# Convex Functions (II)

Lijun Zhang
zlj@nju.edu.cn
http://cs.nju.edu.cn/zlj

# Outline

- The Conjugate Function
- Quasiconvex Functions
- Log-concave and Log-convex Functions
- Convexity with Respect to Generalized Inequalities
- Summary

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# Conjugate Function

- $f: \mathbf{R}^n \to \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \left( y^\top x - f(x) \right)$$

  - $\operatorname{dom} f^* = \{y | f^*(y) < \infty\}$
  - $f^*$ is always convex

# Conjugate Function

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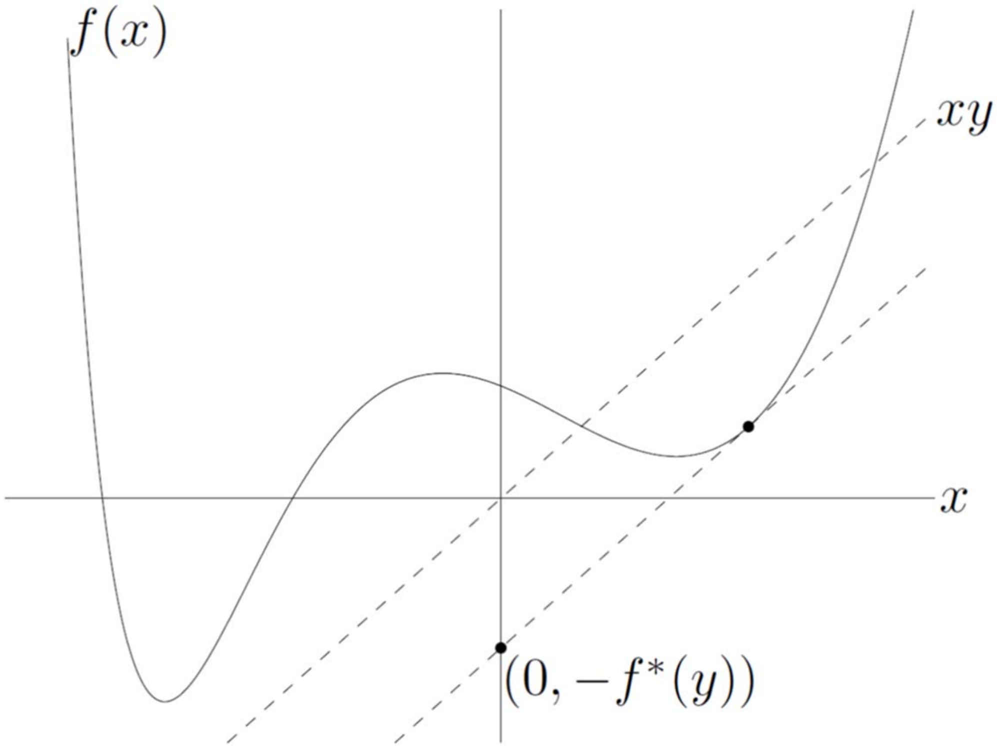

# Conjugate examples

- Affine function
  - $f(x) = ax + b$
  - $f^*(y) = \sup_{x \in \mathbf{R}}(yx - ax - b)$
  - $\mathrm{dom}\, f^* = \{a\}, f^*(a) = -b$

- Negative logarithm
  - $f(x) = -\log x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (yx + \log x)$
  - $\mathrm{dom}\, f^* = -\mathbf{R}_{++}, f^*(y) = -\log(-y) - 1$

# Conjugate examples

- Exponential
  - $f(x) = e^x$
  - $f^*(y) = \sup_{x \in \mathbf{R}} (xy - e^x)$
  - $\mathrm{dom}\, f^* = \mathbf{R}_+, f^*(y) = y \log y - y$

- Negative entropy
  - $f(x) = x \log x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_+} (yx - x \log x)$
  - $\mathrm{dom}\, f^* = \mathbf{R}, f^*(y) = e^{y-1}$

# Conjugate examples

- Inverse
  - $f(x) = 1/x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - 1/x)$
  - $\mathrm{dom}\, f^* = -\mathbf{R}_+, f^*(y) = -2(-y)^{1/2}$

- Strictly convex quadratic function
  - $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (y^\top x - \frac{1}{2} x^\top Q x)$
  - $\mathrm{dom}\, f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2} y^\top Q^{-1} y$

# Conjugate examples

- Log-determinant
  - $f(X) = \log\det X^{-1}, X \in \mathbf{S}_{++}^n$
  - $f^*(Y) = \sup_{X \in \mathbf{S}_{++}^n} (\operatorname{tr}(XY) + \log\det X)$
  - $\operatorname{dom} f^* = -\mathbf{S}_{++}^n, f^*(Y) = \log\det(-Y)^{-1} - n$

- Indicator function
  - $I_S(x) = 0, \operatorname{dom} I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
  - $I_S^*(y) = \sup_{x \in S} y^\top x$
  - $I_S^*(y)$ is the support function of the set $S$

# Conjugate examples

- Norm
  - $f(x) = \|x\|, x \in \mathbf{R}^n$, with dual norm $\|\cdot\|_*$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \|x\|)$
  - $\operatorname{dom} f^* = \{y \mid \|y\|_* \le 1\}, f^*(y) = 0$

- Norm squared
  - $f(x) = \frac{1}{2}\|x\|^2, x \in \mathbf{R}^n$,with dual norm $\|\cdot\|_*$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \frac{1}{2}\|x\|^2)$
  - $\operatorname{dom} f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2}\|y\|_*^2$

# Basic properties

## Fenchel's inequality

- $\forall x \in \mathrm{dom}\, f, y \in \mathrm{dom}\, f^*, f(x) + f^*(y) \geq x^\top y$
- $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$
- $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}^n_{++}$
  $\Rightarrow x^\top y \leq \frac{1}{2} x^\top Q x + \frac{1}{2} y^\top Q^{-1} y$

## Conjugate of the conjugate

- $f$ is convex and closed $\Rightarrow f^{**} = f$

# Basic properties

## Differentiable functions

- $f$ is convex and differentiable, $\operatorname{dom} f = \mathbf{R}^n$

- $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$

- $x^* = \operatorname{argmax}\left(x^\top y - f(x)\right) \Rightarrow \nabla f(x^*) = y$

- $f^*(y) = {x^*}^\top \nabla f(x^*) - f(x^*) = {x^*}^\top y - f(x^*)$
  - ✓ $x^* = \nabla^{-1} f(y)$

# Basic properties

- Scaling with affine transformation
  - $a > 0, b \in \mathbf{R}, g(x) = af(x) + b$
    $\Rightarrow g^*(y) = af^*\left(\frac{y}{a}\right) - b$
  - $A \in \mathbf{R}^{n\times n}$ is nonsingular, $b \in \mathbf{R}^n, g(x) = f(Ax + b) \Rightarrow g^*(y) = f^*(A^{-\top}y) - b^\top A^{-\top}y, \mathrm{dom}\, g^* = A^\top \mathrm{dom}\, f^*$

- Sums of independent functions
  - $f(u, v) = f_1(u) + f_2(v), f_1, f_2$ are convex $\Rightarrow f^*(w, z) = f_1^*(w) + f_2^*(z)$

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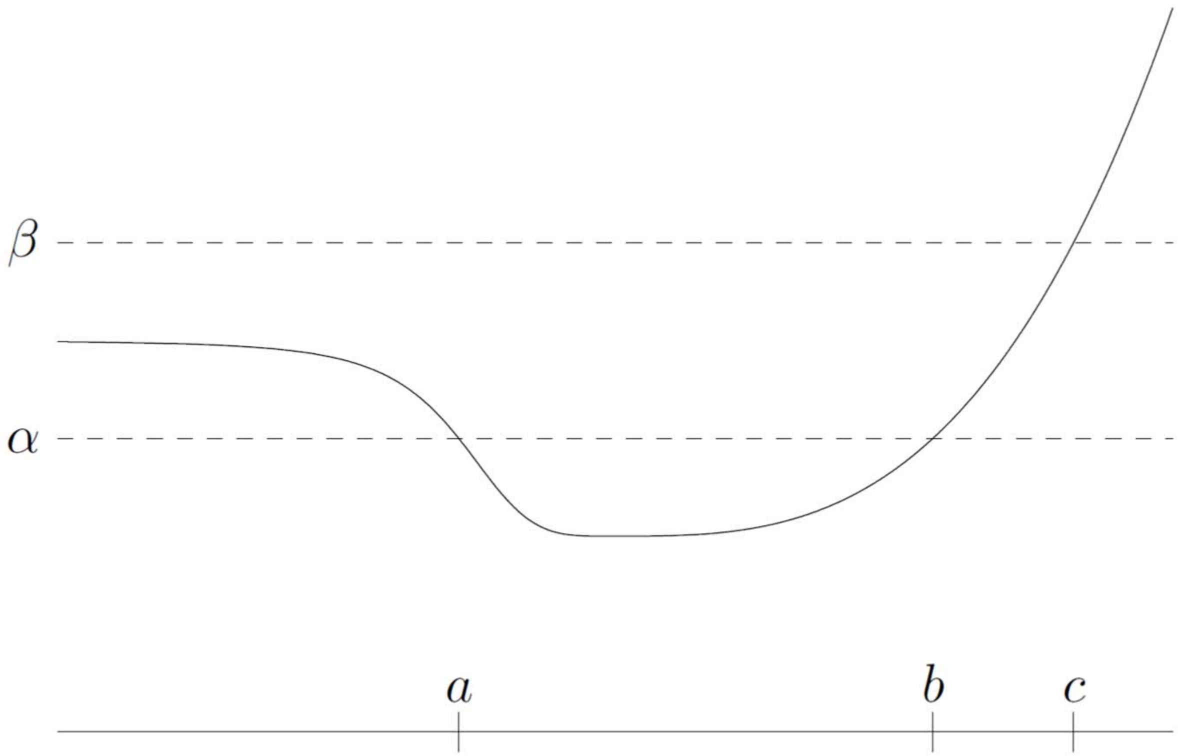
# Quasiconvex functions

- Quasiconvex
  - $f: \mathbf{R}^n \to \mathbf{R}$
  - $S_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \le \alpha\}, \forall \alpha \in \mathbf{R}$ is convex



# Quasiconvex functions

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- Quasiconcave
  - $-f$ is quasiconvex $\Rightarrow f$ is quasiconcave
- Quasilinear
  - $f$ is quasiconvex and quasiconcave $\Rightarrow f$ is quasilinear

# Examples

- Some example on $\mathbf{R}$
  - Logarithm: $\log x$ on $\mathbf{R}_{++}$
  - Ceiling function: $\operatorname{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$

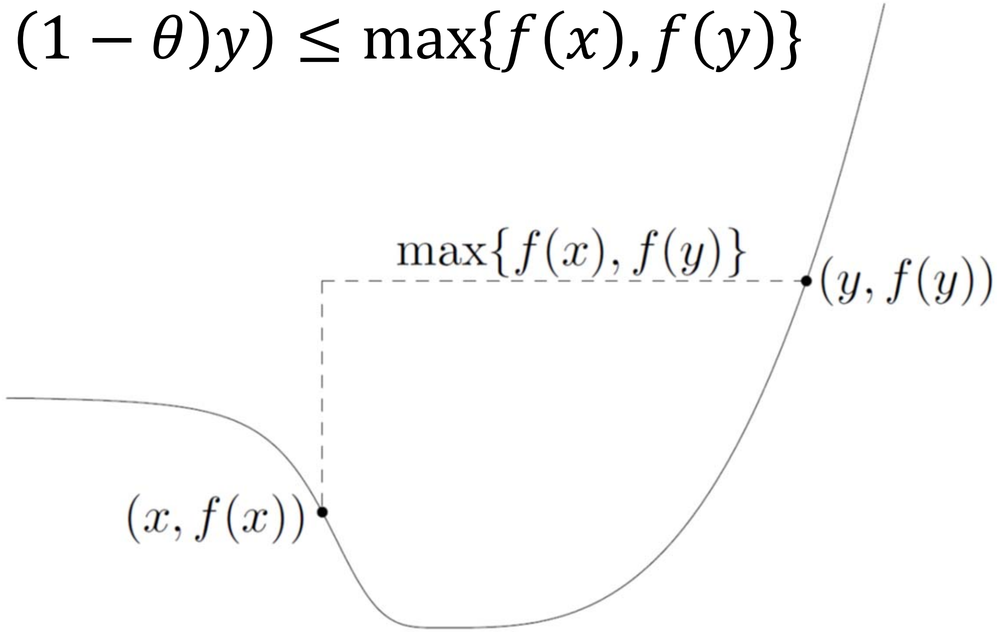
- Linear-fractional function
  - $f(x) = \frac{a^\top x+b}{c^\top x+d}, \operatorname{dom} f = \{x | c^\top x + d > 0\}$
  - $\left\{x \middle| c^\top x + d > 0, \frac{a^\top x+b}{c^\top x+d} \geq \alpha\right\}$ and $\left\{x \middle| c^\top x + d > 0, \frac{a^\top x+b}{c^\top x+d} \leq \alpha\right\}$ is convex

    $\Rightarrow f$ is Quasilinear

# Basic properties

- Jensen's inequality for quasiconvex functions
  - $f$ is quasiconvex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex and $\forall x, y \in \mathrm{dom}\, f, 0 \le \theta \le 1$

$$f(\theta x + (1-\theta)y) \le \max\{f(x), f(y)\}$$



# Basic properties

## Condition

- $f$ is quasiconvex $\Leftrightarrow$ its restriction to any line intersecting its domain is quasiconvex

## Quasiconvex functions on $\mathbf{R}$

- A continuous function $f : \mathbf{R} \to \mathbf{R}$ is quasiconvex $\Leftrightarrow$ one of the following conditions holds
  - $f$ is nondecreasing
  - $f$ is nonincreasing
  - $\exists c \in \operatorname{dom} f, \forall t \in \operatorname{dom} f, t \le c, f$ is nonincreasing, and $t \ge c, f$ is nondecreasing

# Differentiable quasiconvex functions

## First-order conditions

- $f$ is differentiable
- $f$ is quasiconvex $\Leftrightarrow \mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, f(y) \leq f(x) \Rightarrow \nabla f(x)^\top (y - x) \leq 0$
- It is possible that $\nabla f(x) = 0$, but $x$ is not a global minimizer of $f$.

## Second-order conditions

- $f$ is twice differentiable
- $\forall x \in \mathrm{dom}\, f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y > 0 \Rightarrow f$ is quasiconvex

# Operations that preserve quasiconvexity

## Nonnegative weighted maximum

- $f_i$ is quasicovex, $w_i \geq 0 \Rightarrow f = \max\{w_1 f_1, \dots, w_n f_n\}$ is quasiconvex

- $g(x, y)$ is quasiconvex in $x$ for each $y, w(y) \geq 0 \Rightarrow f(x) = \sup_{y \in C}(w(y) f(x, y))$ is quasiconvex

# Operations that preserve quasiconvexity

## Composition

- $g: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex, $h: \mathbf{R} \to \mathbf{R}$ is nondecreasing $\Rightarrow f = h \circ g$ is quasiconvex
- $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(Ax + b)$ is quasiconvex
- $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(\frac{Ax+b}{c^\top x+d})$ is quasiconvex,$\mathrm{dom}\, g = \{x | c^\top x + d > 0, (Ax + b)/(c^\top x + d) \in \mathrm{dom}\, f\}$

## Minimization

- $f(x, y)$ is quasicovex in $x$ and $y$, $C$ is a convex set $\Rightarrow g(x) = \inf_{y \in C} f(x, y)$ is quasiconvex

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# Log-concave and log-convex functions

## Definition

- $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \mathbf{dom}\, f, \log f(x)$ is concave (convex) $\Rightarrow f$ is log-concave (convex)

## Condition

- $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \mathbf{dom}\, f, f$ is log-concave $\Leftrightarrow \forall x, y \in \mathbf{dom}\, f, 0 \leq \theta \leq 1$

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta}$$

# Examples

- $f(x) = a^\top x + b, \mathrm{dom}\, f = \{x | a^\top x + b > 0\}$ is log-concave
- $f(x) = x^a, \mathrm{dom}\, f = \mathbf{R}_{++}, a \le 0 \Rightarrow f$ is log-convex, $a \ge 0 \Rightarrow f$ is log-concave
- $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$ is log-concave
- $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$ is log-convex for $x \ge 1$
- $\det X$ and $\frac{\det X}{\mathrm{tr}\, X}$ are log-concave on $\boldsymbol{S}_{++}^n$

# Properties

- Twice differentiable log convex/concave functions
  - $f$ is twice differentiable, $\mathrm{dom}\, f$ is convex
  - $\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^\top$
  - $f$ is log convex $\Leftrightarrow f(x) \nabla^2 f(x) \succcurlyeq \nabla f(x) \nabla f(x)^\top$
  - $f$ is log concave $\Leftrightarrow f(x) \nabla^2 f(x) \preccurlyeq \nabla f(x) \nabla f(x)^\top$

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# Convexity with respect to a generalized inequality

- $K$-convex
  - $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preccurlyeq_K$
  - $f: \mathbf{R}^n \to \mathbf{R}^m$ is $K$-convex if $\forall x, y \in \operatorname{dom} f, 0 \leq \theta \leq 1$

    $$f(\theta x + (1-\theta)y) \preccurlyeq_K \theta f(x) + (1-\theta) f(y)$$

  - $f: \mathbf{R}^n \to \mathbf{R}^m$ is stricly $K-$convex if $\forall x \neq y \in$ dom $f, 0 < \theta < 1$

    $$f(\theta x + (1-\theta)y) \prec_K \theta f(x) + (1-\theta) f(y)$$

# Examples

## Componentwise Inequality

- $K = \mathbf{R}_+^m$

- $f: \mathbf{R}^n \to \mathbf{R}^m$ is convex with respect to componentwise inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \le \theta \le 1,$
$f(\theta x + (1-\theta)y) \preccurlyeq \theta f(x) + (1-\theta) f(y)$

- Each $f_i$ is a convex function

# Examples

## Matrix Convexity

- $f: \mathbf{R}^n \to \mathbf{S}^m$ is convex with respect to matrix inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \leq \theta \leq 1$

  $$f(\theta x + (1-\theta)y) \preccurlyeq \theta f(x) + (1-\theta) f(y)$$

- $f(X) = XX^\top, X \in \mathbf{R}^{m\times n}$ is matrix convex
- $X^p$ is matrix convex on $\mathbf{S}^n_{++}$ for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$

# Convexity with respect to generalized inequalities

## Dual characterization of $K$-convexity

- A function $f$ is (strictly) $K$-convex $\Leftrightarrow$ For every $w \succcurlyeq_{K^*} 0$, the real-valued function $w^\top f$ is (strictly) convex in the ordinary sense.

## Differentiable $K$-convex functions

- A differentiable function $f$ is $K$-convex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f$,

$$f(y) \succcurlyeq_K f(x) + Df(x)(y - x)$$

- A differentiable function $f$ is strictly $K$-convex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, x \neq y$,

$$f(y) \succ_K f(x) + Df(x)(y - x)$$

# Convexity with respect to generalized inequalities

## Composition theorem

- $g: \mathbf{R}^n \to \mathbf{R}^p$ is $K$-convex, $h: \mathbf{R}^p \to \mathbf{R}$ is convex, and $\tilde{h}$ (the extended-value extension of $h$) is $K$-nondecreasing $\Rightarrow h \circ g$ is convex.

## Example

- $g: \mathbf{R}^{m\times n} \to \mathbf{S}^n, g(X) = X^\top AX + B^\top X + X^\top B + C$ is convex, where $A \succcurlyeq 0, B \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{S}^n$
- $h: \mathbf{S}^n \to \mathbf{R}, h(Y) = -\log\det(-Y)$ is convex and increasing on $\mathrm{dom}\, h = -\mathbf{S}^n_{++}$
- $f(X) = -\log\det(-(X^\top AX + B^\top X + X^\top B + C))$ is convex on $\mathrm{dom}\, f = \{X \in \mathbf{R}^{m\times n} | X^\top AX + B^\top X + X^\top B + C \prec 0\}$

# Monotonicity with respect to a generalized inequality

- $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preccurlyeq_K$
  - $f: \mathbf{R}^n \to \mathbf{R}$ is $K$-nondecreasing if

$$x \preccurlyeq_K y \Rightarrow f(x) \leq f(y)$$

  - $f: \mathbf{R}^n \to \mathbf{R}$ is $K$-increasing if

$$x \preccurlyeq_K y, x \neq y \Rightarrow f(x) < f(y)$$

# Summary

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