Duality (I)

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Outline

- The Lagrange Dual Function
  - The Lagrange Dual Function
  - Lower Bound on Optimal Value
  - The Lagrange Dual Function and Conjugate Functions
- The Lagrange Dual Problem
  - Making Dual Constraints Explicit
  - Weak Duality
  - Strong Duality and Slater’s Constraint Qualification

Optimization Problems

Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \dots, m \qquad (1) \\ & h_i(x) = 0 \qquad i = 1, \dots, p \end{aligned}$$

- Domain is nonempty

$$\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- Denote the optimal value by p^*
- We do not assume the problem is convex

The Lagrangian

- The Lagrangian $L: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  - $\operatorname{dom} L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$
  - λ_i: the Lagrange multiplier associated with the i-th inequality constraint $f_i(x) \leq 0$
  - ν_i: the Lagrange multiplier associated with the i-th equality constraint $h_i(x) = 0$
  - Vectors λ and ν: dual variables or Lagrange multiplier vectors

The Lagrange Dual Function

- $g: \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

$$= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

  - When L is unbounded below in x, $g = -\infty$
  - g is concave
    - ✓ g is the pointwise infimum of a family of affine functions of (λ, ν)
  - It is unconstrained

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Lower Bounds on P^*

- For any $\lambda \succcurlyeq 0$ and any ν

$$g(\lambda, \nu) \leq p^*$$

- Proof
  - $\tilde{x}$ is a feasible point for original problem

$$f_i(\tilde{x}) \leq 0, h_i(\tilde{x}) = 0$$

  - Since $\lambda \succcurlyeq 0$

$$\sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq 0$$

  - Therefore

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$

Lower Bounds on P^*

- For any $\lambda \succcurlyeq 0$ and any ν

$$g(\lambda, \nu) \leq p^*$$

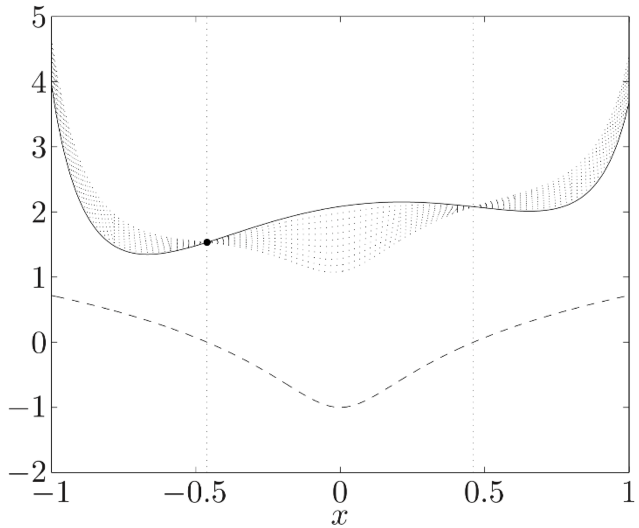
- Proof
  - Hence

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \leq L(\tilde{x}, \lambda, \nu) \leq f_0(\tilde{x})$$

  - Note that $g(\lambda, \nu) \leq f_0(\tilde{x})$ for any feasible $\tilde{x}$
- Discussions
  - The lower bound is vacuous, when $g(\lambda, \nu) = -\infty$
  - It is nontrivial only when $\lambda \succcurlyeq 0, (\lambda, \nu) \in \mathrm{dom}\, g$
  - Dual feasible: (λ, ν) with $\lambda \succcurlyeq 0, (\lambda, \nu) \in \mathrm{dom}\, g$

Example

- A Simple Problem with $x \in \mathbb{R}, m = 1, p = 0$
  - Lower bound from a dual feasible point

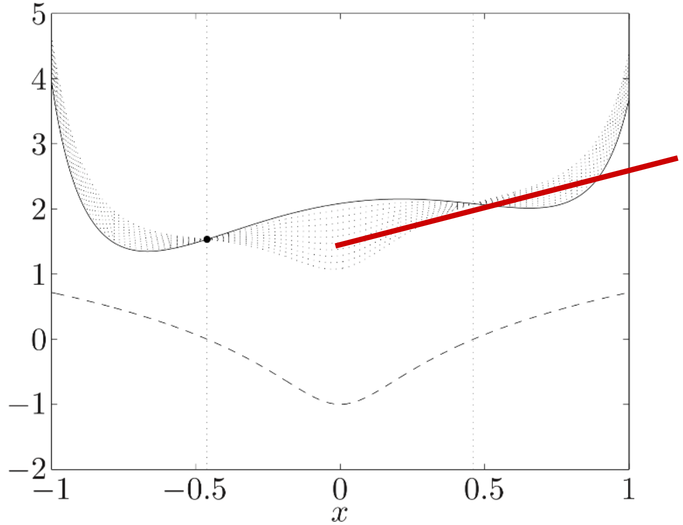


- ✓ Solid curve: objective function f_0
- ✓ Dashed curve: constraint function f_1
- ✓ Feasible set: $[-0.46, 0.46]$ (indicated by the two dotted vertical lines)

Example

A Simple Problem with $x \in \mathbb{R}, m = 1, p = 0$

- Lower bound from a dual feasible point

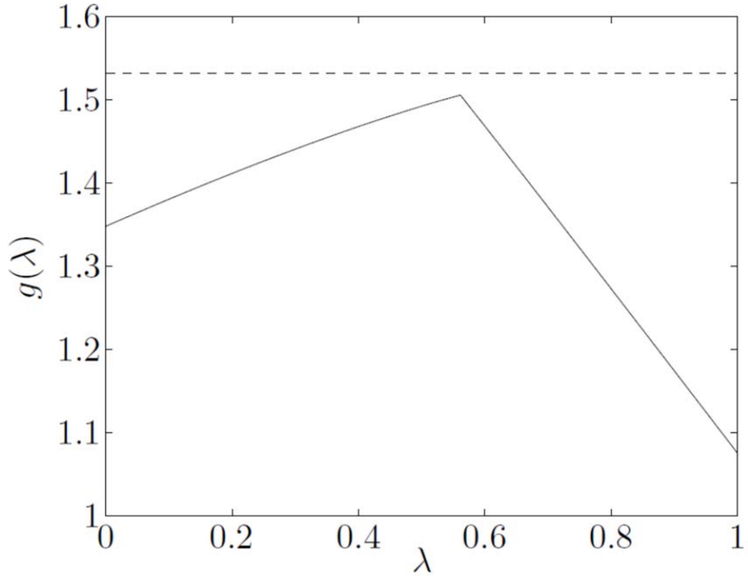


$$g(\lambda) = \inf_{x \in \mathcal{D}} L(x, \lambda)$$
$$\le L(x, \lambda) \le f_0(x)$$

- ✓ Optimal point and value: $x^* = -0.46, p^* = 1.54$
- ✓ Dotted curves: $L(x, \lambda)$ for $\lambda = 0.1, 0.2, \ldots, 1.0$.
  - Each has a minimum value smaller than p^* as on the feasible set (and for $\lambda > 0$), $L(x, \lambda) \le f_0(x)$

Example

□ The dual function g



- Neither f_0 nor f_1 is convex, but the dual function g is concave
- Horizontal dashed line: p^* (the optimal value of the problem)

Linear Approximation Interpretation

- Rewrite (1) as unconstrained problem

$$\min \; f_0(x) + \sum_{i=1}^{m} I_-\big(f_i(x)\big) + \sum_{i=1}^{p} I_0\big(h_i(x)\big) \quad (2)$$

  - $I_-: \mathbb{R} \mapsto \mathbb{R}$ is the indicator function for the nonpositive reals

$$I_-(u) = \begin{cases} 0 & u \leq 0, \\ \infty & u > 0. \end{cases}$$

  - I_0 is the indicator function of $\{0\}$

Linear Approximation Interpretation

- In the formulation (2)
  - $I\_(u)$ expresses our irritation or displeasure associated with a constraint function value $u = f_i(x)$: zero if $f_i(x) \leq 0$, infinite if $f_i(x) > 0$
  - $I_0(u)$ gives our displeasure for an equality constraint value $u = h_i(x)$
  - Our displeasure rises from zero to infinite as $f_i(x)$ transitions from nonpositive to positive

Linear Approximation Interpretation

- In the formulation (2)
  - Suppose we replace $I_-(u)$ with linear function $\lambda_i u$, where $\lambda_i \geq 0$, and $I_0(u)$ with $\nu_i u$
  - Objective becomes the Lagrangian $L(x, \lambda, \nu)$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  - Dual function value $g(\lambda, \nu)$ is optimal value of

$$\min \ f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \quad (3)$$

Linear Approximation Interpretation

- In the formulation (3)
  - We replace I_- and I_0 with linear or "soft" displeasure functions
  - For an inequality constraint, our displeasure is zero when $f_i(x) = 0$, and is positive when $f_i(x) > 0$ (assuming $\lambda_i > 0$)
  - In (2), any nonpositive value of $f_i(x)$ is acceptable
  - In (3), we actually derive pleasure from constraints that have margin, i.e., from $f_i(x) < 0$

Linear Approximation Interpretation

Interpretation of Lower Bound

- The linear function is an underestimator of the indicator function

$$\lambda_i u \leq I_-(u)$$
$$\nu_i u \leq I_0(u)$$

- Lower Bound Property

$$f_0(x) + \sum_{i=1}^{m} I_-\big(f_i(x)\big) + \sum_{i=1}^{p} I_0\big(h_i(x)\big) \geq$$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Example

- Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s.t.} & Ax = b \end{array}$$

  - $A \in \mathbf{R}^{p\times n}$
  - No inequality constraints
  - p (linear) equality constraints

- Lagrangian

$$L(x, \nu) = x^\top x + \nu^\top (Ax - b)$$

  - Domain: $\mathbf{R}^n \times \mathbf{R}^p$

Example

- Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s.t.} & Ax = b \end{array}$$

- Dual Function

$$g(\nu) = \inf_x L(x,\nu) = \inf_x x^\top x + \nu^\top (Ax - b)$$

  - Optimality condition

$$\nabla_x L(x,\nu) = 2x + A^\top \nu = 0 \Rightarrow x = -(1/2)A^\top \nu$$

Example

- Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- Dual Function

$$\Rightarrow g(\nu) = L(-(1/2)A^\top \nu, \nu) = -(1/4)\nu^\top A A^\top \nu - b^\top \nu$$

  - Concave Function

- Lower Bound Property

$$-(1/4)\nu^\top A A^\top \nu - b^\top \nu \le \inf \{x^\top x \mid Ax = b\}$$

Example

Standard Form LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

- Inequality constraints: $f_i(x) = -x_i, i = 1, \dots, n$

Lagrangian

$$\begin{aligned} L(x, \lambda, \nu) &= c^\top x - \sum_{i=1}^{n} \lambda_i x_i + \nu^\top (Ax - b) \\ &= -b^\top \nu + (c + A^\top \nu - \lambda)^\top x \end{aligned}$$

Dual Function

$$\begin{aligned} g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\ &= -b^\top \nu + \inf_x (c + A^\top \nu - \lambda)^\top x \end{aligned}$$

Example

Standard Form LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \succcurlyeq 0 \end{aligned}$$

- Inequality constraints: $f_i(x) = -x_i, i = 1, \dots, n$

Dual Function

$$g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

- The lower bound is nontrivial only when λ and ν satisfy $\lambda \succcurlyeq 0$ and $A^\top \nu - \lambda + c = 0$

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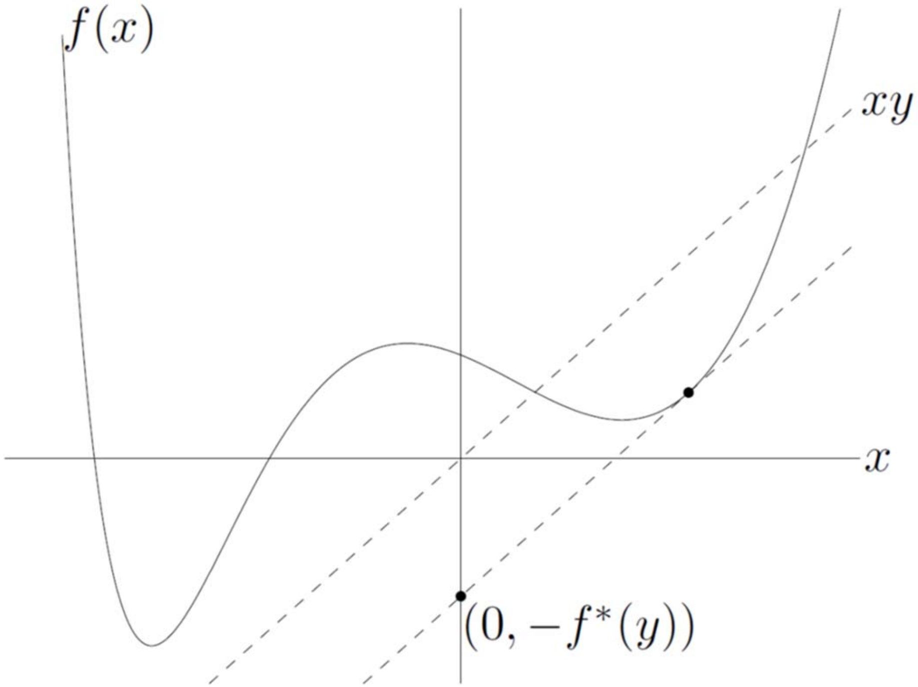
Conjugate Function

- $f: \mathbf{R}^n \to \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^\top x - f(x) \right)$$

  - $\mathrm{dom}\, f^* = \{y \mid f^*(y) < \infty\}$
  - f^* is always convex



The Lagrange Dual Function and Conjugate Functions

A Simple Example

$$\begin{aligned} \min \quad & f(x) \\ \mathrm{s.t.} \quad & x = 0 \end{aligned}$$

Lagrangian

$$L(x, \nu) = f(x) + \nu^\top x$$

Dual Function

$$\begin{aligned} g(\nu) &= \inf_x \left(f(x) + \nu^\top x\right) \\ &= -\sup_x \big((-\nu)^\top x - f(x)\big) = -f^*(-\nu) \end{aligned}$$

The Lagrange Dual Function and Conjugate Functions

A More General Example

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax \preccurlyeq b \\ & Cx = d \end{aligned}$$

Lagrangian

$$L(x, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

Dual Function

$$\begin{aligned} g(\nu) &= \inf_x \left(f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d) \right) \\ &= -b^\top \lambda - d^\top \nu + \inf_x \left(f_0(x) + (A^\top \lambda + C^\top \nu)^\top x \right) \\ &= -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu) \end{aligned}$$

The Lagrange Dual Function and Conjugate Functions

□ A More General Example

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax \preccurlyeq b \\ & Cx = d \end{aligned}$$

□ Lagrangian

$$L(x, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

□ Dual Function

$$g(\nu) = -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu)$$

- $\mathrm{dom}\, g = \{(\lambda, \nu) \mid -A^\top \lambda - C^\top \nu \in \mathrm{dom}\, f_0^*\}$

Example

- Equality Constrained Norm Minimization

$$\begin{array}{ll} \min & \|x\| \\ \text{s.t.} & Ax = b \end{array}$$

- Conjugate of $f_0 = \|\cdot\|$

$$f_0^*(y) = \begin{cases} 0 & \|y\|_* \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

- The Dual Function

$$g(\nu) = -b^\top \nu - f_0^*(-A^\top \nu) = \begin{cases} -b^\top \nu & \|A^\top \nu\|_* \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Example

□ Entropy Maximization

$$\begin{aligned} \min \quad & f_0(x) = \sum_{i=1}^{n} x_i \log x_i \\ \text{s.t.} \quad & Ax \preccurlyeq b \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

□ Conjugate of f_0

$$f_0^*(y) = \sum_{i=1}^{n} e^{y_i - 1}$$

□ The Dual Function

$$\begin{aligned} g(\lambda, \nu) &= -b^\top \lambda - \nu - f_0^*(-A^\top \lambda - \nu \mathbf{1}) \\ &= -b^\top \lambda - \nu - \textstyle\sum_{i=1}^{n} e^{-a_i^\top \lambda - \nu - 1} \\ &= -b^\top \lambda - \nu - e^{-\nu - 1} \textstyle\sum_{i=1}^{n} e^{-a_i^\top \lambda} \end{aligned}$$

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The Lagrange Dual Problem

- For any $\lambda \succcurlyeq 0$ and any ν

$$g(\lambda, \nu) \leq p^*$$

  - What is the best lower bound?

- Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, \nu) \\ \text{s.t.} & \lambda \succcurlyeq 0 \end{array}$$

- Primal Problem

$$\begin{array}{lll} \min & f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0 & i = 1, \ldots, p \end{array} \quad (1)$$

The Lagrange Dual Problem

- For any $\lambda \succcurlyeq 0$ and any ν

$$g(\lambda, \nu) \leq p^*$$

  - What is the best lower bound?

- Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, \nu) \\ \text{s.t.} & \lambda \succcurlyeq 0 \end{array}$$

  - Dual feasible: (λ, ν) with $\lambda \succcurlyeq 0, g(\lambda, \nu) > -\infty$
  - Dual optimal or optimal Lagrange multipliers: (λ^*, ν^*)
  - A convex optimization problem

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Making Dual Constraints Explicit

- Motivation
  - The $\operatorname{dom} g = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}$ may have dimension $\leq m + p$
  - Identify the equality constraints that are 'hidden' or 'implicit' in g
- Standard Form LP

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \succcurlyeq 0 \end{array}$$

- Dual Function

$$g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Example

Lagrange Dual of Standard Form LP

- Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda,\nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \succcurlyeq 0 \end{aligned}$$

- An Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \nu \\ \text{s.t.} \quad & A^\top \nu - \lambda + c = 0 \\ & \lambda \succcurlyeq 0 \end{aligned}$$

✓ Make equality constraints explicit

Example

Lagrange Dual of Standard Form LP

- Lagrange Dual Problem

$$
\begin{aligned}
\max \quad & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\
\text{s.t.} \quad & \lambda \succcurlyeq 0
\end{aligned}
$$

- Another Equivalent Problem

$$
\begin{aligned}
\max \quad & -b^\top \nu \\
\text{s.t.} \quad & A^\top \nu + c \succcurlyeq 0
\end{aligned}
$$

  - ✓ An LP in inequality form

Standard Form LP → (Lagrange Dual) → Inequality Form LP

Example

Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \preccurlyeq b \end{aligned}$$

- Lagrangian

$$L(x,\lambda) = c^\top x + \lambda^\top (Ax - b) = -b^\top \lambda + (A^\top \lambda + c)^\top x$$

- Lagrange dual function

$$g(\lambda) = \inf_x L(x,\lambda) = -b^\top \lambda + \inf_x (A^\top \lambda + c)^\top x$$

$$= \begin{cases} -b^\top \lambda & A^\top \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Example

Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \preccurlyeq b \end{aligned}$$

- Lagrange Dual Problem

$$\max \quad g(\lambda,\nu) = \begin{cases} -b^\top\lambda & A^\top\lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

$$\text{s.t.} \quad \lambda \succcurlyeq 0$$

- An Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top\lambda \\ \text{s.t.} \quad & A^\top\lambda + c = 0 \\ & \lambda \succcurlyeq 0 \end{aligned}$$

Example

Lagrange Dual of Inequality Form LP

- Inequality form LP (Primal Problem)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \preccurlyeq b \end{aligned}$$

- An Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda + c = 0 \\ & \lambda \succcurlyeq 0 \end{aligned}$$

  - ✓ An LP in standard form

Inequality Form LP $\xrightarrow{\text{Lagrange Dual}}$ Standard Form LP

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Weak Duality

- For any $\lambda \succcurlyeq 0$ and any ν

$$g(\lambda, \nu) \leq p^*$$

  - What is the best lower bound?
- Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, \nu) \\ \text{s.t.} \quad & \lambda \succcurlyeq 0 \end{aligned}$$

  - Optimal value d^*
- Weak Duality

$$d^* \leq p^*$$

Does not rely
on convexity!

Weak Duality

- Weak Duality

$$d^* \leq p^*$$

  - If the primal problem is unbounded below, i.e., $p^* = -\infty$, we must have $d^* = -\infty$, i.e., the Lagrange dual problem is infeasible
  - If $d^* = \infty$, we must have $p^* = \infty$

- Optimal duality gap

$$p^* - d^*$$

  - Nonegative

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Strong Duality

Strong Duality

$$d^* = p^*$$

- The optimal duality gap is zero
- The best bound that can be obtained from the Lagrange dual function is tight
- In general, does not hold

Usually hold for convex optimization

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- $f_0, \ldots, f_m$ are convex

Slater's Constraint Qualification

- Constraint Qualifications
  - Sufficient conditions for strong duality
- Slater's condition
  - $\exists x \in \operatorname{relint} \mathcal{D}$ such that

$$f_i(x) < 0, \qquad i = 1, \ldots, m, \qquad Ax = b$$

  - Such a point x is called strictly feasible
- If Slater's condition holds and the problem is convex
  - Strong duality holds
  - Dual optimal value is attained when $d^* > -\infty$

Slater's Constraint Qualification

- Constraint Qualifications
  - Sufficient conditions for strong duality
- Slater's condition (weaker form)
  - If the first k constraint functions are affine
  - $\exists x \in \operatorname{relint} \mathcal{D}$ such that

$$
\begin{aligned}
&f_i(x) \leq 0, \quad i = 1, \ldots, k \\
&f_i(x) < 0, \quad i = k+1, \ldots, m \\
&Ax = b
\end{aligned}
$$

  - When constraints are all linear equalities and inequalities, and $\operatorname{dom} f_0$ is open
    - ✓ Reduce to feasibility

Example

- Least-squares Solution of Linear Equations

$$\begin{aligned} \min \quad & x^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- Dual Problem

$$\max \; -(1/4)\nu^\top A A^\top \nu - b^\top \nu$$

- Slater's condition
  - The primal problem is feasible, i.e., $b \in \mathcal{R}(A)$
- Strong duality always holds
  - Even when $b \notin \mathcal{R}(A)$

Example

- Lagrange dual of LP

  Standard Form LP ⟺ (Lagrange Dual) ⟺ Inequality Form LP

- Strong duality holds for any LP
  - If the primal problem is feasible or the dual problem is feasible

- Strong duality may fail
  - If both the primal and dual problems are infeasible

Example

QCQP (Primal Problem)

$$\begin{aligned} \min \quad & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{s.t.} \quad & (1/2)x^\top P_i x + q_i^\top x + r_i \leq 0, \qquad i = 1, \dots, m \end{aligned}$$

- $P_0 \in \mathbf{S}_{++}^n$ and $P_i \in \mathbf{S}_+^n, i = 1, \dots, m$

Dual Problem

$$\begin{aligned} \max \quad & -(1/2)q(\lambda)^\top P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{s.t.} \quad & \lambda \succcurlyeq 0 \end{aligned}$$

- $P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i$
- $r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i$

Slater's condition

- $\exists x\,,\ (1/2)x^\top P_i x + q_i^\top x + r_i < 0, i = 1, \dots, m$

Example

A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \le 1 \end{aligned}$$

- $A \in \mathbf{S}^n, A \nsucceq 0$ and $b \in \mathbf{R}^n$
- Lagrangian

$$\begin{aligned} L(x,\lambda) &= x^\top A x + 2b^\top x + \lambda(x^\top x - 1) \\ &= x^\top (A + \lambda I)x + 2b^\top x - \lambda \end{aligned}$$

- Dual Function

$$g(\lambda) = \begin{cases} -b^\top (A+\lambda I)^\dagger b - \lambda & A + \lambda I \succcurlyeq 0, b \in \mathcal{R}(A+\lambda I) \\ -\infty & \text{otherwise} \end{cases}$$

Example

- A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{array}{ll} \min & x^\top A x + 2b^\top x \\ \text{s.t.} & x^\top x \leq 1 \end{array}$$

  - $A \in \mathbf{S}^n, A \nsucceq 0$ and $b \in \mathbf{R}^n$

- Dual Problem

$$\begin{array}{ll} \max & -b^\top (A + \lambda I)^\dagger b - \lambda \\ \text{s.t.} & A + \lambda I \succeq 0, b \in \mathcal{R}(A + \lambda I) \end{array}$$

  - A convex optimization problem

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$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \le 1 \end{aligned}$$

  - $A \in \mathbf{S}^n, A \nsucceq 0$ and $b \in \mathbf{R}^n$

Strong duality holds

- Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n \left(q_i^\top b\right)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s.t.} \quad & \lambda \ge -\lambda_{\min}(A) \end{aligned}$$

  - A convex optimization problem
  - λ_i and q_i: eigenvalues and corresponding (orthonormal) eigenvectors of A

Example

- A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \le 1 \end{aligned}$$

  - $A \in \mathbb{S}^n_+, A \nsucceq 0$ and $b \in \mathbb{R}^n$

Strong duality holds for any optimization problem with quadratic objective and one quadratic inequality constraint, provided Slater's condition holds

- Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n \left(q_i^\top b\right)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s.t.} \quad & \lambda \ge -\lambda_{\min}(A) \end{aligned}$$

  - A convex optimization problem
  - λ_i and q_i: eigenvalues and corresponding (orthonormal) eigenvectors of A

Summary

- The Lagrange Dual Function
  - The Lagrange Dual Function
  - Lower Bound on Optimal Value
  - The Lagrange Dual Function and Conjugate Functions
- The Lagrange Dual Problem
  - Making Dual Constraints Explicit
  - Weak Duality
  - Strong Duality and Slater's Constraint Qualification