Convex Optimization Problems (I)

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Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions

- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization
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Basic Terminology

- **Optimization Problems**

\[
\begin{align*}
\text{min } & \quad f_0(x) \\
\text{s.t. } & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- Optimization variable: \( x \in \mathbb{R}^n \)
- Objective function: \( f_0: \mathbb{R}^n \rightarrow \mathbb{R} \)
- Inequality constraints: \( f_i(x) \leq 0 \)
- Inequality constraint functions: \( f_i: \mathbb{R}^n \rightarrow \mathbb{R} \)
- Equality constraints: \( h_i(x) = 0 \)
- Equality constraint functions: \( h_i: \mathbb{R}^n \rightarrow \mathbb{R} \)

Unconstrained when \( m = p = 0 \)
Basic Terminology

- Optimization Problems

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]  \hspace{1cm} (1)

- Domain

\[\mathcal{D} = \bigcap_{i=0}^{m} \text{dom } f_i \cap \bigcap_{i=1}^{p} \text{dom } h_i\]

- \(x \in \mathcal{D}\) is feasible if it satisfies all the constraints

- The problem is feasible if there exists at least one feasible point
Basic Terminology

- **Optimal Value** $p^*$
  
  $p^* = \inf \{f_0(x)|f_i(x) \leq 0, i = 1,\ldots,m, h_i(x) = 0, i = 1,\ldots,p\}$

  - Infeasible problem: $p^* = \infty$
  - Unbounded below: if there exist $x_k$ with $f_0(x_k) \to -\infty$ as $k \to \infty$, then $p^* = -\infty$

- **Optimal Points**
  
  - $x^*$ is feasible and $f_0(x^*) = p^*$

- **Optimal Set**

  $X_{opt} = \{x|f_i(x) \leq 0, i = 1,\ldots,m, h_i(x) = 0, i = 1,\ldots,p, f_0(x) = p^*\}$

- **$p^*$ is achieved if $X_{opt}$ is nonempty**
Basic Terminology

- **ε-suboptimal Points**
  - a feasible $x$ with $f_0(x) \leq p^* + \varepsilon$

- **ε-suboptimal Set**
  - the set of all ε-suboptimal points

- **Locally Optimal Points**
  \[
  \begin{align*}
  \min & \quad f_0(z) \\
  \text{s.t.} & \quad f_i(z) \leq 0, \quad i = 1, \ldots, m \\
  & \quad h_i(z) = 0, \quad i = 1, \ldots, p \\
  & \quad \|z - x\|_2 \leq R
  \end{align*}
  \]
  - $x$ is feasible and solves this above problem

- **Globally Optimal Points**
Basic Terminology

☐ **Types of Constraints**

- If $f_i(x) = 0$, $f_i(x) \leq 0$ is **active** at $x$
- If $f_i(x) < 0$, $f_i(x) \leq 0$ is **inactive** at $x$
- $h_i(x) = 0$ is active at all feasible points
- **Redundant** constraint: deleting it does not change the feasible set

☐ **Examples on** $x \in \mathbb{R}$ and $\text{dom } f_0 = \mathbb{R}_{++}$

- $f_0(x) = 1/x : p^* = 0$, the optimal value is not achieved
- $f_0(x) = -\log x : p^* = -\infty$, unbounded blow
- $f_0(x) = x \log x : p^* = -1/e$, $x^* = 1/e$ is optimal
Basic Terminology

- **Feasibility Problems**

  find \( x \)
  
  s.t.
  
  \[ f_i(x) \leq 0, \quad i = 1, \ldots, m \]
  
  \[ h_i(x) = 0, \quad i = 1, \ldots, p \]

  - Determine whether constraints are consistent

- **Maximization Problems**

  \[ \max f_0(x) \]
  
  s.t.
  
  \[ f_i(x) \leq 0, \quad i = 1, \ldots, m \]
  
  \[ h_i(x) = 0, \quad i = 1, \ldots, p \]

  - It can be solved by minimizing \(-f_0\)

  - Optimal Value \( p^* \)

  \[ p^* = \sup \{ f_0(x) | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p \} \]
Basic Terminology

- **Standard Form**
  \[
  \min \quad f_0(x) \\
  \text{s.t.} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  \quad h_i(x) = 0, \quad i = 1, \ldots, p
  \]

- **Box constraints**
  \[
  \min \quad f_0(x) \\
  \text{s.t.} \quad l_i \leq x_i \leq u_i, \quad i = 1, \ldots, n
  \]

- **Reformulation**
  \[
  \min \quad f_0(x) \\
  \text{s.t.} \quad l_i - x_i \leq 0, \quad i = 1, \ldots, n \\
  \quad x_i - u_i \leq 0, \quad i = 1, \ldots, n
  \]
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Equivalent Problems

- Two Equivalent Problems
  - If from a solution of one, a solution of the other is readily found, and vice versa

- A Simple Example
  
  \[
  \begin{align*}
  \min & \quad \tilde{f}(x) = \alpha_0 f_0(x) \\
  \text{s.t.} & \quad \tilde{f}_i(x) = \alpha_i f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  & \quad \tilde{h}_i(x) = \beta_i h_i(x) = 0, \quad i = 1, \ldots, p
  \end{align*}
  \]

  - \(\alpha_i > 0, i = 0, \ldots, m\)
  - \(\beta_i \neq 0, i = 1, \ldots, p\)
  - Equivalent to the problem (1)
Change of Variables

- $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one and $\phi(\text{dom } \phi) \supseteq \mathcal{D}$, and define

\[
\tilde{f}_i(z) = f_i(\phi(z)), \quad i = 0, \ldots, m \\
\tilde{h}_i(z) = h_i(\phi(z)), \quad i = 1, \ldots, p
\]

- An Equivalent Problem

\[
\min \quad \tilde{f}_0(z) \\
\text{s.t.} \quad \tilde{f}_i(z) \leq 0, \quad i = 1, \ldots, m \\
\tilde{h}_i(z) = 0, \quad i = 1, \ldots, p
\]

- If $z$ solves it, $x = \phi(z)$ solves the problem (1)
- If $x$ solves (1), $z = \phi^{-1}(x)$ solves it
Transformation of Functions

- $\psi_0 : \mathbb{R} \to \mathbb{R}$ is monotone increasing
- $\psi_1, \ldots, \psi_m : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$
- $\psi_{m+1}, \ldots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$
- Define $\tilde{f}_i(x) = \psi_i(f_i(x)), \quad i = 0, \ldots, m$
  $\tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \quad i = 1, \ldots, p$
- An Equivalent Problem

$$\min \tilde{f}_0(x)$$
$$\text{s. t. } \tilde{f}_i(x) \leq 0, \quad i = 1, \ldots, m$$
$$\tilde{h}_i(x) = 0, \quad i = 1, \ldots, p$$
Example

- **Least-norm Problems**
  \[
  \min \quad \|Ax - b\|_2
  \]
  - Not differentiable at any \( x \) with \( Ax - b = 0 \)

- **Least-norm-squared Problems**
  \[
  \min \quad \|Ax - b\|_2^2 = (Ax - b)^T(Ax - b)
  \]
  - Differentiable for all \( x \)
slack variables

- $f_i(x) \leq 0$ if and only if there is an $s_i \geq 0$ that satisfies $f_i(x) + s_i = 0$

- An Equivalent Problem

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad s_i \geq 0, \quad i = 1, \ldots, m \\
& \quad f_i(x) + s_i = 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- $s_i$ is the slack variable associated with the inequality constraint $f_i(x) \leq 0$

- $x$ is optimal for the problem (1) if and only if $(x, s)$ is optimal for the above problem, where $s_i = -f_i(x)$
Eliminating Equality Constraints

- Assume $\phi: \mathbb{R}^k \to \mathbb{R}^n$ is such that $x$ satisfies
  \[ h_i(x) = 0, \quad i = 1, \ldots, p \]
  if and only if there is some $z \in \mathbb{R}^k$ such that
  \[ x = \phi(z) \]

- An Equivalent Problem
  \[
  \begin{align*}
  \min & \quad \tilde{f}_0(z) = f_0(\phi(z)) \\
  \text{s.t.} & \quad \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]
  - If $z$ is optimal for this problem, $x = \phi(z)$ is optimal for the problem (1)
  - If $x$ is optimal for (1), there is at least one $z$ which is optimal for this problem
Eliminating linear equality constraints

- Assume the equality constraints are all linear $Ax = b$, and $x_0$ is one solution

- Let $F \in \mathbb{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbb{R}^k\}$$

- An Equivalent Problem ($x = Fz + x_0$)

$$\min \ f_0(Fz + x_0)$$
$$\text{s.t.} \ f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m$$

$k = n - \text{rank}(A)$
Introducing Equality Constraints

Consider the problem
\[
\begin{align*}
\min & \quad f_0(A_0x + b_0) \\
\text{s.t.} & \quad f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]
\[x \in \mathbb{R}^n, A_i \in \mathbb{R}^{k_i \times n} \text{ and } f_i : \mathbb{R}^{k_i} \to \mathbb{R}\]

An Equivalent Problem
\[
\begin{align*}
\min & \quad f_0(y_0) \\
\text{s.t.} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
& \quad y_i = A_ix + b_i, \quad i = 0, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]
\[\text{Introduce } y_i \in \mathbb{R}^{k_i} \text{ and } y_i = A_ix + b_i\]
Optimizing over Some Variables

- Suppose \( x \in \mathbb{R}^n \) is partitioned as \( x = (x_1, x_2) \), with \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \) and \( n_1 + n_2 = n \)

- Consider the problem

\[
\begin{align*}
\min & \quad f_0(x_1, x_2) \\
\text{s.t.} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1 \\
& \quad \tilde{f}_i(x_2) \leq 0, \quad i = 1, \ldots, m_2
\end{align*}
\]

- An Equivalent Problem

\[
\begin{align*}
\min & \quad \tilde{f}_0(x_1) \\
\text{s.t.} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1
\end{align*}
\]

where

\[
\tilde{f}_0(x_1) = \inf \{ f_0(x_1, z) | \tilde{f}_i(z) \leq 0, i = 1, \ldots, m_2 \}
\]
Example

- **Minimize a Quadratic Function**

\[
\begin{align*}
\min & \quad x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\
\text{s.t.} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

- **Minimize over** \(x_2\)

\[
\begin{align*}
\inf_{x_2} \left( x_1^\top P_{11} x_1 + 2x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \right) \\
= x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1
\end{align*}
\]

- **An Equivalent Problem**

\[
\begin{align*}
\min & \quad x_1^\top (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) x_1 \\
\text{s.t.} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]
Epigraph Problem Form

- **Epigraph Form**

  \[
  \begin{align*}
  \text{min} & \quad t \\
  \text{s.t.} & \quad f_0(x) - t \leq 0 \\
  & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  & \quad h_i(x) = 0, \quad i = 1, \ldots, p
  \end{align*}
  \]

- Introduce a variable \( t \in \mathbb{R} \)
- \((x, t)\) is optimal for this problem if and only if \(x\) is optimal for (1) and \(t = f_0(x)\)
- The objective function of the epigraph form problem is a **linear function** of \(x, t\)
Epigraph Problem Form

- Geometric Interpretation

Find the point in the epigraph that minimizes $t$
Making Constraints Implicit

- Unconstrained problem

\[
\min F(x)
\]

- \(\text{dom } F = \{x \in \text{dom } f_0 | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}\)

- \(F(x) = f_0(x) \text{ for } x \in \text{dom } F\)

- It has not made the problem any easier

- It could make the problem more difficult, because \(F\) is probably not differentiable
Making Constraints Explicit

A Unconstrained Problem

\[ \min f(x) \]

where

\[ f(x) = \begin{cases} 
  x^\top x & Ax = b \\
  \infty & \text{otherwise} 
\end{cases} \]

An implicit equality constraint \( Ax = b \)

An Equivalent Problem

\[ \min x^\top x \]

s.t. \( Ax = b \)

Objective and constraint functions are differentiable
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Problem Descriptions

- **Parameter Problem Description**
  - Functions have some analytical or closed form
  - Example: $f_0(x) = x^T P x + q^T x + r$, where $P \in S^n, q \in \mathbb{R}^n$ and $r \in \mathbb{R}$
  - Give the values of the parameters

- **Oracle Model (Black-box Model)**
  - Can only query the objective and constraint functions by oracle
  - Evaluate $f(x)$ and its gradient $\nabla f(x)$
  - Know some prior information
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Convex Optimization Problems

- **Standard Form**

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

- The objective function must be convex
- The inequality constraint functions must be convex
- The equality constraint functions \( h_i(x) = a_i^T x - b_i \) must be affine
Convex Optimization Problems

Properties

- Feasible set of a convex optimization problem is convex

\[ \bigcap_{i=0}^{m} \text{dom } f_i \cap \bigcap_{i=1}^{m} \{x | f_i(x) \leq 0\} \cap \bigcap_{i=1}^{p} \{x | a_i^T x = b_i\} \]

- Minimize a convex function over a convex set
- $\varepsilon$-suboptimal set is convex
- The optimal set is convex
- If the objective is strictly convex, then the optimal set contains at most one point
Concave Maximization Problems

- **Standard Form**

  \[
  \begin{align*}
  \text{max} & \quad f_0(x) \\
  \text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  & \quad a_i^T x = b_i, \quad i = 1, \ldots, p
  \end{align*}
  \]

- It is referred as a convex optimization problem if \( f_0 \) is concave and \( f_1, \ldots, f_m \) are convex

- It is readily solved by minimizing the convex objective function \(-f_0\)
Abstract From Convex Optimization Problem

- **Consider the Problem**

  \[
  \begin{align*}
  \min & \quad f_0(x) = x_1^2 + x_2^2 \\
  \text{s.t.} & \quad f_1(x) = x_1/(1 + x_2^2) \leq 0 \\
  & \quad h_1(x) = (x_1 + x_2)^2 = 0
  \end{align*}
  \]

  - Not a convex optimization problem
    - \( f_1 \) is not convex and \( h_1 \) is not affine
  - But the **feasible set** is indeed convex
  - Abstract convex optimization problem

- **An Equivalent Convex Problem**

  \[
  \begin{align*}
  \min & \quad f_0(x) = x_1^2 + x_2^2 \\
  \text{s.t.} & \quad f_1(x) = x_1 \leq 0 \\
  & \quad h_1(x) = x_1 + x_2 = 0
  \end{align*}
  \]
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Local and Global Optima

- Any locally optimal point of a convex problem is also (globally) optimal

- Proof by Contradiction
  - $x$ is locally optimal implies:
    \[ f_0(x) = \inf \{ f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R \} \]
    for some $R$
  - Suppose $x$ is not globally optimal, i.e., there exists $f_0(y) < f_0(x)$ and $\|y - x\|_2 > R$
  - Define
    \[ z = (1 - \theta)x + \theta y, \theta = \frac{R}{2\|y - x\|_2} \in (0,1) \]
Local and Global Optima

- By convexity of the feasible set, $z$ is feasible.

- It is easy to check

$$\|z - x\|_2 = \|\theta(y - x)\|_2 = \left\| \frac{R(y - x)}{2\|y - x\|_2} \right\|_2 = \frac{R}{2} < R$$

- By convexity of $f_0$

$$f_0(z) \leq (1 - \theta)f_0(x) + \theta f_0(y) < f_0(x)$$

which contradicts

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \leq R\}$$
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An Optimality Criterion for Differentiable $f_0$

- Suppose $f_0$ is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x), \forall x, y \in \text{dom } f_0$$
An Optimality Criterion for Differentiable $f_0$

- Suppose $f_0$ is differentiable
  \[ f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x), \forall x, y \in \text{dom } f_0 \]

- Let $X$ denote the feasible set
  \[ X = \{ x | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p \} \]

- $x$ is optimal if and only if $x \in X$ and
  \[ \nabla f_0(x)^T (y - x) \geq 0 \text{ for all } y \in X \]
An Optimality Criterion for Differentiable $f_0$

- $x$ is optimal if and only if $x \in X$ and 
  $$\nabla f_0(x)^T(y - x) \geq 0 \text{ for all } y \in X$$

- $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at $x$
Proof of Optimality Condition

**Sufficient Condition**

\[ \nabla f_0(x)^T (y - x) \geq 0 \]
\[ f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x) \]

\[ \Rightarrow f_0(y) \geq f_0(x) \]

**Necessary Condition**

- Suppose \( x \) is optimal but \( \exists y \in X, \nabla f_0(x)^T (y - x) < 0 \)
- Define \( z(t) = ty + (1 - t)x, t \in [0,1] \)
  \[ f_0(z(0)) = f_0(x), \quad \frac{d}{dt} f_0(z(t)) \bigg|_{t=0} = \nabla f_0(x)^T (y - x) < 0 \]
- So, for small positive \( t \), \( f_0(z(t)) < f_0(x) \)
Unconstrained Problems

- $x$ is optimal if and only if $\nabla f_0(x) = 0$
  - Consider $y = x - t\nabla f_0(x)$ and $t > 0$
  - When $t$ is small, $y$ is feasible
    \[
    \nabla f_0(x)^\top(y - x) = -t\|\nabla f_0(x)\|^2 \geq 0 \iff \nabla f_0(x) = 0
    \]

- Unconstrained Quadratic Optimization
  \[
  \min f_0(x) = \frac{1}{2}x^\top P x + q^\top x + r, \quad \text{where } P \in S^n_+
  \]
  - $x$ is optimal if and only if $\nabla f_0(x) = Px + q = 0$
  - If $q \notin \mathcal{R}(P)$, no solution, $f_0$ is unbound below
  - If $P > 0$, unique minimizer $x^* = -P^{-1}q$
  - If $P$ is singular, but $q \in \mathcal{R}(P)$, $X_{\text{opt}} = -P^\dagger q + \mathcal{N}(P)$
Problems with Equality Constraints Only

☐ Consider the Problem

\[ \min f_0(x) \]
\[ \text{s.t. } Ax = b \]

☐ \( x \) is optimal if and only if

\[ \forall f_0(x)^T(y - x) \geq 0, \forall Ay = b \]
Problems with Equality Constraints Only

Consider the Problem

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad Ax = b
\end{align*}
\]

\(x\) is optimal if and only if

\(\nabla f_0(x)^\top(y - x) \geq 0, \forall Ay = b\)

\(\{y | Ay = b\} = \{x + v | v \in \mathcal{N}(A)\}\)

\(\iff \nabla f_0(x)^\top v \geq 0, \forall v \in \mathcal{N}(A)\)

\(\iff \nabla f_0(x)^\top v = 0, \forall v \in \mathcal{N}(A)\)

\(\iff \nabla f_0(x) \perp \mathcal{N}(A) \iff \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\top)\)

\(\iff \exists v \in \mathbb{R}^p, \nabla f_0(x) + A^\top v = 0\)

Lagrange Multiplier
Optimality Condition

\(Ax = b\)

\(\nabla f_0(x) + A^\top v = 0\)
Minimization over the Nonnegative Orthant

Consider the Problem

\[
\min f_0(x) \\
\text{s.t. } x \succeq 0
\]

\(x\) is optimal if and only if

\[
\nabla f_0(x)^T(y - x) \succeq 0, \forall y \succeq 0
\]

\(\iff\)

\[
\begin{cases}
\nabla f_0(x) \succeq 0 \\
-\nabla f_0(x)^T x \succeq 0
\end{cases}
\iff
\begin{cases}
\nabla f_0(x) \succeq 0 \\
\nabla f_0(x)^T x = 0
\end{cases}
\]

The Optimality Condition

\(x \succeq 0, \nabla f_0(x) \succeq 0, x_i (\nabla f_0(x))_i = 0, i = 1, \ldots, n\)

The last condition is called complementarity
Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions

- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization
Equivalent Convex Problems

- **Standard Form**
  \[
  \begin{align*}
  \text{min} & \quad f_0(x) \\
  \text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  & \quad a_i^T x = b_i, \quad i = 1, \ldots, p
  \end{align*}
  \]

- **Eliminating Equality Constraints**
  \[
  \begin{align*}
  \text{min} & \quad f_0(Fz + x_0) \\
  \text{s.t.} & \quad f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]
  - \( A = [a_1^T; \ldots; a_p^T], \quad b = (b_1; \ldots; b_p) \)
  - \( Ax_0 = b, \quad \mathcal{R}(F) = \mathcal{N}(A) \)
  - The composition of a convex function with an affine function is convex
Equivalent Convex Problems

- **Introducing Equality Constraints**
  - If an objective or constraint function has the form $f_i(A_i x + b_i)$, where $A_i \in \mathbb{R}^{k_i \times n}$, we can replace it with $f_i(y_i)$ and add the constraint $y_i = A_i x + b_i$, where $y_i \in \mathbb{R}^{k_i}$

- **Slack Variables**
  - Introduce new constraint $f_i(x) + s_i = 0$ and requiring that $f_i$ is affine
  - Introduce slack variables for linear inequalities preserves convexity of a problem

- **Minimizing over Some Variables**
  - It preserves convexity. $f_0(x_1, x_2)$ needs to be jointly convex in $x_1$ and $x_2$
Equivalent Convex Problems

- **Epigraph Problem Form**

  \[
  \begin{align*}
  \text{min} & \quad t \\
  \text{s.t.} & \quad f_0(x) - t \leq 0 \\
 & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & \quad a_i^\top x = b_i, \quad i = 1, \ldots, p
  \end{align*}
  \]

- The objective is linear (hence convex)
- The new constraint function \( f_0(x) - t \) is also convex in \( (x, t) \)
- This problem is convex
- Any convex optimization problem is readily transformed to one with linear objective.
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Quasiconvex Optimization

**Standard Form**

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( f_0 \) is quasiconvex and \( f_1, \ldots, f_m \) are convex
- Have locally optimal solutions that are not (globally) optimal
Quasiconvex Optimization

- **Standard Form**

  \[
  \min \quad f_0(x) \\
  \text{s.t.} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  Ax = b
  \]

  - \( f_0 \) is quasiconvex and \( f_1, \ldots, f_m \) are convex
  - Have locally optimal solutions that are not (globally) optimal

- **Optimality Conditions for Differentiable \( f_0 \)**

  - Let \( X \) denote the feasible set, \( x \) is optimal if
    \[
    x \in X, \quad \nabla f_0(x)^T(y - x) > 0 \quad \text{for all} \quad y \in X \setminus \{x\}
    \]
  1. Only a sufficient condition
  2. Requires \( \nabla f_0(x) \) to be nonzero
Representation via family of convex functions

- Represent the sublevel sets of a quasiconvex function $f$ via inequalities of convex functions.
  - $\phi_t : \mathbb{R}^n \to \mathbb{R}$ is convex, $t \in \mathbb{R}$
    \[ f(x) \leq t \iff \phi_t(x) \leq 0 \]
  - $\phi_t$ is a nonincreasing function of $t$
  - Examples
    \[ \phi_t(x) = \begin{cases} 
    0 & f(x) \leq t \\
    \infty & \text{otherwise} 
  \end{cases} \]
    \[ \phi_t(x) = \text{dist}(x, \{z | f(z) \leq t\}) \]
Let $\phi_t : \mathbb{R}^n \to \mathbb{R}, t \in \mathbb{R}$, be a family of convex functions such that

$$f_0(x) \leq t \iff \phi_t(x) \leq 0$$

and for each $x$, $\phi_s(x) \leq \phi_t(x)$ whenever $s \geq t$

- Let $p^*$ be the optimal value of quasiconvex problem
- Consider the feasibility problem

\[
\begin{align*}
\text{find} & \quad x \\
\text{s. t.} & \quad \phi_t(x) \leq 0 \\
& \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- If it is feasible, $p^* \leq t$. Conversely, $p^* \geq t$
Bisection for Quasiconvex Optimization

Algorithm

given \( l \leq p^*, u \geq p^* \), tolerance \( \epsilon > 0 \)

repeat
1. \( t := (l + u)/2 \)
2. Solve the convex feasibility problem
3. if it is feasible, \( u := t \); else \( l := t \)

until \( u - l \leq \epsilon \)

- The interval \([l, u]\) is guaranteed to contain \( p^* \)
- The length of the interval after \( k \) iterations is \( 2^{-k}(u - l) \)
- \( \lceil \log_2((u - l)/\epsilon) \rceil \) iterations are required
Bisection for Quasiconvex Optimization

- An $\epsilon$-suboptimal Solution

- $l \leq p^* \leq u$
- $u - l \leq \epsilon$
- $u - p^* \leq \epsilon$

\[
\begin{align*}
\text{find} & \quad x \\
\text{s.t.} & \quad \phi_u(x) \leq 0 \\
& \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- $f_0(x) \leq u = p^* + u - p^* \leq p^* + \epsilon$
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