

# Duality (I)

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# Outline

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- The Lagrange Dual Function
  - The Lagrange Dual Function
  - Lower Bound on Optimal Value
  - The Lagrange Dual Function and Conjugate Functions
- The Lagrange Dual Problem
  - Making Dual Constraints Explicit
  - Weak Duality
  - Strong Duality and Slater's Constraint Qualification



# Optimization Problems

## □ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{aligned} \quad (1)$$

- Domain is nonempty

$$\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- Denote the optimal value by  $p^*$
- We do **not** assume the problem is convex



# The Lagrangian

□ The Lagrangian  $L: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x)$$

- $\text{dom } L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$
- $\lambda_i$ : the Lagrange multiplier associated with the  $i$ -th inequality constraint  $f_i(x) \leq 0$
- $v_i$ : the Lagrange multiplier associated with the  $i$ -th equality constraint  $h_i(x) = 0$
- Vectors  $\lambda$  and  $v$ : **dual variables** or Lagrange multiplier vectors



# The Lagrange Dual Function

$$\square g: \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$$

$$\begin{aligned} g(\lambda, v) &= \inf_{x \in \mathcal{D}} L(x, \lambda, v) \\ &= \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x) \right) \end{aligned}$$

- When  $L$  is unbounded below in  $x$ ,  $g = -\infty$
- $g$  is **concave**
  - ✓  $g$  is the pointwise infimum of a family of affine functions of  $(\lambda, v)$
- It is **unconstrained**



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## □ The Lagrange Dual Problem

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# Lower Bounds on $p^*$

□ For any  $\lambda \geq 0$  and any  $\nu$

$$g(\lambda, \nu) \leq p^*$$

□ Proof

■  $\tilde{x}$  is a feasible point for original problem

$$f_i(\tilde{x}) \leq 0, h_i(\tilde{x}) = 0$$

■ Since  $\lambda \geq 0$

$$\sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \leq 0$$

■ Therefore

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$



# Lower Bounds on $p^*$

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□ For any  $\lambda \geq 0$  and any  $\nu$

$$g(\lambda, \nu) \leq p^*$$

□ Proof

■ Hence

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

■ Note that  $g(\lambda, \nu) \leq f_0(\tilde{x})$  for any feasible  $\tilde{x}$

□ Discussions

■ The lower bound is vacuous, when  $g(\lambda, \nu) = -\infty$

■ It is nontrivial only when  $\lambda \geq 0, (\lambda, \nu) \in \text{dom } g$

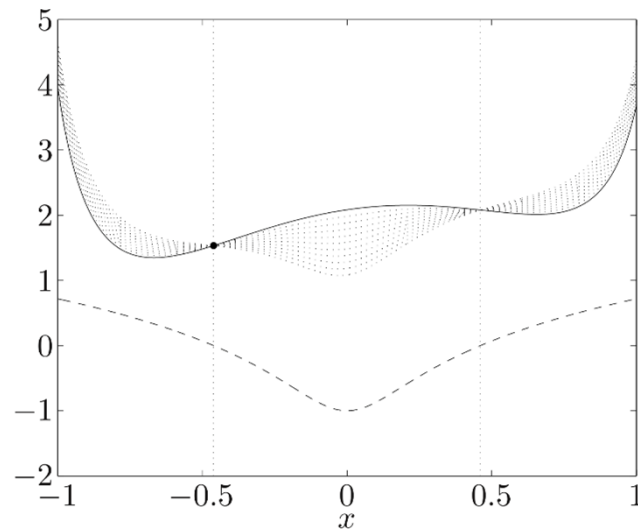
■ Dual feasible:  $(\lambda, \nu)$  with  $\lambda \geq 0, (\lambda, \nu) \in \text{dom } g$





# Example

- A Simple Problem with  $x \in \mathbb{R}, m = 1, p = 0$ 
  - Lower bound from a dual feasible point

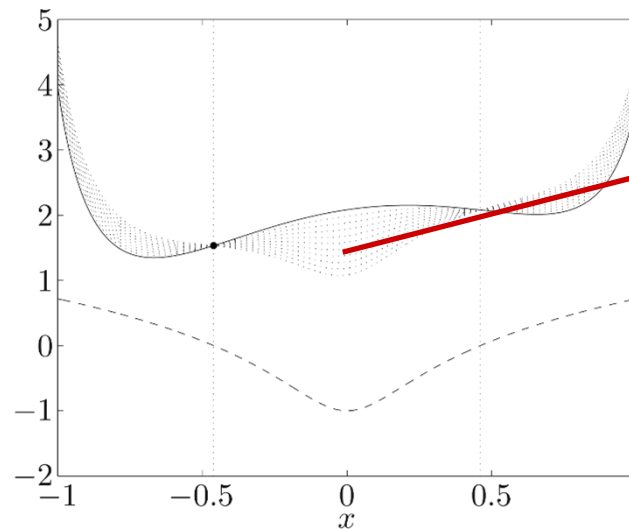


- ✓ Solid curve: objective function  $f_0$
- ✓ Dashed curve: constraint function  $f_1$
- ✓ Feasible set:  $[-0.46, 0.46]$  (indicated by the two dotted vertical lines)



# Example

- A Simple Problem with  $x \in \mathbb{R}, m = 1, p = 0$ 
  - Lower bound from a dual feasible point



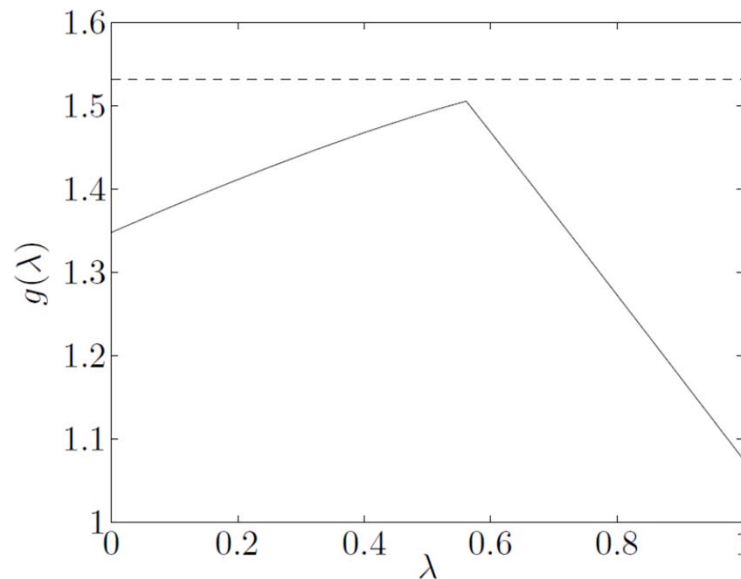
$$g(\lambda) = \inf_{x \in \mathcal{D}} L(x, \lambda) \\ \leq L(x, \lambda) \leq f_0(x)$$

- ✓ Optimal point and value:  $x^* = -0.46, p^* = 1.54$
- ✓ Dotted curves:  $L(x, \lambda)$  for  $\lambda = 0.1, 0.2, \dots, 1.0$ .
  - Each has a minimum value smaller than  $p^*$  as on the feasible set (and for  $\lambda > 0$ ),  $L(x, \lambda) \leq f_0(x)$



# Example

## □ The dual function $g$



- Neither  $f_0$  nor  $f_1$  is convex, but the dual function  $g$  is **concave**
- Horizontal dashed line:  $p^*$  (the optimal value of the problem)



# Example

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## □ Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

- $A \in \mathbf{R}^{p \times n}$
- No inequality constraints
- $p$  (linear) equality constraints

## □ Lagrangian

$$L(x, v) = x^\top x + v^\top (Ax - b)$$

- Domain:  $\mathbf{R}^n \times \mathbf{R}^p$



# Example

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## □ Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

## □ Dual Function

$$g(v) = \inf_x L(x, v) = \inf_x x^\top x + v^\top (Ax - b)$$

### ■ Optimality condition

$$\nabla_x L(x, v) = 2x + A^\top v = 0 \Rightarrow x = -(1/2)A^\top v$$



# Example

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## □ Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

## □ Dual Function

$$\Rightarrow g(v) = L(-(1/2)A^\top v, v) = -(1/4)v^\top AA^\top v - b^\top v$$

### ■ Concave Function

## □ Lower Bound Property

$$-(1/4)v^\top AA^\top v - b^\top v \leq \inf \{x^\top x \mid Ax = b\}$$



# Example

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## □ Standard Form LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s. t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Inequality constraints:  $f_i(x) = -x_i, i = 1, \dots, n$

## □ Lagrangian

$$\begin{aligned} L(x, \lambda, \nu) &= c^T x - \sum_{i=1}^n \lambda_i x_i + \nu^T (Ax - b) \\ &= -b^T \nu + (c + A^T \nu - \lambda)^T x \end{aligned}$$

## □ Dual Function

$$\begin{aligned} g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\ &= -b^T \nu + \inf_x (c + A^T \nu - \lambda)^T x \end{aligned}$$



# Example

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## □ Standard Form LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s. t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Inequality constraints:  $f_i(x) = -x_i, i = 1, \dots, n$

## □ Dual Function

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

- The lower bound is nontrivial only when  $\lambda$  and  $\nu$  satisfy  $\lambda \geq 0$  and  $A^T \nu - \lambda + c = 0$





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## □ The Lagrange Dual Problem

- Making Dual Constraints Explicit
- Weak Duality
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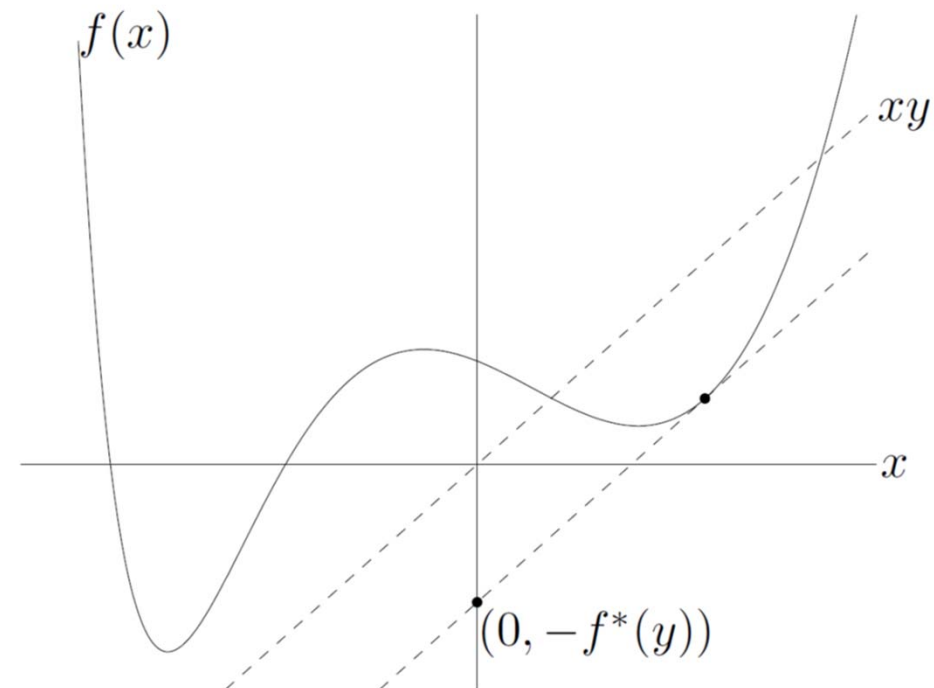


# Conjugate Function

□  $f: \mathbf{R}^n \rightarrow \mathbf{R}$ . Its conjugate function is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^\top x - f(x))$$

- $\text{dom } f^* = \{y | f^*(y) < \infty\}$
- $f^*$  is always convex



# The Lagrange Dual Function and Conjugate Functions



## □ A Simple Example

$$\begin{array}{ll} \min & f(x) \\ \text{s. t.} & x = 0 \end{array}$$

## □ Lagrangian

$$L(x, v) = f(x) + v^T x$$

## □ Dual Function

$$\begin{aligned} g(v) &= \inf_x (f(x) + v^T x) \\ &= - \sup_x ((-v)^T x - f(x)) = -f^*(-v) \end{aligned}$$

# The Lagrange Dual Function and Conjugate Functions



## □ A More General Example

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & Ax \preceq b \\ & Cx = d \end{array}$$

## □ Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

## □ Dual Function

$$\begin{aligned} g(\lambda, \nu) &= \inf_x (f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)) \\ &= -b^\top \lambda - d^\top \nu + \inf_x (f_0(x) + (A^\top \lambda + C^\top \nu)^\top x) \\ &= -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu) \end{aligned}$$

# The Lagrange Dual Function and Conjugate Functions



## □ A More General Example

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & Ax \preceq b \\ & Cx = d \end{array}$$

## □ Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

## □ Dual Function

$$g(\lambda, \nu) = -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu)$$

$$\blacksquare \text{ dom } g = \{(\lambda, \nu) \mid -A^\top \lambda - C^\top \nu \in \text{dom } f_0^*\}$$



# Example

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## □ Equality Constrained Norm Minimization

$$\begin{aligned} \min \quad & \|x\| \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

## □ Conjugate of $f_0 = \|\cdot\|$

$$f_0^*(y) = \begin{cases} 0 & \|y\|_* \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

## □ The Dual Function

$$g(v) = -b^\top v - f_0^*(-A^\top v) = \begin{cases} -b^\top v & \|A^\top v\|_* \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$



# Example

## □ Entropy Maximization

$$\begin{aligned} \min \quad & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \text{s. t.} \quad & Ax \preceq b \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

## □ Conjugate of $f_0$

$$f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

## □ The Dual Function

$$\begin{aligned} g(\lambda, v) &= -b^\top \lambda - v - f_0^*(-A^\top \lambda - v\mathbf{1}) \\ &= -b^\top \lambda - v - \sum_{i=1}^n e^{-a_i^\top \lambda - v - 1} \\ &= -b^\top \lambda - v - e^{-v-1} \sum_{i=1}^n e^{-a_i^\top \lambda} \end{aligned}$$



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# The Lagrange Dual Problem

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□ For any  $\lambda \geq 0$  and any  $v$

$$g(\lambda, v) \leq p^*$$

■ What is the best lower bound?

□ Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, v) \\ \text{s. t.} & \lambda \geq 0 \end{array}$$

□ Primal Problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array} \quad (1)$$



# The Lagrange Dual Problem

---

□ For any  $\lambda \geq 0$  and any  $\nu$

$$g(\lambda, \nu) \leq p^*$$

■ What is the best lower bound?

□ Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, \nu) \\ \text{s. t.} & \lambda \geq 0 \end{array}$$

■ Dual feasible:  $(\lambda, \nu)$  with  $\lambda \geq 0, g(\lambda, \nu) > -\infty$

■ Dual optimal or optimal Lagrange multipliers:  $(\lambda^*, \nu^*)$

■ A convex optimization problem



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# Making Dual Constraints Explicit

## □ Motivation

- The  $\text{dom } g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}$  may have dimension  $\leq m + p$
- Identify the equality constraints that are 'hidden' or 'implicit' in  $g$

## □ Standard Form LP

$$\begin{array}{ll} \min & c^\top x \\ \text{s. t.} & Ax = b \\ & x \geq 0 \end{array}$$

## □ Dual Function

$$g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$



# Example

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## □ Lagrange Dual of Standard Form LP

### ■ Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, \nu) = \begin{cases} -b^\top \nu - A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

### ■ An Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \nu \\ \text{s.t.} \quad & A^\top \nu - \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

- ✓ Make equality constraints explicit



# Example

## □ Lagrange Dual of Standard Form LP

### ■ Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

### ■ Another Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \nu \\ \text{s.t.} \quad & A^\top \nu + c \geq 0 \end{aligned}$$

✓ An LP in inequality form

Standard Form LP

Lagrange Dual  
→

Inequality Form LP



# Example

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## □ Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{array}{ll} \min & c^T x \\ \text{s. t.} & Ax \preceq b \end{array}$$

- Lagrangian

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b) = -b^T \lambda + (A^T \lambda + c)^T x$$

- Lagrange dual function

$$\begin{aligned} g(\lambda) &= \inf_x L(x, \lambda) = -b^T \lambda + \inf_x (A^T \lambda + c)^T x \\ &= \begin{cases} -b^T \lambda & A^T \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$



# Example

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## □ Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

- Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda) = \begin{cases} -b^T \lambda & A^T \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \succeq 0 \end{aligned}$$

- An Equivalent Problem

$$\begin{aligned} \max \quad & -b^T \lambda \\ \text{s.t.} \quad & A^T \lambda + c = 0 \\ & \lambda \succeq 0 \end{aligned}$$





# Example

## □ Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

- An Equivalent Problem

$$\begin{aligned} \max \quad & -b^T \lambda \\ \text{s.t.} \quad & A^T \lambda + c = 0 \\ & \lambda \succeq 0 \end{aligned}$$

- ✓ An LP in standard form

Inequality Form LP

Lagrange Dual  
→

Standard Form LP



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- Making Dual Constraints Explicit
- **Weak Duality**
- Strong Duality and Slater's Constraint Qualification



# Weak Duality

- For any  $\lambda \geq 0$  and any  $v$

$$g(\lambda, v) \leq p^*$$

- What is the best lower bound?

- Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, v) \\ \text{s. t.} \quad & \lambda \geq 0 \end{aligned}$$

- Optimal value  $d^*$

- Weak Duality

$$d^* \leq p^*$$



Does not rely  
on convexity!



# Weak Duality

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## □ Weak Duality

$$d^* \leq p^*$$

- If the primal problem is unbounded below, i.e.,  $p^* = -\infty$ , we must have  $d^* = -\infty$ , i.e., the Lagrange dual problem is infeasible
- If  $d^* = \infty$ , we must have  $p^* = \infty$

## □ Optimal duality gap

$$p^* - d^*$$

- Nonegative



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# Strong Duality

## □ Strong Duality

$$d^* = p^*$$

- The optimal duality gap is zero
- The best bound that can be obtained from the Lagrange dual function is tight
- In general, does not hold

## □ Usually hold for convex optimization

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s. t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

- $f_0, \dots, f_m$  are convex



# Slater's Constraint Qualification

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## □ Constraint Qualifications

- Sufficient conditions for strong duality

## □ Slater's condition

- $\exists x \in \text{relint } \mathcal{D}$  such that

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

- Such a point  $x$  is called strictly feasible

## □ If Slater's condition holds and the problem is convex

- Strong duality holds
- Dual optimal value is attained when  $d^* > -\infty$



# Slater's Constraint Qualification

---

## □ Constraint Qualifications

- Sufficient conditions for strong duality

## □ Slater's condition (weaker form)

- If the first  $k$  constraint functions are affine
- $\exists x \in \text{relint } \mathcal{D}$  such that

$$f_i(x) \leq 0, \quad i = 1, \dots, k$$

$$f_i(x) < 0, \quad i = k + 1, \dots, m$$

$$Ax = b$$

- When constraints are all linear equalities and inequalities, and  $\text{dom } f_0$  is open
  - ✓ Reduce to feasibility





# Example

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- Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s. t.} \quad & Ax = b \end{aligned}$$

- Dual Problem

$$\max \quad -(1/4)v^\top AA^\top v - b^\top v$$

- Slater's condition

- The primal problem is feasible, i.e.,  $b \in \mathcal{R}(A)$

- Strong duality always holds

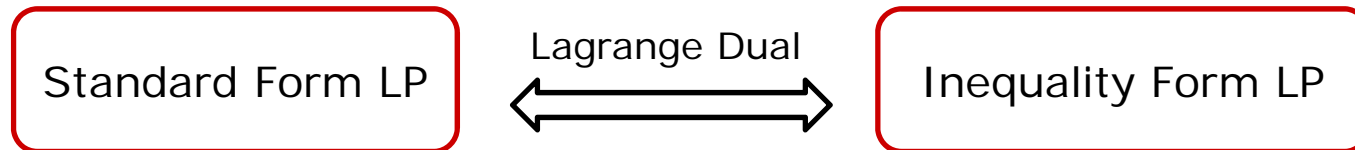
- Even when  $b \notin \mathcal{R}(A)$



# Example

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## □ Lagrange dual of LP



## □ Strong duality holds for any LP

- If the primal problem is feasible **or** the dual problem is feasible

## □ Strong duality may fail

- If **both** the primal and dual problems are infeasible



# Example

## □ QCQP (Primal Problem)

$$\begin{aligned} \min \quad & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{s.t.} \quad & (1/2)x^\top P_i x + q_i^\top x + r_i \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- $P_0 \in \mathbf{S}_{++}^n$  and  $P_i \in \mathbf{S}_+^n, i = 1, \dots, m$

## □ Dual Problem

$$\begin{aligned} \max \quad & -(1/2)q(\lambda)^\top P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

- $P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i$

- $r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i$

## □ Slater's condition

- $\exists x, (1/2)x^\top P_i x + q_i^\top x + r_i < 0, i = 1, \dots, m$



# Example

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## □ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top Ax + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

■  $A \in \mathbf{S}^n, A \not\geq 0$  and  $b \in \mathbf{R}^n$

■ Lagrangian

$$\begin{aligned} L(x, \lambda) &= x^\top Ax + 2b^\top x + \lambda(x^\top x - 1) \\ &= x^\top (A + \lambda I)x + 2b^\top x - \lambda \end{aligned}$$

■ Dual Function

$$g(\lambda) = \begin{cases} -b^\top (A + \lambda I)^\dagger b - \lambda & A + \lambda I \geq 0, b \in \mathcal{R}(A + \lambda I) \\ -\infty & \text{otherwise} \end{cases}$$



# Example

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## □ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

- $A \in \mathbf{S}^n, A \not\geq 0$  and  $b \in \mathbf{R}^n$

## □ Dual Problem

$$\begin{aligned} \max \quad & -b^\top (A + \lambda I)^\dagger b - \lambda \\ \text{s.t.} \quad & A + \lambda I \geq 0, b \in \mathcal{R}(A + \lambda I) \end{aligned}$$

- A convex optimization problem



# Example

## □ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

- $A \in \mathbf{S}^n, A \not\equiv 0$  and  $b \in \mathbf{R}^n$

Strong duality  
holds

## □ Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n (q_i^\top b)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s.t.} \quad & \lambda \geq -\lambda_{\min}(A) \end{aligned}$$

- A convex optimization problem
- $\lambda_i$  and  $q_i$ : eigenvalues and corresponding (orthonormal) eigenvectors of  $A$



# Example

## □ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

- $A \in \mathbf{S}^n, A \not\equiv 0$  and  $b \in \mathbf{R}^n$

Strong duality holds for any optimization problem with quadratic objective and one quadratic inequality constraint, provided Slater's condition holds

## □ Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n (q_i^\top b)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s.t.} \quad & \lambda \geq -\lambda_{\min}(A) \end{aligned}$$

- A convex optimization problem
- $\lambda_i$  and  $q_i$ : eigenvalues and corresponding (orthonormal) eigenvectors of  $A$



# Summary

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- The Lagrange Dual Function
  - The Lagrange Dual Function
  - Lower Bound on Optimal Value
  - The Lagrange Dual Function and Conjugate Functions
- The Lagrange Dual Problem
  - Making Dual Constraints Explicit
  - Weak Duality
  - Strong Duality and Slater's Constraint Qualification