# Convex Optimization Problems (I)

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# Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions
- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization

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# Basic Terminology

Unconstrained when $m = p = 0$

## Optimization Problems

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \qquad (1) \\ & h_i(x) = 0, \qquad i = 1, \ldots, p \end{aligned}$$

- Optimization variable: $x \in \mathbf{R}^n$
- Objective function: $f_0: \mathbf{R}^n \rightarrow \mathbf{R}$
- Inequality constraints: $f_i(x) \leq 0$
- Inequality constraint functions: $f_i: \mathbf{R}^n \rightarrow \mathbf{R}$
- Equality constraints: $h_i(x) = 0$
- Equality constraint functions: $h_i: \mathbf{R}^n \rightarrow \mathbf{R}$

# Basic Terminology

## Optimization Problems

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \qquad (1) \\ & h_i(x) = 0, \qquad i = 1, \ldots, p \end{aligned}$$

- Domain

$$\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- $x \in \mathcal{D}$ is feasible if it satisfies all the constraints
- The problem is feasible if there exists at least one feasible point

# Basic Terminology

- Optimal Value $p^\star$

$$p^\star = \inf\{f_0(x) | f_i(x) \le 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

  - Infeasible problem: $p^\star = \infty$
  - Unbounded below: if there exist $x_k$ with $f_0(x_k) \to -\infty$ as $k \to \infty$, then $p^\star = -\infty$
- Optimal Points
  - $x^\star$ is feasible and $f_0(x^\star) = p^\star$
- Optimal Set

$$X_{\text{opt}} = \{x | f_i(x) \le 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, f_0(x) = p^*\}$$

- $p^\star$ is achieved if $X_{\text{opt}}$ is nonempty

# Basic Terminology

- $\varepsilon$-suboptimal Points
  - a feasible $x$ with $f_0(x) \leq p^\star + \varepsilon$
- $\varepsilon$-suboptimal Set
  - the set of all $\varepsilon$-suboptimal points
- Locally Optimal Points

$$
\begin{array}{lll}
\min & f_0(z) & \\
\text{s.t.} & f_i(z) \leq 0, & i = 1, \ldots, m \\
& h_i(z) = 0, & i = 1, \ldots, p \\
& \|z - x\|_2 \leq R &
\end{array}
$$

  - $x$ is feasible and solves the above problem
- Globally Optimal Points

# Basic Terminology

## Types of Constraints

- If $f_i(x) = 0$, $f_i(x) \leq 0$ is active at $x$
- If $f_i(x) < 0$, $f_i(x) \leq 0$ is inactive at $x$
- $h_i(x) = 0$ is active at all feasible points
- Redundant constraint: deleting it does not change the feasible set

## Examples on $x \in \mathbf{R}$ and $\mathrm{dom}\, f_0 = \mathbf{R}_{++}$

- $f_0(x) = 1/x : p^\star = 0$, the optimal value is not achieved
- $f_0(x) = -\log x : p^\star = -\infty$, unbounded blow
- $f_0(x) = x \log x : p^\star = -1/e$, $x^\star = 1/e$ is optimal

# Basic Terminology

- Feasibility Problems

$$\begin{array}{lll} \text{find} & x & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

  - Determine whether constraints are consistent

- Maximization Problems

$$\begin{array}{lll} \max & f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

  - It can be solved by minimizing $-f_0$
  - Optimal Value $p^\star$

$$p^\star = \sup \{ f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p \}$$

# Basic Terminology

## Standard Form

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\
& h_i(x) = 0, \qquad i = 1, \ldots, p
\end{aligned}
$$

## Box constraints

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & l_i \le x_i \le u_i, \qquad i = 1, \ldots, n
\end{aligned}
$$

## Reformulation

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & l_i - x_i \le 0, \qquad i = 1, \ldots, n \\
& x_i - u_i \le 0, \qquad i = 1, \ldots, n
\end{aligned}
$$

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# Equivalent Problems

- Two Equivalent Problems
  - If from a solution of one, a solution of the other is readily found, and vice versa
- A Simple Example

$$\begin{array}{lll} \min & \tilde{f}(x) = \alpha_0 f_0(x) & \\ \text{s.t.} & \tilde{f}_i(x) = \alpha_i f_i(x) \le 0, & i = 1, \ldots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

  - $\alpha_i > 0, i = 0, \ldots, m$
  - $\beta_i \neq 0, i = 1, \ldots, p$
  - Equivalent to the problem $(1)$

# Change of Variables

- $\phi: \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and $\phi(\mathrm{dom}\,\phi) \supseteq \mathcal{D}$, and define

$$\tilde{f}_i(z) = f_i(\phi(z)), \qquad i = 0, \ldots, m$$
$$\tilde{h}_i(z) = h_i(\phi(z)), \qquad i = 1, \ldots, p$$

- An Equivalent Problem

$$\begin{array}{lll} \min & \tilde{f}_0(z) & \\ \mathrm{s.t.} & \tilde{f}_i(z) \le 0, & i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, & i = 1, \ldots, p \end{array}$$

  - If $z$ solves it, $x = \phi(z)$ solves the problem $(1)$
  - If $x$ solves $(1)$, $z = \phi^{-1}(x)$ solves it

# Transformation of Functions

- $\psi_0: \mathbf{R} \to \mathbf{R}$ is monotone increasing
- $\psi_1, \ldots, \psi_m: \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$
- $\psi_{m+1}, \ldots, \psi_{m+p}: \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$
- Define $\tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, \ldots, m$
  $\tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \quad i = 1, \ldots, p$
- An Equivalent Problem

$$
\begin{array}{lll}
\min & \tilde{f}_0(x) & \\
\text{s.t.} & \tilde{f}_i(x) \leq 0, & i = 1, \ldots, m \\
 & \tilde{h}_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

# Example

## Least-norm Problems

$$\min \quad \|Ax - b\|_2$$

- Not differentiable at any $x$ with $Ax - b = 0$

## Least-norm-squared Problems

$$\min \quad \|Ax - b\|_2^2 = (Ax - b)^\top (Ax - b)$$

- Differentiable for all $x$

# Slack Variables

- $f_i(x) \le 0$ if and only if there is an $s_i \ge 0$ that satisfies $f_i(x) + s_i = 0$
- An Equivalent Problem

$$
\begin{array}{lll}
\min & f_0(x) & \\
\text{s.t.} & s_i \ge 0, & i = 1, \ldots, m \\
& f_i(x) + s_i = 0, & i = 1, \ldots, m \\
& h_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

  - $s_i$ is the slack variable associated with the inequality constraint $f_i(x) \le 0$
  - $x$ is optimal for the problem $(1)$ if and only if $(x, s)$ is optimal for the above problem, where $s_i = -f_i(x)$

# Eliminating Equality Constraints

- Assume $\phi: \mathbf{R}^k \rightarrow \mathbf{R}^n$ is such that $x$ satisfies

$$h_i(x) = 0, \qquad i = 1, \dots, p$$

if and only if there is some $z \in \mathbf{R}^k$ such that

$$x = \phi(z)$$

- An Equivalent Problem

$$\begin{array}{ll} \min & \tilde{f}_0(z) = f_0(\phi(z)) \\ \text{s.t.} & \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \qquad i = 1, \dots, m \end{array}$$

  - If $z$ is optimal for this problem, $x = \phi(z)$ is optimal for the problem $(1)$
  - If $x$ is optimal for $(1)$, there is at least one $z$ which is optimal for this problem

# Eliminating linear equality constraints

- Assume the equality constraints are all linear $Ax = b$, and $x_0$ is one solution
- Let $F \in \mathbf{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbf{R}^k\}$$

- An Equivalent Problem ($x = Fz + x_0$)

$$\begin{array}{ll} \min & f_0(Fz + x_0) \\ \mathrm{s.t.} & f_i(Fz + x_0) \leq 0, \qquad i = 1, \ldots, m \end{array}$$

  - $k = n - \mathrm{rank}(A)$

# Linear algebra

## Range and nullspace

- Let $A \in \mathbf{R}^{m\times n}$, the range of $A$, denoted $\mathcal{R}(A)$, is the set of all vectors in $\mathbf{R}^m$ that can be written as linear combinations of the columns of A:

$$\mathcal{R}(A) = \{Ax | x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$$

- The nullspace (or kernel) of A, denoted $\mathcal{N}(A)$, is the set of all vectors $x$ mapped into zero by A:

$$\mathcal{N}(A) = \{x | Ax = 0\} \subseteq \mathbf{R}^n$$

- if $\mathcal{V}$ is a subspace of $\mathbf{R}^n$, its orthogonal complement, denoted $\mathcal{V}^\perp$, is defined as:

$$\mathcal{V}^\perp = \{x | z^\top x = 0 \text{ for all } z \in \mathcal{V}\}$$

# Introducing Equality Constraints

- Consider the problem

$$\begin{array}{lll} \min & f_0(A_0x+b_0) & \\ \text{s.t.} & f_i(A_ix+b_i)\le 0, & i=1,\dots,m \\ & h_i(x)=0, & i=1,\dots,p \end{array}$$

  - $x\in\mathbf{R}^n, A_i\in\mathbf{R}^{k_i\times n}$ and $f_i:\mathbf{R}^{k_i}\to\mathbf{R}$

- An Equivalent Problem

$$\begin{array}{lll} \min & f_0(y_0) & \\ \text{s.t.} & f_i(y_i)\le 0, & i=1,\dots,m \\ & y_i=A_ix+b_i, & i=0,\dots,m \\ & h_i(x)=0, & i=1,\dots,p \end{array}$$

  - Introduce $y_i\in\mathbf{R}^{k_i}$ and $y_i=A_ix+b_i$

# Optimizing over Some Variables

- Suppose $x \in \mathbf{R}^n$ is partitioned as $x = (x_1, x_2)$, with $x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}$ and $n_1 + n_2 = n$
- Consider the problem

$$\begin{array}{lll} \min & f_0(x_1, x_2) & \\ \text{s.t.} & f_i(x_1) \leq 0, & i = 1, \ldots, m_1 \\ & \tilde{f}_i(x_2) \leq 0, & i = 1, \ldots, m_2 \end{array}$$

- An Equivalent Problem

$$\begin{array}{lll} \min & \tilde{f}_0(x_1) & \\ \text{s.t.} & f_i(x_1) \leq 0, & i = 1, \ldots, m_1 \end{array}$$

  - where

$$\tilde{f}_0(x_1) = \inf\{f_0(x_1, z) | \tilde{f}_i(z) \leq 0, i = 1, \ldots, m_2\}$$

# Example

- Minimize a Quadratic Function

$$\begin{array}{ll} \min & x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\ \text{s.t.} & f_i(x_1) \le 0, \qquad i = 1, \ldots, m \end{array}$$

- Minimize over $x_2$

$$\begin{aligned} \inf_{x_2} \left( x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \right) \\ = x_1^\top \left( P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1 \end{aligned}$$

- An Equivalent Problem

$$\begin{array}{ll} \min & x_1^\top \left( P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1 \\ \text{s.t.} & f_i(x_1) \le 0, \qquad i = 1, \ldots, m \end{array}$$

# Epigraph Problem Form

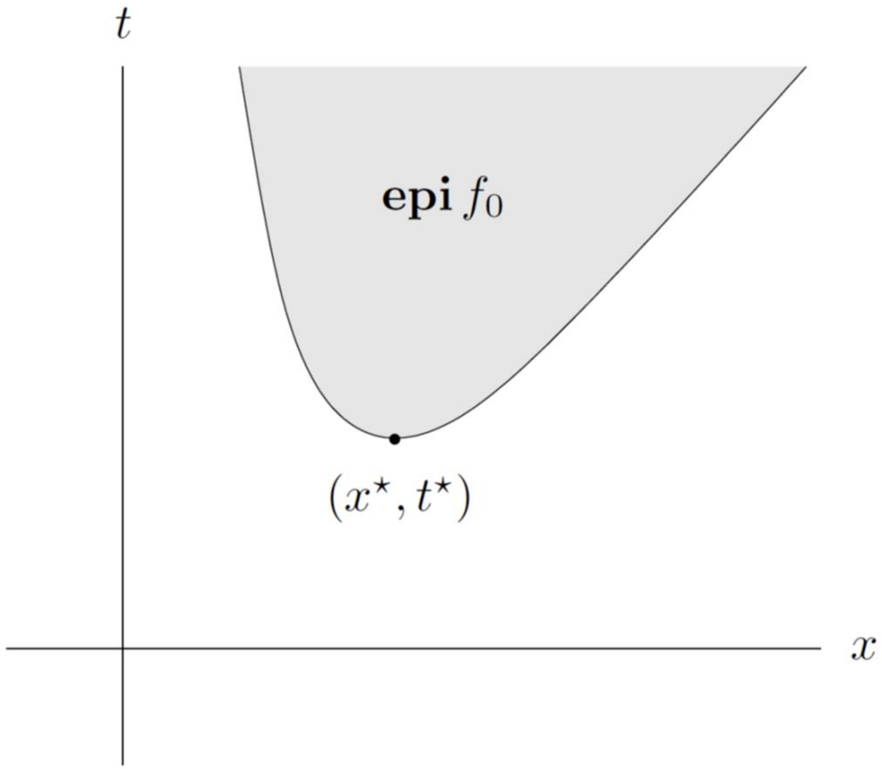
## Epigraph Form

$$\begin{array}{ll} \min & t \\ \text{s.t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & h_i(x) = 0, \qquad i = 1, \ldots, p \end{array}$$

- Introduce a variable $t \in \mathbf{R}$
- $(x, t)$ is optimal for this problem if and only if $x$ is optimal for $(1)$ and $t = f_0(x)$
- The objective function of the epigraph form problem is a linear function of $x, t$

# Epigraph Problem Form

## □ Geometric Interpretation




- Find the point in the epigraph that minimizes $t$

# Making Constraints Implicit

## □ Unconstrained problem

$$\min F(x)$$

- $\mathrm{dom}\, F = \{x \in \mathrm{dom}\, f_0 | f_i(x) \leq 0, i = 1, \ldots, m,$
  $h_i(x) = 0, i = 1, \ldots, p\}$
- $F(x) = f_0(x)$ for $x \in \mathrm{dom}\, F$
- It has not make the problem any easier
- It could make the problem more difficult, because $F$ is probably not differentiable

# Making Constraints Explicit

- A Unconstrained Problem

$$\min f(x)$$

  - where

$$f(x) = \begin{cases} x^\top x & Ax = b \\ \infty & \text{otherwise} \end{cases}$$

  - An implicit equality constraint $Ax = b$

- An Equivalent Problem

$$\begin{aligned} \min \quad & x^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

  - Objective and constraint functions are differentiable

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# Problem Descriptions

- Parameter Problem Description
  - Functions have some analytical or closed form
  - Example: $f_0(x) = x^\top P x + q^\top x + r$, where $P \in \mathbf{S}^n, q \in \mathbf{R}^n$ and $r \in \mathbf{R}$
  - Give the values of the parameters
- Oracle Model (Black-box Model)
  - Can only query the objective and constraint functions by oracle
  - Evaluate $f(x)$ and its gradient $\nabla f(x)$
  - Know some prior information (convexity)

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# Convex Optimization Problems

## □ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \mathrm{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & a_i^\top x = b_i, \qquad i = 1, \ldots, p \end{aligned}$$

- The objective function must be convex
- The inequality constraint functions must be convex
- The equality constraint functions $h_i(x) = a_i^\top x - b_i$ must be affine

# Convex Optimization Problems

## Properties

- Feasible set of a convex optimization problem is convex

$$\bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{m} \{x \mid f_i(x) \leq 0\} \cap \bigcap_{i=1}^{p} \{x \mid a_i^\top x = b_i\}$$

  - ✓ Minimize a convex function over a convex set
- $\varepsilon$-suboptimal set is convex
- The optimal set is convex
- If the objective is strictly convex, then the optimal set contains at most one point

# Concave Maximization Problems

## Standard Form

$$
\begin{array}{lll}
\max & f_0(x) & \\
\text{s.t.} & f_i(x) \le 0, & i = 1, \ldots, m \\
& a_i^\top x = b_i, & i = 1, \ldots, p
\end{array}
$$

- It is referred as a convex optimization problem if $f_0$ is concave and $f_1, \ldots, f_m$ are convex
- It is readily solved by minimizing the convex objective function $-f_0$

# Abstract From Convex Optimization Problem

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1/(1+x_2^2) \le 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

  - Not a convex optimization problem
    - ✓ $f_1$ is not convex and $h_1$ is not affine
  - But the feasible set is indeed convex
  - Abstract convex optimization problem
- An Equivalent Convex Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1 \le 0 \\ & h_1(x) = x_1 + x_2 = 0 \end{aligned}$$

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# Local and Global Optima

- Any locally optimal point of a convex problem is also (globally) optimal
- Proof by Contradiction
  - $x$ is locally optimal implies
    $$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \le R\}$$
    for some $R$
  - Suppose $x$ is not globally optimal, i.e., there exists $f_0(y) < f_0(x)$ and $\|y - x\|_2 > R$
  - Define
    $$z = (1-\theta)x + \theta y, \theta = \frac{R}{2\|y - x\|_2} \in (0,1)$$

# Local and Global Optima

- By convexity of the feasible set

$$z \text{ is feasible}$$

- It is easy to check

$$\|z - x\|_2 = \|\theta(y - x)\|_2 = \left\| \frac{R(y - x)}{2\|y - x\|_2} \right\|_2 = \frac{R}{2} < R$$

- By convexity of $f_0$

$$f_0(z) \le (1 - \theta) f_0(x) + \theta f_0(y) < f_0(x)$$

which contradicts

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \le R\}$$

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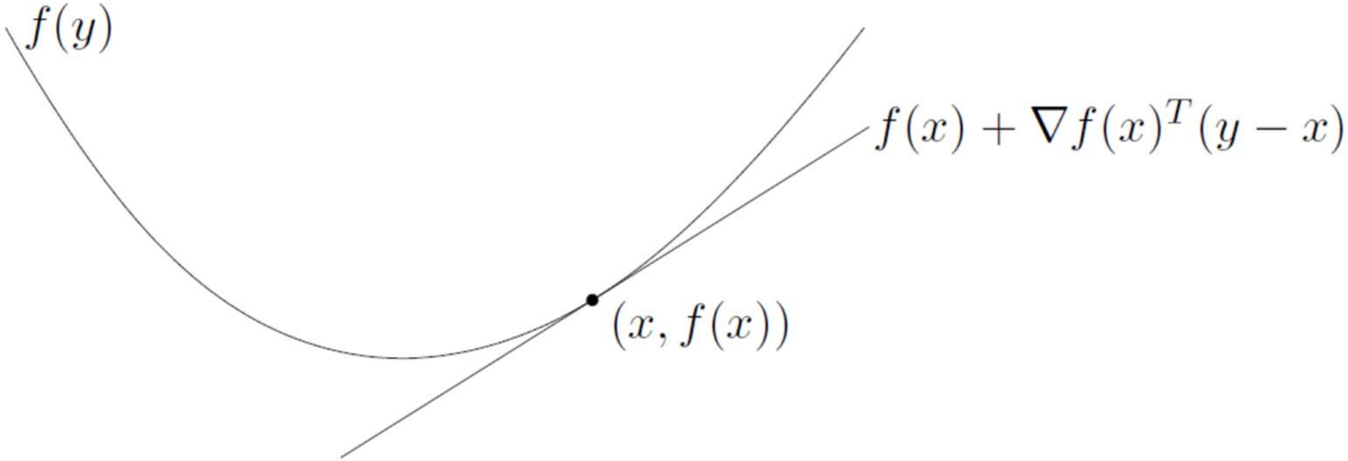
# An Optimality Criterion for Differentiable $f_0$

- Suppose $f_0$ is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \mathrm{dom}\, f_0$$



# An Optimality Criterion for Differentiable $f_0$

- Suppose $f_0$ is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \mathrm{dom}\, f_0$$

- Let $X$ denote the feasible set

$$X = \{x | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$$

- $x$ is optimal if and only if $x \in X$ and

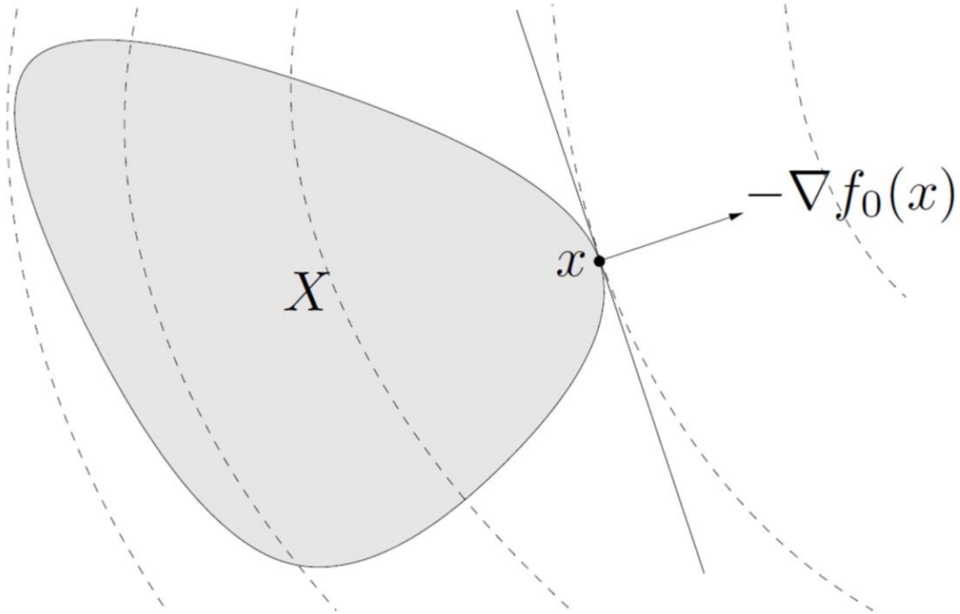
$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

# An Optimality Criterion for Differentiable $f_0$

- $x$ is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

- $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at $x$



# Proof of Optimality Condition

- Sufficient Condition

$$\left.\begin{array}{c}\nabla f_0(x)^\top(y-x) \geq 0 \\ f_0(y) \geq f_0(x) + \nabla f_0(x)^\top(y-x)\end{array}\right\} \Rightarrow f_0(y) \geq f_0(x)$$

- Necessary Condition
  - Suppose $x$ is optimal but

$$\exists y \in X, \nabla f_0(x)^\top(y-x) < 0$$

  - Define $z(t) = ty + (1-t)x, t \in [0,1]$

$$f_0\big(z(0)\big) = f_0(x), \qquad \frac{d}{dt} f_0\big(z(t)\big)\Big|_{t=0} = \nabla f_0(x)^\top(y-x) < 0$$

  - So, for small positive $t$, $f_0\big(z(t)\big) < f_0(x)$

# Unconstrained Problems

- $x$ is optimal if and only if $\nabla f_0(x) = 0$
  - Consider $y = x - t\nabla f_0(x)$ and $t > 0$
  - When $t$ is small, $y$ is feasible

$$\nabla f_0(x)^\top (y - x) = -t\|\nabla f_0(x)\|_2^2 \geq 0 \Leftrightarrow \nabla f_0(x) = 0$$

- Unconstrained Quadratic Optimization

$$\min \ f_0(x) = (1/2)x^\top Px + q^\top x + r, \qquad \text{where } P \in \mathbf{S}_+^n$$

  - $x$ is optimal if and only if $\nabla f_0(x) = Px + q = 0$
  - If $q \notin \mathcal{R}(P)$, no solution, $f_0$ is unbound below
  - If $P \succ 0$, unique minimizer $x^\star = -P^{-1}q$
  - If $P$ is singular, but $q \in \mathcal{R}(P)$, $X_{\text{opt}} = -P^\dagger q + \mathcal{N}(P)$

# Problems with Equality Constraints Only

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- $x$ is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall\, Ay = b$$

# Problems with Equality Constraints Only

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- $x$ is optimal if and only if

**Lagrange Multiplier Optimality Condition**

$$Ax = b$$
$$\nabla f_0(x) + A^\top v = 0$$

$$\left.\begin{array}{c} \nabla f_0(x)^\top (y - x) \geq 0, \forall\, Ay = b \\ \{y | Ay = b\} = \{x + v | v \in \mathcal{N}(A)\} \end{array}\right\}$$

$$\Leftrightarrow \nabla f_0(x)^\top v \geq 0, \forall\, v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x)^\top v = 0, \forall\, v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x) \perp \mathcal{N}(A) \Leftrightarrow \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$$

$$\Leftrightarrow \exists v \in \mathbf{R}^p, \nabla f_0(x) + A^\top v = 0$$

# Minimization over the Nonnegative Orthant

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & x \succcurlyeq 0 \end{aligned}$$

- $x$ is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall y \succcurlyeq 0$$

$$\Leftrightarrow \begin{cases} \nabla f_0(x) \succcurlyeq 0 \\ -\nabla f_0(x)^\top x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \nabla f_0(x) \succcurlyeq 0 \\ \nabla f_0(x)^\top x = 0 \end{cases}$$

- The Optimality Condition

$$x \succcurlyeq 0, \qquad \nabla f_0(x) \succcurlyeq 0, \qquad x_i \big(\nabla f_0(x)\big)_i = 0, i = 1, \dots, n$$

  - The last condition is called complementarity

# Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions
- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization

# Equivalent Convex Problems

- Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & a_i^\top x = b_i, \qquad i = 1, \ldots, p \end{aligned}$$

- Eliminating Equality Constraints

$$\begin{aligned} \min \quad & f_0(Fz + x_0) \\ \text{s.t.} \quad & f_i(Fz + x_0) \leq 0, \qquad i = 1, \ldots, m \end{aligned}$$

  - $A = [a_1^\top; \ldots; a_p^\top]$, $b = (b_1; \ldots; b_p)$
  - $Ax_0 = b$, $\mathcal{R}(F) = \mathcal{N}(A)$
  - The composition of a convex function with an affine function is convex

# Equivalent Convex Problems

- Introducing Equality Constraints
  - If an objective or constraint function has the form $f_i(A_i x + b_i)$, where $A_i \in \mathbf{R}^{k_i \times n}$, we can replace it with $f_i(y_i)$ and add the constraint $y_i = A_i x + b_i$, where $y_i \in \mathbf{R}^{k_i}$
- Slack Variables
  - Introduce new constraint $f_i(x) + s_i = 0$ and requiring that $f_i$ is affine
  - Introduce slack variables for linear inequalities preserves convexity of a problem
- Minimizing over Some Variables
  - It preserves convexity. $f_0(x_1, x_2)$ needs to be jointly convex in $x_1$ and $x_2$

# Equivalent Convex Problems

## Epigraph Problem Form

$$\begin{array}{ll} \min & t \\ \text{s.t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & a_i^\top x = b_i, \qquad i = 1, \ldots, p \end{array}$$

- The objective is linear (hence convex)
- The new constraint function $f_0(x) - t$ is also convex in $(x, t)$
- This problem is convex
- Any convex optimization problem is readily transformed to one with linear objective.

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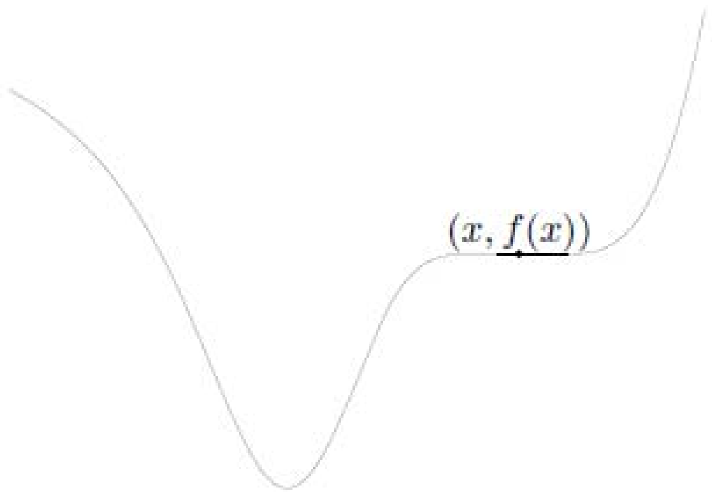
# Quasiconvex Optimization

- Standard Form

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\
& Ax = b
\end{aligned}
$$

  - $f_0$ is quasiconvex and $f_1, \ldots, f_m$ are convex
  - Have locally optimal solutions that are not (globally) optimal



# Quasiconvex Optimization

- Standard Form

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  - $f_0$ is quasiconvex and $f_1, \ldots, f_m$ are convex
  - Have locally optimal solutions that are not (globally) optimal

- Optimality Conditions for Differentiable $f_0$
  - Let $X$ denote the feasible set, $x$ is optimal if

$$x \in X, \qquad \nabla f_0(x)^\top (y - x) > 0 \text{ for all } y \in X \setminus \{x\}$$

1. Only a sufficient condition
2. Requires $\nabla f_0(x)$ to be nonzero

# Representation via family of convex functions

- Rerpesent the sublevel sets of a quasiconvex function $f$ via inequalities of convex functions.
  - $\phi_t : \mathbf{R}^n \rightarrow \mathbf{R}$ is convex, $t \in \mathbf{R}$

$$f(x) \leq t \Leftrightarrow \phi_t(x) \leq 0$$

  - $\phi_t$ is a nonincreasing function of $t$
  - Examples

$$\phi_t(x) = \begin{cases} 0 & f(x) \leq t \\ \infty & \text{otherwise} \end{cases}$$

$$\phi_t(x) = \operatorname{dist}(x, \{z | f(z) \leq t\})$$

# Quasiconvex Optimization via Convex Feasibility Problems

- Let $\phi_t: \mathbf{R}^n \to \mathbf{R}, t \in \mathbf{R}$, be a family of convex functions such that

$$f_0(x) \le t \Leftrightarrow \phi_t(x) \le 0$$

and for each $x$, $\phi_s(x) \le \phi_t(x)$ whenever $s \ge t$

  - Let $p^\star$ be the optimal value of quasiconvex problem
  - Consider the feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{s.t.} & \phi_t(x) \le 0 \\ & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  - If it is feasible, $p^\star \le t$. Conversely, $p^\star \ge t$

# Bisection for Quasiconvex Optimization

## Algorithm

**given** $l \le p^\star, u \ge p^\star$, tolerance $\epsilon > 0$

**repeat**

1. $t \coloneqq (l+u)/2$
2. Solve the convex feasibility problem
3. **if** it is feasible, $u \coloneqq t$;  **else** $l \coloneqq t$

**until** $u - l \le \epsilon$

- The interval $[l, u]$ is guaranteed to contain $p^\star$
- The length of the interval after $k$ iterations is $2^{-k}(u-l)$
- $\lceil \log_2((u-l)/\epsilon) \rceil$ iterations are required

# Bisection for Quasiconvex Optimization

- An $\epsilon-$suboptimal Solution
  - $l \leq p^* \leq u$
  - $u - l \leq \epsilon$
  - $u - p^* \leq \epsilon$

$$\begin{array}{ll} \text{find} & x \\ \text{s.t.} & \phi_u(x) \leq 0 \\ & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  - $f_0(x) \leq u = p^* + u - p^* \leq p^* + \epsilon$

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