Introduction to Optimization Methods **Fall 2024**

Homework 3

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Notice

- The submission email is: **optfall24@163.com**
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Linear optimization problem

Consider minimizing a linear function over an affine set.

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b. \end{aligned}$$

Provide the specific form of the optimal value $p_\star$.

Problem 2: Strong duality and Slater's condition

Consider the optimization problem

$$\begin{aligned} \min \quad & e^{-x} \\ \text{s.t.} \quad & x^2/y \leq 0 \end{aligned}$$

with variables x and y, and domain $\mathcal{D} = \{(x, y) | y > 0\}$.

(1) Verify that this is a convex optimization problem and find the optimal value.

(2) Give the Lagrange dual problem. Find the optimal solution λ^* and optimal value d^* of the dual problem. What is the optimal duality gap?

(3) Does Slater's condition hold for this problem?

Problem 3: KKT conditions

Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \end{aligned}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$.

(1) Write the Lagrangian for this problem.

(2) Does strong duality hold in this problem?

(3) Write the KKT conditions for this optimization problem.

Problem 4: Equality Constrained Least-squares

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|Ax-b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m\times n}$ with $\textbf{rank}\ A = n$, and $G \in \mathbf{R}^{p\times n}$ with $\textbf{rank}\ G = p$.

(1) Derive the Lagrange dual problem with Lagrange multiplier vector v.

(2) Derive expressions for the primal solution $x^\star$ and the dual solution $v^\star$.

Problem 5: Negative-entropy Regularization

Please show how to compute

$$\underset{x\in\Delta^n}{\operatorname{argmin}} \quad b^\top x + c\cdot\sum_{i=1}^{n} x_i \ln x_i$$

where $\Delta^n = \{x | \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \cdots, n\}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Problem 6: Support Vector Machines
Consider the following optimization problem

$$\text{minimize} \qquad \textstyle\sum_{i=1}^n \max\left(0, 1 - y_i(w^T x_i + b)\right) + \frac{\lambda}{2}\|w\|_2^2$$

where $x_i \in \mathbf{R}^d, y_i \in \mathbf{R}, i = 1, \cdots, n$ are given, and $w \in \mathbf{R}^d, b \in \mathbf{R}$ are the variables.

(1) Derive an equivalent problem by introducing new variables $u_i, i = 1, \cdots, n$ and equality constraints

$$u_i = y_i(w^T x_i + b), i = 1, \cdots, n.$$

(2) Derive the Lagrange dual problem of the above equivalent problem.

(3) Give the Karush-Kuhn-Tucker conditions.

Hint: Let $\ell(x) = \max(0, 1-x)$. *Its conjugate function* $\ell^*(y) = \sup_x(yx - \ell(x)) = \begin{cases} y, & -1 \leq y \leq 0 \\ \infty, & \textit{otherwise} \end{cases}$