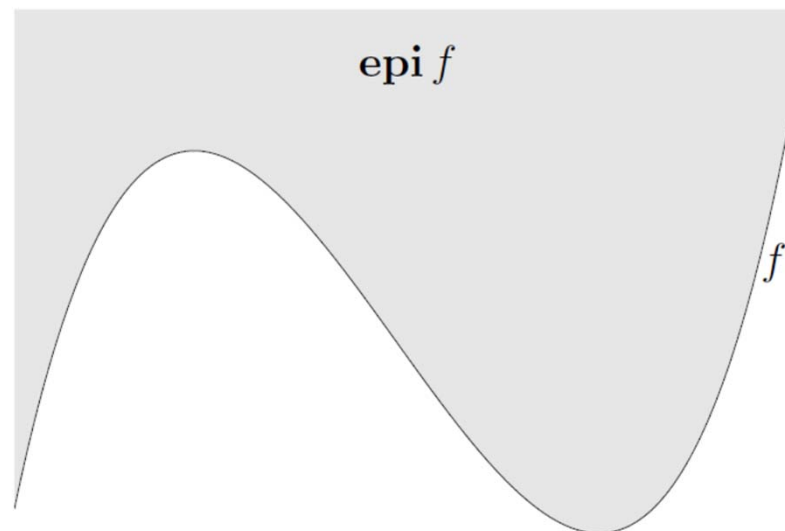
# Epigraph

□ Graph of function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$

■  $\{(x, f(x)) | x \in \text{dom } f\}$

□ Epigraph of function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$

■  $\text{epi } f = \{(x, t) | x \in \text{dom } f, f(x) \leq t\}$







# Epigraph

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## □ Epigraph of function $f: \mathbf{R}^n \rightarrow \mathbf{R}$

- $\text{epi } f = \{(x, t) | x \in \text{dom } f, f(x) \leq t\}$

## □ Hypograph

- $\text{hypo } f = \{(x, t) | x \in \text{dom } f, t \leq f(x)\}$

## □ Conditions

- $f(x)$  is convex  $\Leftrightarrow$   $\text{epi } f$  is convex

- $f(x)$  is concave  $\Leftrightarrow$   $\text{hypo } f$  is convex



# Example

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## □ Matrix Fractional Function

$$f(x, Y) = x^T Y^{-1} x, \text{ dom } f = \mathbf{R}^n \times \mathbf{S}_{++}^n$$

■ Quadratic-over-linear:  $f(x, y) = x^2 / y$

■  $\text{epi } f = \{(x, Y, t) \mid Y \succ 0, x^T Y^{-1} x \leq t\}$

$$= \left\{ (x, Y, t) \mid \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succcurlyeq 0, Y \succ 0 \right\}$$

✓ Schur complement condition

■  $\text{epi } f$  is convex

✓ Linear matrix inequality

✓ Recall Example 2.10 in the book



# Example

## □ Matrix Fractional Function

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$$= \left\{ (x, Y, t) \mid \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succcurlyeq 0, Y \succ 0 \right\}$$

✓ Schur complement condition

## □ Linear Matrix Inequality

$$A(x) = x_1 A_1 + \cdots + x_n A_n$$

$$\{x \mid A(x) \preceq B\} = \{x \mid B - A(x) \in \mathbf{S}_+^m\}$$



# Application of Epigraph

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## □ First order Condition

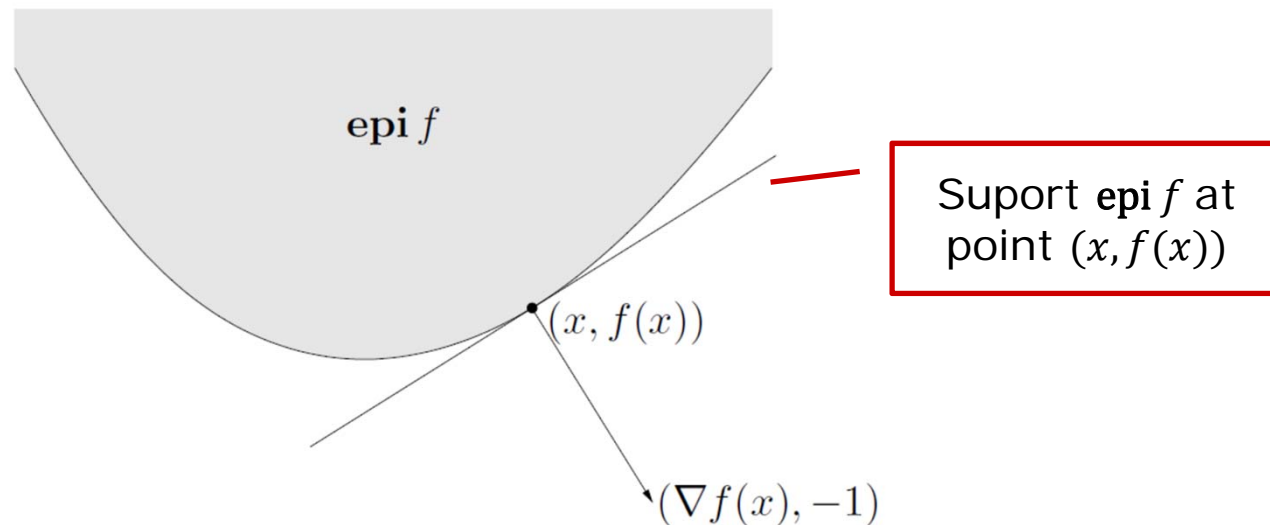
- $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$
- $(y, t) \in \text{epi } f \Rightarrow t \geq f(y) \geq f(x) + \nabla f(x)^\top (y - x)$



# Application of Epigraph

## □ First order Condition

- $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$
- $(y, t) \in \text{epi } f \Rightarrow t \geq f(x) + \nabla f(x)^\top (y - x)$
- $(y, t) \in \text{epi } f \Rightarrow \begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix}^\top \left( \begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0$





# Jensen's Inequality

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## □ Basic inequality

- $\theta \in [0,1]$

- $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$

## □ $K$ points

- $\theta_i \in [0,1], \theta_1 + \dots + \theta_k = 1$

- $f(\theta_1 x_1 + \dots + \theta_k x_k) \leq \theta_1 f(x_1) + \dots + \theta_k f(x_k)$



# Jensen's Inequality

## □ Infinite points

- $p(x) \geq 0, S \subseteq \text{dom } f, \int_S p(x) dx = 1$
- $f\left(\int_S p(x)x dx\right) \leq \int_S f(x)p(x) dx$
- $f(\mathbf{E}x) \leq \mathbf{E}f(x)$ 
  - ✓  $f(x) \leq \mathbf{E}f(x+z), z$  is a zero-mean noisy

## □ Hölder's inequality

- $\frac{1}{p} + \frac{1}{q} = 1, p > 1$
- $\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}}$



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## □ Summary





# Nonnegative Weighted Sums

## □ Finite sums

- $w_i \geq 0, f_i$  is convex
- $f = w_1 f_1 + \dots + w_m f_m$  is convex

The set of convex functions is itself a convex cone

## □ Infinite sums

- $f(x, y)$  is **convex in  $x$** ,  $\forall y \in \mathcal{A}, w(y) \geq 0$
- $g(x) = \int_{\mathcal{A}} f(x, y) w(y) dy$  is convex

## □ Epigraph interpretation

- $\mathbf{epi}(wf) = \{(x, t) | wf(x) \leq t\}$
- $\begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \mathbf{epi}(f) = \{(x, wt) | f(x) \leq t\}$
- $\mathbf{epi}(wf) = \begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \mathbf{epi}(f)$



# Composition with an affine mapping

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□  $f: \mathbf{R}^n \rightarrow \mathbf{R}$

□  $A \in \mathbf{R}^{n \times m}, b \in \mathbf{R}^n$

□ Affine Mapping

$$g(x) = f(Ax + b)$$

■ If  $f$  is convex, so is  $g$ .

■ If  $f$  is concave, so is  $g$ .



# Pointwise Maximum

□  $f_1, f_2$  is convex

$$f(x) = \max\{f_1(x), f_2(x)\}$$

is convex with  $\text{dom } f = \text{dom } f_1 \cap \text{dom } f_2$

■  $f(\theta x + (1 - \theta)y)$

$$= \max\{f_1(\theta x + (1 - \theta)y), f_2(\theta x + (1 - \theta)y)\}$$

$$\leq \max\{\theta f_1(x) + (1 - \theta)f_1(y), \theta f_2(x) + (1 - \theta)f_2(y)\}$$

$$\leq \theta \max\{f_1(x), f_2(x)\} + (1 - \theta) \max\{f_1(y), f_2(y)\}$$

$$= \theta f(x) + (1 - \theta)f(y)$$

■  $f_1, \dots, f_m$  is convex  $\Rightarrow f(x) = \max\{f_1(x), \dots, f_m(x)\}$



# Examples

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## □ Piecewise-linear functions

- $f(x) = \max\{a_1^\top x + b_1, \dots, a_L^\top x + b_L\}$

## □ Sum of $r$ largest components

- $x \in \mathbf{R}^n, x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$

- $f(x) = \sum_{i=1}^r x_{[i]}$  is convex

$$= \max\{x_{i_1} + \dots + x_{i_r} \mid 1 \leq i_1 < \dots < i_r \leq n\}$$

- Pointwise maximum of  $\frac{n!}{r!(n-r)!}$  linear functions



# Pointwise Supremum

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- $\forall y \in \mathcal{A}, f(x, y)$  is convex in  $x$

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex with  $\text{dom } g = \{x \mid (x, y) \in \text{dom } f, \forall y \in \mathcal{A}, \sup_{y \in \mathcal{A}} f(x, y) < \infty\}$

- Epigraph interpretation

- $\text{epi } g = \bigcap_{y \in \mathcal{A}} \text{epi } f(\cdot, y)$

- Intersection of convex sets is convex

- Pointwise infimum of a set of concave functions is concave



# Examples

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## □ Support function of a set

- $C \subseteq \mathbf{R}^n, C \neq \emptyset$
- $S_C(x) = \sup\{x^\top y \mid y \in C\}$
- $\text{dom } S_C = \{x \mid \sup_{y \in C} x^\top y < \infty\}$

## □ Distance to farthest point of a set

- $C \subseteq \mathbf{R}^n$
- $f(x) = \sup_{y \in C} \|x - y\|$



# Examples

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## □ Maximum eigenvalue of a symmetric matrix

- $f(X) = \lambda_{\max}(X)$ ,  $\text{dom } f = \mathbf{S}^m$
- $f(X) = \sup\{y^\top X y \mid \|y\|_2 = 1\}$

## □ Norm of a matrix

- $f(X) = \|X\|_2$  is maximum singular value of  $X$
- $\text{dom } f = \mathbf{R}^{p \times q}$
- $f(X) = \sup\{u^\top X v \mid \|u\|_2 = 1, \|v\|_2 = 1\}$

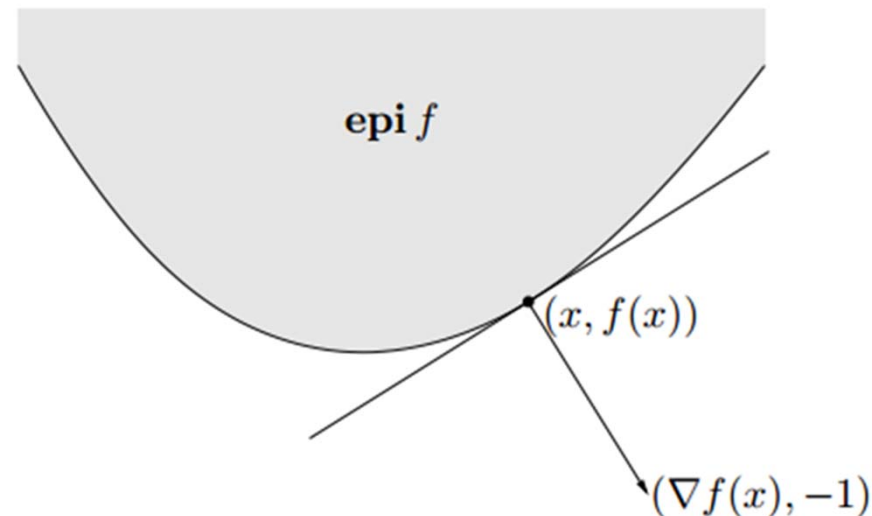


# Representation

- Almost every convex function can be expressed as the pointwise supremum of a family of affine functions.

$f: \mathbf{R}^n \rightarrow \mathbf{R}$  is convex and  $\text{dom } f = \mathbf{R}^n$

$$\Rightarrow f(x) = \sup\{g(x) \mid g \text{ affine}, g(z) \leq f(z) \forall z\}$$







# Compositions

---

## □ Definition

■  $h: \mathbf{R}^k \rightarrow \mathbf{R}, g: \mathbf{R}^n \rightarrow \mathbf{R}^k$

■  $f = h \circ g: \mathbf{R}^n \rightarrow \mathbf{R}$

$$f(x) = h(g(x))$$

■  $\text{dom } f = \{x \in \text{dom } g \mid g(x) \in \text{dom } h\}$

## □ Chain Rule

■  $h: \mathbf{R} \rightarrow \mathbf{R}, g: \mathbf{R}^n \rightarrow \mathbf{R}$

$$\nabla^2 f(x) = h'(g(x))\nabla^2 g(x) + h''(g(x))\nabla g(x)\nabla g(x)^\top$$



# Scalar Composition

□  $h: \mathbf{R} \rightarrow \mathbf{R}, g: \mathbf{R} \rightarrow \mathbf{R}$

■  $h$  and  $g$  are twice differentiable

■  $\text{dom } g = \text{dom } h = \mathbf{R}$

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

■  $f$  is convex, if  $f''(x) \geq 0$

■  $h'' \geq 0, h' \geq 0, g'' \geq 0$

✓  $h$  is convex and nondecreasing,  $g$  is convex

■  $h'' \geq 0, h' \leq 0, g'' \leq 0$

✓  $h$  is convex and nonincreasing,  $g$  is concave



# Scalar Composition

---

□  $h: \mathbf{R} \rightarrow \mathbf{R}, g: \mathbf{R} \rightarrow \mathbf{R}$

■  $h$  and  $g$  are twice differentiable

■  $\text{dom } g = \text{dom } h = \mathbf{R}$

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

■  $f$  is concave, if  $f''(x) \leq 0$

■  $h'' \leq 0, h' \geq 0, g'' \leq 0$

✓  $h$  is concave and nondecreasing,  $g$  is concave

■  $h'' \leq 0, h' \leq 0, g'' \geq 0$

✓  $h$  is concave and nonincreasing,  $g$  is convex



# Scalar Composition

□  $h: \mathbf{R} \rightarrow \mathbf{R}, g: \mathbf{R}^n \rightarrow \mathbf{R}$

- Without differentiability assumption
  - Without domain condition
  - $h(x) = 0$  with  $\text{dom } h = [1, 2]$ , which is convex and nondecreasing
  - $g(x) = x^2$  with  $\text{dom } g = \mathbf{R}$ , which is convex
- $$f(x) = h(g(x)) = 0$$
- $\text{dom } f = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$



# Scalar Composition

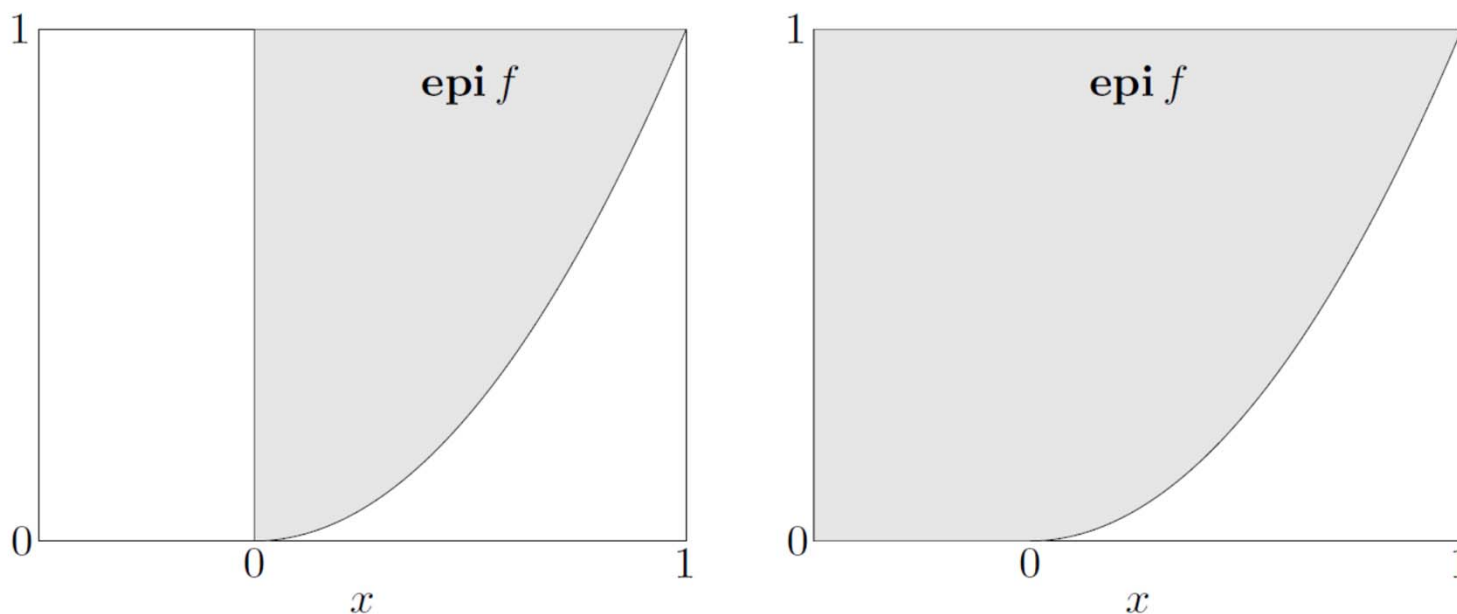
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□  $h: \mathbf{R} \rightarrow \mathbf{R}, g: \mathbf{R}^n \rightarrow \mathbf{R}$

- Without differentiability assumption
- Without domain condition
- $h$  is convex,  $\tilde{h}$  is nondecreasing, and  $g$  is convex  $\Rightarrow f$  is convex
- $h$  is convex,  $\tilde{h}$  is nonincreasing, and  $g$  is concave  $\Rightarrow f$  is convex
- The conditions for concave are similar



# Extended-value Extensions



**Figure 3.7** *Left.* The function  $x^2$ , with domain  $\mathbf{R}_+$ , is convex and nondecreasing on its domain, but its extended-value extension is *not* nondecreasing. *Right.* The function  $\max\{x, 0\}^2$ , with domain  $\mathbf{R}$ , is convex, and its extended-value extension is nondecreasing.



# Examples

---

- $g$  is convex  $\Rightarrow \exp g(x)$  is convex
- $g$  is concave and positive  $\Rightarrow \log g(x)$  is concave
- $g$  is concave and positive  $\Rightarrow 1/g(x)$  is convex
- $g$  is convex and nonnegative and  $p \geq 1 \Rightarrow g(x)^p$  is convex
- $g$  is convex  $\Rightarrow -\log(-g(x))$  is convex on  $\{x | g(x) < 0\}$



# Vector Composition

---

□  $h: \mathbf{R}^k \rightarrow \mathbf{R}, g_i: \mathbf{R} \rightarrow \mathbf{R}$

$$f = h \circ g = h(g_1(x), \dots, g_k(x))$$

■  $h$  and  $g$  are twice differentiable

■  $\text{dom } g_i = \mathbf{R}, \text{dom } h = \mathbf{R}^k$

$$f'(x) = \nabla h(g(x))^{\top} g'(x)$$

$$f''(x) = g'(x)^{\top} \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^{\top} g''(x)$$





# Vector Composition

□  $h: \mathbf{R}^k \rightarrow \mathbf{R}, g_i: \mathbf{R} \rightarrow \mathbf{R}$

$$f = h \circ g = h(g_1(x), \dots, g_k(x))$$

■  $h$  and  $g$  are twice differentiable

■  $\text{dom } g_i = \mathbf{R}, \text{dom } h = \mathbf{R}^k$

$$f''(x) = g'(x)^\top \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^\top g''(x)$$

■  $f$  is convex, if  $f''(x) \geq 0$

✓  $h$  is convex,  $h$  is nondecreasing in each argument, and  $g_i$  are convex

✓  $h$  is convex,  $h$  is nonincreasing in each argument, and  $g_i$  are concave



# Vector Composition

□  $h: \mathbf{R}^k \rightarrow \mathbf{R}, g_i: \mathbf{R} \rightarrow \mathbf{R}$

$$f = h \circ g = h(g_1(x), \dots, g_k(x))$$

■  $h$  and  $g$  are twice differentiable

■  $\text{dom } g_i = \mathbf{R}, \text{dom } h = \mathbf{R}^k$

$$f''(x) = g'(x)^\top \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^\top g''(x)$$

■  $f$  is concave, if  $f''(x) \leq 0$

✓  $h$  is concave,  $h$  is nondecreasing in each argument, and  $g_i$  are concave

□ The general case is similar



# Examples

---

- $h(z) = z_{[1]} + \cdots + z_{[r]}, z \in \mathbf{R}^k, g_1, \dots, g_k$  are convex  $\Rightarrow h \circ g$  is convex
- $h(z) = \log(\sum_{i=1}^k e^{z_i}), g_1, \dots, g_k$  are convex  $\Rightarrow h \circ g$  is convex
- $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$  on  $\mathbf{R}_+^k$  is concave for  $0 \leq p \leq 1$ , and its extension is nondecreasing. If  $g_i$  is concave and nonnegative  $\Rightarrow h \circ g = (\sum_{i=1}^k g_i(x)^p)^{1/p}$  is concave
- Suppose  $p \geq 1$ , and  $g_1, \dots, g_k$  are convex and nonnegative. Then the function  $(\sum_{i=1}^k g_i(x)^p)^{1/p}$  is convex



# Minimization

---

- $f$  is convex in  $(x, y)$ ,  $C$  is convex ( $C \neq \emptyset$ )
  - $g(x) = \inf_{y \in C} f(x, y)$  is convex if  $g(x) > -\infty, \forall x \in \text{dom } g$
  - $\text{dom } g = \{x \mid (x, y) \in \text{dom } f \text{ for some } y \in C\}$
  
- Proof by Epigraph
  - $\text{epi } g = \{(x, t) \mid (x, y, t) \in \text{epi } f \text{ for some } y \in C\}$
  - The projection of a convex set is convex.



# Minimization

---

- $f$  is convex in  $(x, y)$ ,  $C$  is convex ( $C \neq \emptyset$ )
  - $g(x) = \inf_{y \in C} f(x, y)$  is convex if  $g(x) > -\infty, \forall x \in \text{dom } g$
  - $\text{dom } g = \{x | (x, y) \in \text{dom } f \text{ for some } y \in C\}$

# Pointwise Supremum

---

- $\forall y \in \mathcal{A}, f(x, y)$  is convex in  $x$   
$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$
  
is convex with  $\text{dom } g = \{x | (x, y) \in \text{dom } f, \forall y \in \mathcal{A}, \sup_{y \in \mathcal{A}} f(x, y) < \infty\}$



# Pointwise Supremum

---

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is convex with  $\text{dom } g = \{x \mid (x, y) \in \text{dom } f, \forall y \in \mathcal{A}, \sup_{y \in \mathcal{A}} f(x, y) < \infty\}$

- Epigraph interpretation

- $\text{epi } g = \bigcap_{y \in \mathcal{A}} \text{epi } f(\cdot, y)$

- Intersection of convex sets is convex

- Pointwise infimum of a set of concave functions is concave



# Examples

## □ Schur complement

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$
- $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, A, C$  is symmetric  $\Rightarrow f(x, y)$  is convex
- $g(x) = \inf_y f(x, y) = x^T (A - B C^\dagger B^T) x$  is convex  
 $\Rightarrow A - B C^\dagger B^T \succeq 0, C^\dagger$  is the pseudo-inverse of  $C$

## □ Distance to a set

- $S$  is a **convex** nonempty set,  $f(x, y) = \|x - y\|$  is convex in  $(x, y)$
- $g(x) = \text{dist}(x, S) = \inf_{y \in S} \|x - y\|$



# Examples

---

## □ Distance to farthest point of a set

- $C \subseteq \mathbf{R}^n$
- $f(x) = \sup_{y \in C} \|x - y\|$

## □ Distance to a set

- $S$  is a **convex** nonempty set,  $f(x, y) = \|x - y\|$  is convex in  $(x, y)$
- $g(x) = \text{dist}(x, S) = \inf_{y \in S} \|x - y\|$





# Examples

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## □ Affine domain

- $h(y)$  is convex
- $g(x) = \inf \{h(y) | Ay = x\}$  is convex

## □ Proof

- $f(x, y) = \begin{cases} h(y) & \text{if } Ay - x = 0 \\ \infty & \text{otherwise} \end{cases}$
- $f(x, y)$  is convex in  $(x, y)$
- $g$  is the minimum of  $f$  over  $y$



# Perspective of a function

□  $f: \mathbf{R}^n \rightarrow \mathbf{R}, g: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$  defined as

$$g(x, t) = tf(x/t)$$

is the perspective of  $f$

- $\text{dom } g = \{(x, t) | x/t \in \text{dom } f, t > 0\}$
- $f$  is convex  $\Rightarrow g$  is convex

□ Proof

$$\begin{aligned}(x, t, s) \in \text{epi } g &\Leftrightarrow tf\left(\frac{x}{t}\right) \leq s \\ &\Leftrightarrow f\left(\frac{x}{t}\right) \leq \frac{s}{t} \\ &\Leftrightarrow (x/t, s/t) \in \text{epi } f\end{aligned}$$

- Perspective mapping preserve convexity



# Perspective Functions

---

□ Perspective function  $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$

$$P(z, t) = \frac{z}{t}, \text{ dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$$

□ If  $C \in \text{dom } P$  is convex, then its image

$$P(C) = \{P(x) | x \in C\}$$

is convex

□ If  $C \in \mathbf{R}^n$  is convex, the inverse image

$$P^{-1}(C) = \left\{ (x, t) \in \mathbf{R}^{n+1} \mid \frac{x}{t} \in C, t \geq 0 \right\}$$

is convex



# Example

---

## □ Euclidean norm squared

- $f(x) = x^\top x$

- $g(x, t) = t \left(\frac{x}{t}\right)^\top \left(\frac{x}{t}\right) = \frac{x^\top x}{t}, t > 0$

## □ Composition with an Affine function

- $f: \mathbf{R}^m \rightarrow \mathbf{R}$  is convex

- $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m, c \in \mathbf{R}^n, d \in \mathbf{R}$

- $\text{dom } g = \left\{ x \mid c^\top x + d > 0, \frac{Ax+b}{c^\top x+d} \in \text{dom } f \right\}$

- $g(x) = (c^\top x + d)f\left(\frac{Ax+b}{c^\top x+d}\right)$  is convex



# Outline

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## □ Basic Properties

- Definition
- First-order Conditions, Second-order Conditions
- Jensen's inequality and extensions
- Epigraph

## □ Operations That Preserve Convexity

- Nonnegative Weighted Sums
- Composition with an affine mapping
- Pointwise maximum and supremum
- Composition
- Minimization
- Perspective of a function

## □ Summary



# Summary

---

## □ Basic Properties

- Definition
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