Introduction to Optimization Methods **Fall 2024**

Homework 1

Instructor: Lijun Zhang *Name:* Student name, *StudentId:* Student id

Notice

- The submission email is: **optfall24@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Inequalities.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\cdot\|$ denote the Euclidean norm.

a) Prove the Schwarz's inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

b) Prove the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Notice: You cannot use $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta$ to complete the proof in problem a).

Problem 2: Convex sets

a) Show that a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m\}$ is convex.

b) Show that if $S \subseteq \mathbb{R}^n$ is convex, and $A \in \mathbb{R}^{m\times n}$, then $A(S) = \{Ax : x \in S\}$, is convex.

c) Show that if $S \subseteq \mathbb{R}^m$ is convex, and $A \in \mathbb{R}^{m\times n}$, then $A^{-1}(S) = \{x : Ax \in S\}$, is convex.

Problem 3: Hyperplane

What is the distance between two parallel hyperplanes, i.e.,$\{x|a^\top x = b\}$ and $\{x|a^\top x = c\}$?

Problem 4: Examples

Let $S \subseteq \mathbb{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

a) For $a \geq 0$ we define S_a as $\{x| \operatorname{dist}(x, S) \leq a\}$, where $\operatorname{dist}(x, S) = \inf_{y\in S} \|x - y\|$. We refer to S_a as S expanded or extended by a. Show that if S is convex, then S_a is convex.

b) For $a \geq 0$ we define S_{-a} as $\{x|B(x, a) \subseteq S\}$, where $B(x, a)$ is the ball (in the norm $\|\cdot\|$), with radius a. We refer to S_{-a} as S shrunk or restricted by a, since S_{-a} consists of all points that are at least a distance a from $\mathbb{R}^n\backslash S$. Show that if S is convex, then S_{-a} is convex.

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following,

a) K^* is indeed a convex cone.

b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.