Convex Functions (II)

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Outline

- The Conjugate Function
- Quasiconvex Functions
- Log-concave and Log-convex Functions
- Convexity with Respect to Generalized Inequalities
- Summary

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Conjugate Function

- $f: \mathbf{R}^n \to \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^\top x - f(x) \right)$$

  - $\mathrm{dom}\, f^* = \{y | f^*(y) < \infty\}$
  - f^* is always convex

Conjugate Function

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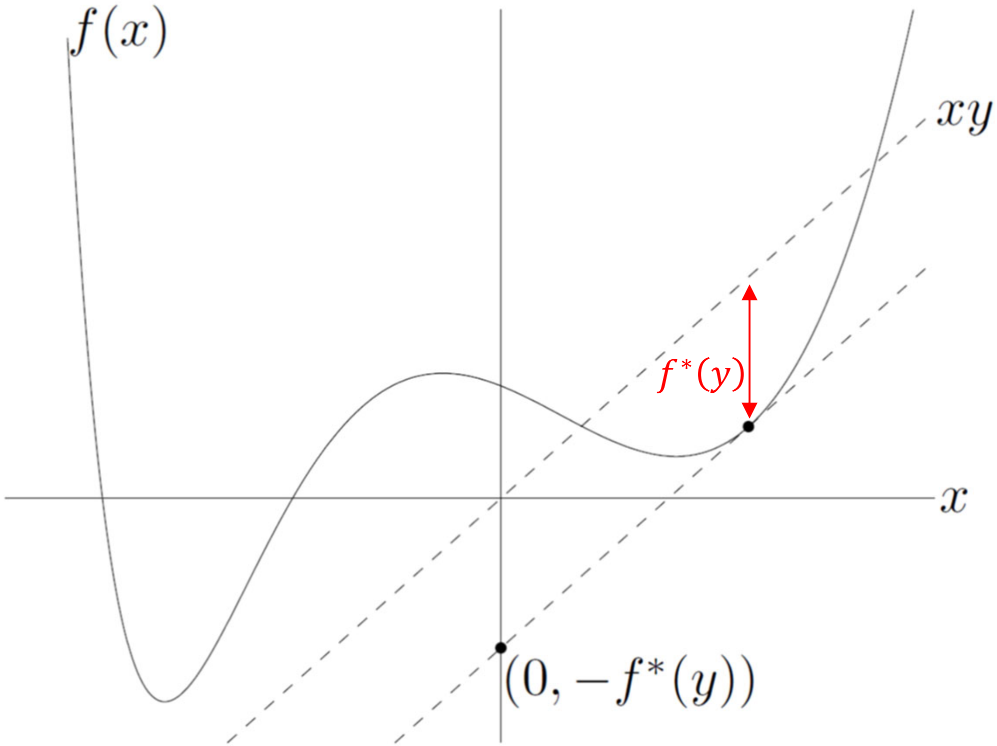

Conjugate examples

- Affine function
  - $f(x) = ax + b$
  - $f^*(y) = \sup_{x \in \mathbf{R}} (yx - ax - b)$
  - $\mathrm{dom}\, f^* = \{a\}, f^*(a) = -b$

- Negative logarithm
  - $f(x) = -\log x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (yx + \log x)$
  - $\mathrm{dom}\, f^* = -\mathbf{R}_{++}, f^*(y) = -\log(-y) - 1$

Conjugate examples

- Exponential
  - $f(x) = e^x$
  - $f^*(y) = \sup_{x \in \mathbf{R}} (xy - e^x)$
  - $\operatorname{dom} f^* = \mathbf{R}_+, f^*(y) = y \log y - y$

- Negative entropy
  - $f(x) = x \log x$
  - $f^*(y) = \sup_{x \in \mathbf{R}_+} (yx - x \log x)$
  - $\operatorname{dom} f^* = \mathbf{R}, f^*(y) = e^{y-1}$

Conjugate examples

- Inverse
  - $f(x) = 1/x, x \in \mathbf{R}_{++}$
  - $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - 1/x)$
  - $\operatorname{dom} f^* = -\mathbf{R}_+, f^*(y) = -2(-y)^{1/2}$

- Strictly convex quadratic function
  - $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (y^\top x - \frac{1}{2} x^\top Q x)$
  - $\operatorname{dom} f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2} y^\top Q^{-1} y$

Conjugate examples

- Log-determinant
  - $f(X) = \log\det X^{-1}, X \in \mathbf{S}^n_{++}$
  - $f^*(Y) = \sup_{X \in \mathbf{S}^n_{++}} (\mathrm{tr}(XY) + \log\det X)$
  - $\mathrm{dom}\, f^* = -\mathbf{S}^n_{++}, f^*(Y) = \log\det(-Y)^{-1} - n$

- Indicator function
  - $I_S(x) = 0, \mathrm{dom}\, I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
  - $I_S^*(y) = \sup_{x \in S} y^\top x$
  - $I_S^*(y)$ is the support function of the set S

Conjugate examples

- Support function of a set
  - $C \subseteq \mathbf{R}^n, C \neq \emptyset$
  - $S_C(x) = \sup\{x^\top y | y \in C\}$
  - $\operatorname{dom} S_C = \{x | \sup_{y \in C} x^\top y < \infty\}$

- Indicator function
  - $I_S(x) = 0, \operatorname{dom} I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
  - $I_S^*(y) = \sup_{x \in S} y^\top x$
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Conjugate examples

- Norm
  - $f(x) = \|x\|, x \in \mathbf{R}^n$, with dual norm $\|\cdot\|_*$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \|x\|)$
  - $\mathrm{dom}\, f^* = \{y \mid \|y\|_* \le 1\}, f^*(y) = 0$

- Norm squared
  - $f(x) = \frac{1}{2}\|x\|^2, x \in \mathbf{R}^n$, with dual norm $\|\cdot\|_*$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \frac{1}{2}\|x\|^2)$
  - $\mathrm{dom}\, f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2}\|y\|_*^2$

Basic properties

- Fenchel's inequality
  - $\forall x \in \operatorname{dom} f, y \in \operatorname{dom} f^*,\ f(x) + f^*(y) \geq x^\top y$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$
  - $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
    $\Rightarrow x^\top y \leq \frac{1}{2} x^\top Q x + \frac{1}{2} y^\top Q^{-1} y$

- Conjugate of the conjugate
  - f is convex and closed $\Rightarrow f^{**} = f$

Basic properties

- Differentiable functions
  - f is convex and differentiable, $\mathrm{dom}\, f = \mathbf{R}^n$
  - $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$
  - $x^* = \mathrm{argmax}\big(x^\top y - f(x)\big) \Rightarrow \nabla f(x^*) = y$
  - $f^*(y) = x^{*\top} \nabla f(x^*) - f(x^*) = x^{*\top} y - f(x^*)$
    - ✓ $x^* = \nabla^{-1} f(y)$

Basic properties

- Scaling, affine transformation
  - $a > 0, b \in \mathbf{R}, g(x) = af(x) + b$
    $\Rightarrow g^*(y) = af^*\left(\frac{y}{a}\right) - b$
  - $A \in \mathbf{R}^{n\times n}$ is nonsingular, $b \in \mathbf{R}^n, g(x) = f(Ax + b) \Rightarrow g^*(y) = f^*(A^{-\top}y) - b^\top A^{-\top}y, \mathrm{dom}\, g^* = A^\top \mathrm{dom}\, f^*$

- Sums of independent functions
  - $f(u, v) = f_1(u) + f_2(v), f_1, f_2$ are convex $\Rightarrow$
    $f^*(w, z) = f_1^*(w) + f_2^*(z)$

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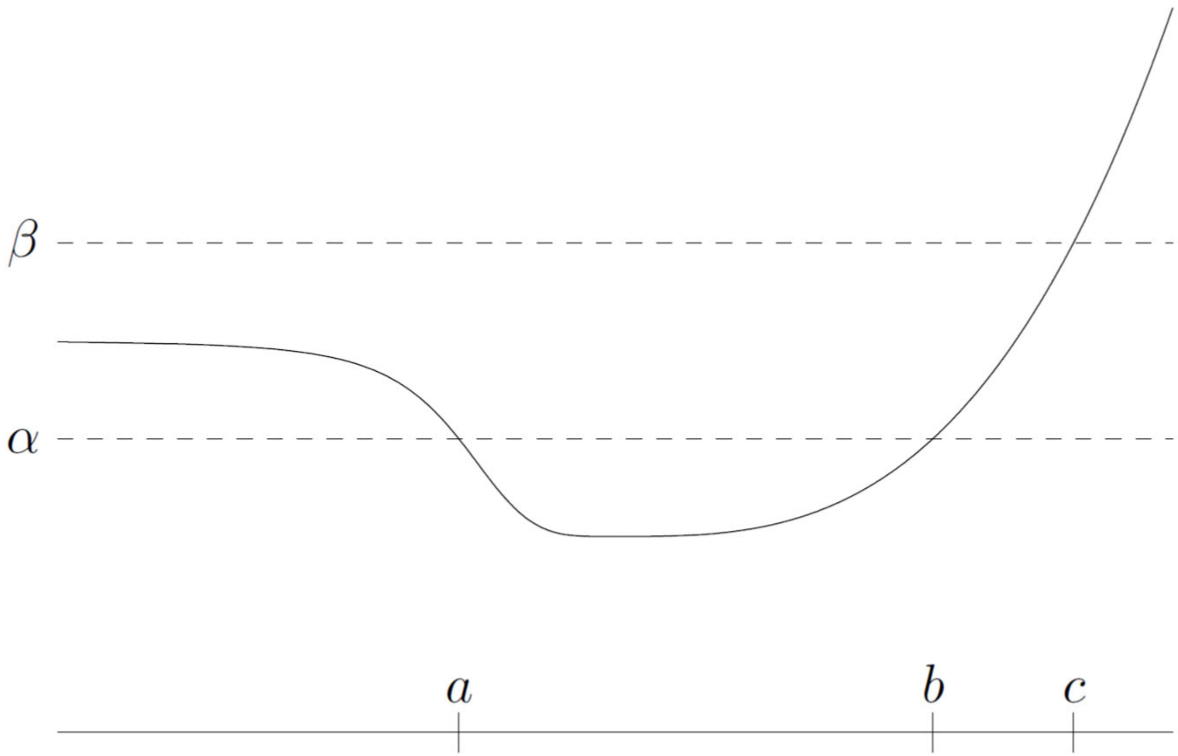
Quasiconvex functions

- Quasiconvex
  - $f: \mathbf{R}^n \to \mathbf{R}$
  - $S_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \leq \alpha\}, \forall \alpha \in \mathbf{R}$ is convex



Quasiconvex functions

- Quasiconvex
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- Quasiconcave
  - $-f$ is quasiconvex $\Rightarrow f$ is quasiconcave

- Quasilinear
  - f is quasiconvex and quasiconcave $\Rightarrow f$ is quasilinear

Examples

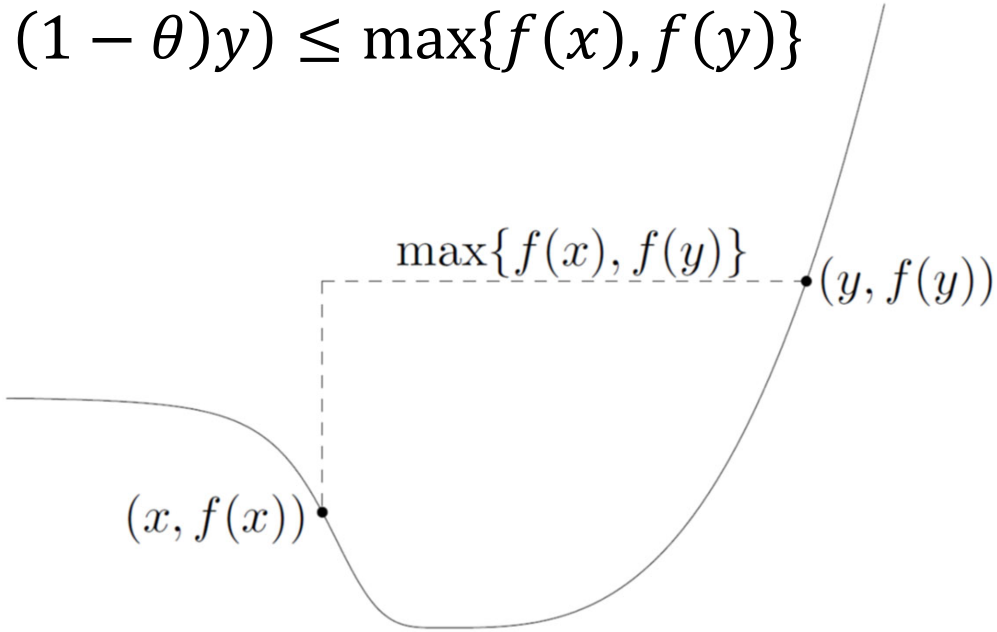
- Some example on $\mathbf{R}$
  - Logarithm: $\log x$ on $\mathbf{R}_{++}$
    - ✓ Concave, quasiconvex, quasiconcave
  - Ceiling function: $\text{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$
    - ✓ quasiconvex, quasiconcave
- Linear-fractional function
  - $f(x) = \frac{a^\top x+b}{c^\top x+d}, \text{dom}\, f = \{x | c^\top x + d > 0\}$
  - $\left\{x \middle| c^\top x + d > 0, \frac{a^\top x+b}{c^\top x+d} \geq \alpha\right\}$ and $\left\{x \middle| c^\top x + d > 0, \frac{a^\top x+b}{c^\top x+d} \leq \alpha\right\}$ is convex

    $\Rightarrow f$ is Quasilinear

Basic properties

- Jensen's inequality for quasiconvex functions
  - f is quasiconvex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex and $\forall x, y \in \mathrm{dom}\, f, 0 \leq \theta \leq 1$

$$f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$$



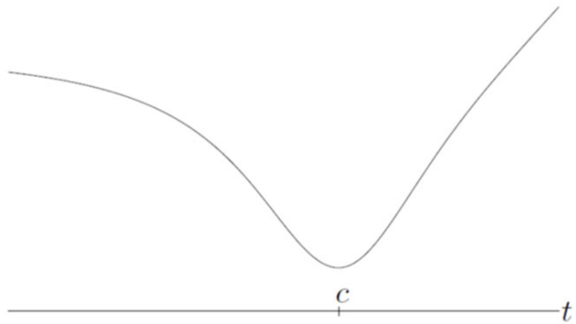
Basic properties

- **Condition**
  - f is quasiconvex $\Leftrightarrow$ its restriction to any line intersecting its domain is quasiconvex
- **Quasiconvex functions on $\mathbf{R}$**
  - A continuous function $f: \mathbf{R} \to \mathbf{R}$ is quasiconvex $\Leftrightarrow$ one of the following conditions holds

✓ f is nondecreasing

✓ f is nonincreasing

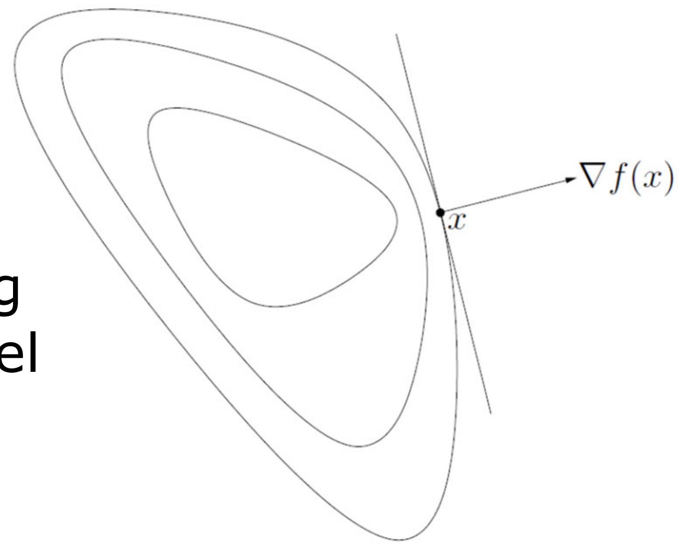
✓ $\exists c \in \mathrm{dom}\, f, \forall t \in \mathrm{dom}\, f, t \leq c, f$ is nonincreasing, and $t \geq c, f$ is nondecreasing



Differentiable quasiconvex functions

First-order conditions

- f is differentiable
- f is quasiconvex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, f(y) \leq f(x) \Rightarrow \nabla f(x)^{\top}(y-x) \leq 0$



$\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{y | f(y) \leq f(x)\}$ at x

Differentiable quasiconvex functions

First-order conditions

- f is differentiable
- f is quasiconvex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, f(y) \le f(x) \Rightarrow \nabla f(x)^\top (y - x) \le 0$
- It is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f.

Second-order conditions

- f is twice differentiable
- f is quasiconvex $\Rightarrow$ $\forall x \in \mathrm{dom}\, f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y \ge 0$

Differentiable quasiconvex functions

First-order conditions

- f is differentiable
- f is quasiconvex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, f(y) \le f(x) \Rightarrow \nabla f(x)^\top (y - x) \le 0$
- It is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f.

Second-order conditions

- f is twice differentiable
- $\forall x \in \mathrm{dom}\, f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y > 0 \Rightarrow f$ is quasiconvex

Operations that preserve quasiconvexity

Nonnegative weighted maximum

- f_i is quasicovex, $w_i \geq 0 \Rightarrow f = \max\{w_1 f_1, \ldots, w_n f_n\}$ is quasiconvex

- $g(x, y)$ is quasiconvex in x for each $y, w(y) \geq 0 \Rightarrow f(x) = \sup_{y \in C}(w(y) g(x, y))$ is quasiconvex

Operations that preserve quasiconvexity

Composition

- $g: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex, $h: \mathbf{R} \to \mathbf{R}$ is nondecreasing $\Rightarrow f = h \circ g$ is quasiconvex
- $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(Ax + b)$ is quasiconvex
- $f: \mathbf{R}^n \to \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(\frac{Ax+b}{c^\top x+d})$ is quasiconvex,$\mathrm{dom}\ g = \{x | c^\top x + d > 0, (Ax + b)/(c^\top x + d) \in \mathrm{dom}\ f\}$

Minimization

- $f(x, y)$ is quasicovex in x and y, C is a convex set $\Rightarrow g(x) = \inf_{y \in C} f(x, y)$ is quasiconvex

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Log-concave and log-convex functions

- Definition
  - $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \operatorname{dom} f, \log f(x)$ is concave (convex) $\Rightarrow f$ is log-concave (convex)
  - A log-convex function is convex
  - A nonnegative concave function is log-concave
- Condition
  - $f: \mathbf{R}^n \to \mathbf{R}, f(x) > 0, \forall x \in \operatorname{dom} f, f$ is log-concave $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \le \theta \le 1$

$$f(\theta x + (1-\theta)y) \ge f(x)^{\theta} f(y)^{1-\theta}$$

Examples

- $f(x) = a^\top x + b, \operatorname{dom} f = \{x \mid a^\top x + b > 0\}$ is log-concave
- $f(x) = x^a, \operatorname{dom} f = \mathbf{R}_{++}, a \le 0 \Rightarrow f$ is log-convex, $a \ge 0 \Rightarrow f$ is log-concave
- $f(x) = e^{ax}$ is log-convex and log-concave
- $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$ is log-concave
- $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$ is log-convex for $x \ge 1$
- $\det X$ and $\frac{\det X}{\operatorname{tr} X}$ are log-concave on $\boldsymbol{S}_{++}^n$

Properties

- Twice differentiable log-convex/concave functions
  - f is twice differentiable, $\mathrm{dom}\, f$ is convex
  - $\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^\top$
  - f is log-convex $\Leftrightarrow f(x) \nabla^2 f(x) \succcurlyeq \nabla f(x) \nabla f(x)^\top$
  - f is log-concave $\Leftrightarrow f(x) \nabla^2 f(x) \preccurlyeq \nabla f(x) \nabla f(x)^\top$

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Convexity with respect to a generalized inequality

- K-convex
  - $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preccurlyeq_K$
  - $f: \mathbf{R}^n \to \mathbf{R}^m$ is K-convex if $\forall x, y \in$ $\mathrm{dom}\, f, 0 \leq \theta \leq 1$

    $$f(\theta x + (1-\theta)y) \preccurlyeq_K \theta f(x) + (1-\theta)f(y)$$

  - $f: \mathbf{R}^n \to \mathbf{R}^m$ is $\text{stricly}\ K-$convex if $\forall x \neq y \in$ dom $f, 0 < \theta < 1$

    $$f(\theta x + (1-\theta)y) \prec_K \theta f(x) + (1-\theta)f(y)$$

Examples

- Componentwise Inequality
  - $K = \mathbf{R}_+^m$
  - $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is convex with respect to componentwise inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \leq \theta \leq 1,$
    $f(\theta x + (1-\theta)y) \preccurlyeq \theta f(x) + (1-\theta) f(y)$
  - Each f_i is a convex function

Examples

- Matrix Convexity
  - $f: \mathbf{R}^n \to \mathbf{S}^m$ is convex with respect to matrix inequality $\Leftrightarrow \forall x, y \in \operatorname{dom} f, 0 \leq \theta \leq 1$

    $$f(\theta x + (1-\theta)y) \preccurlyeq \theta f(x) + (1-\theta) f(y)$$

  - $f(X) = XX^\top, X \in \mathbf{R}^{m \times n}$ is matrix convex
  - X^p is matrix convex on $\mathbf{S}^n_{++}$ for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$

Convexity with respect to generalized inequalities

- Dual characterization of K-convexity
  - A function f is (strictly) K-convex $\Leftrightarrow$ For every $w \succcurlyeq_{K^*} 0$, the real-valued function $w^\top f$ is (strictly) convex in the ordinary sense.

- We refer to the generalized inequality $\preccurlyeq_{K^*}$ as the dual of $\preccurlyeq_K$
  - $x \preccurlyeq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$
  - $x \prec_K y$ if and only if $\lambda^\top x < \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$, $\lambda \neq 0$

Convexity with respect to generalized inequalities

Dual characterization of K-convexity

- A function f is (strictly) K-convex $\Leftrightarrow$ For every $w \succcurlyeq_{K^*} 0$, the real-valued function $w^\top f$ is (strictly) convex in the ordinary sense.

Differentiable K-convex functions

- A differentiable function f is K-convex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f$,

$$f(y) \succcurlyeq_K f(x) + Df(x)(y - x)$$

- A differentiable function f is strictly K-convex $\Leftrightarrow$ $\mathrm{dom}\, f$ is convex, $\forall x, y \in \mathrm{dom}\, f, x \neq y$,

$$f(y) \succ_K f(x) + Df(x)(y - x)$$

Convexity with respect to generalized inequalities

Composition theorem

- $g: \mathbf{R}^n \to \mathbf{R}^p$ is K-convex, $h: \mathbf{R}^p \to \mathbf{R}$ is convex, and $\tilde{h}$ (the extended-value extension of h) is K-nondecreasing $\Rightarrow h \circ g$ is convex.

Example

- $g: \mathbf{R}^{m\times n} \to \mathbf{S}^n, g(X) = X^\top AX + B^\top X + X^\top B + C$ is convex, where $A \succcurlyeq 0, B \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{S}^n$
- $h: \mathbf{S}^n \to \mathbf{R}, h(Y) = -\log\det(-Y)$ is convex and increasing on $\operatorname{dom} h = -\mathbf{S}^n_{++}$
- $f(X) = -\log\det(-(X^\top AX + B^\top X + X^\top B + C))$ is convex on $\operatorname{dom} f = \{X \in \mathbf{R}^{m\times n} | X^\top AX + B^\top X + X^\top B + C \prec 0\}$

Monotonicity with respect to a generalized inequality

- $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preccurlyeq_K$
  - $f: \mathbf{R}^n \to \mathbf{R}$ is K-nondecreasing if

$$x \preccurlyeq_K y \Rightarrow f(x) \leq f(y)$$

  - $f: \mathbf{R}^n \to \mathbf{R}$ is K-increasing if

$$x \preccurlyeq_K y, x \neq y \Rightarrow f(x) < f(y)$$

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