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| **Introduction to Optimization Methods** | **Fall 2024** |

# Homework 2

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**Notice**

- The submission email is: **optfall24@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

**Problem 1: Convex functions**

a) Prove that a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $x, y \in \mathbb{R}^n$,

$$\int_0^1 f(x + \lambda(y - x))d\lambda \leq \frac{f(x) + f(y)}{2}.$$

b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, with $\mathbb{R}_+ \subseteq dom(f)$. Prove that its running average $F$, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \qquad dom(F) = \mathbb{R}_{++},$$

is convex.

**Problem 2: Concave function**

Suppose $p < 1, p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$$

with $dom(f) = \mathbb{R}_{++}$ is concave.

**Problem 3: Convexity**

Let $\psi : \Omega \mapsto \mathbb{R}$ be a strictly convex and continuously differentiable function. We define

$$\Delta_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla\psi(y), x - y\rangle, \quad \forall x, y \in \Omega.$$

a) Prove that $\Delta_\psi(x, y) \geqslant 0$, $\forall x, y \in \Omega$ and the equality holds only when $x = y$.

b) Let $L$ be a convex and differentiable function defined on $\Omega$ and $C \subset \Omega$ be a convex set. Let $x_0 \in \Omega - C$ and define

$$x^* = \underset{x \in C}{\arg\min}\, L(x) + \Delta_\psi(x, x_0).$$

Prove that for any $y \in C$,

$$L(y) + \Delta_\psi(y, x_0) \geqslant L(x^*) + \Delta_\psi(x^*, x_0) + \Delta_\psi(y, x^*).$$

**Problem 4: Projection**

For any point $y$, the projection onto a nonempty and closed convex set $X$ is defined as

$$\Pi_X(y) = \arg\min_{x \in X} \frac{1}{2}\|x - y\|_2^2.$$

a) Prove that $\|\Pi_X(x) - \Pi_X(y)\|_2^2 \leqslant \langle \Pi_X(x) - \Pi_X(y), x - y \rangle$.

b) Prove that $\|\Pi_X(x) - \Pi_X(y)\|_2 \leqslant \|x - y\|_2$.

**Problem 5: Conjugate Function**

Derive the conjugates of the following functions.

a) $f(x) = \max\{0, 1 - x\}$.

b) $f(x) = \ln(1 + e^{-x})$.

c) $f(x) = x^p$ over $\mathbb{R}_{++}$ where $p > 1$.