Convex Optimization Problems (I)

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Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions
- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization

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Basic Terminology

Unconstrained
when $m = p = 0$

Optimization Problems

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \qquad (1) \\
& h_i(x) = 0, \qquad i = 1, \ldots, p
\end{aligned}
$$

- Optimization variable: $x \in \mathbf{R}^n$
- Objective function: $f_0 : \mathbf{R}^n \to \mathbf{R}$
- Inequality constraints: $f_i(x) \le 0$
- Inequality constraint functions: $f_i : \mathbf{R}^n \to \mathbf{R}$
- Equality constraints: $h_i(x) = 0$
- Equality constraint functions: $h_i : \mathbf{R}^n \to \mathbf{R}$

Basic Terminology

Optimization Problems

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{aligned} \quad (1)$$

- Domain

$$\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- $x \in \mathcal{D}$ is feasible if it satisfies all the constraints
- The problem is feasible if there exists at least one feasible point

Basic Terminology

- Optimal Value $p^\star$

$$p^\star = \inf\{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$$

  - Infeasible problem: $p^\star = \infty$
  - Unbounded below: if there exist x_k with $f_0(x_k) \to -\infty$ as $k \to \infty$, then $p^\star = -\infty$

- Optimal Points
  - $x^\star$ is feasible and $f_0(x^\star) = p^\star$

- Optimal Set

$$X_{\text{opt}} = \{x | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, f_0(x) = p^*\}$$

- $p^\star$ is achieved if X_{opt} is nonempty

Basic Terminology

- ε-suboptimal Points
  - a feasible x with $f_0(x) \leq p^\star + \varepsilon$
- ε-suboptimal Set
  - the set of all ε-suboptimal points
- Locally Optimal Points

$$\begin{array}{lll} \min & f_0(z) & \\ \text{s.t.} & f_i(z) \leq 0, & i = 1, \ldots, m \\ & h_i(z) = 0, & i = 1, \ldots, p \\ & \|z - x\|_2 \leq R & \end{array}$$

  - x is feasible and solves the above problem
- Globally Optimal Points

Basic Terminology

- Types of Constraints
  - If $f_i(x) = 0$, $f_i(x) \leq 0$ is active at x
  - If $f_i(x) < 0$, $f_i(x) \leq 0$ is inactive at x
  - $h_i(x) = 0$ is active at all feasible points
  - Redundant constraint: deleting it does not change the feasible set
- Examples on $x \in \mathbf{R}$ and $\operatorname{dom} f_0 = \mathbf{R}_{++}$
  - $f_0(x) = 1/x : p^\star = 0$, the optimal value is not achieved
  - $f_0(x) = -\log x : p^\star = -\infty$, unbounded blow
  - $f_0(x) = x \log x : p^\star = -1/e$, $x^\star = 1/e$ is optimal

Basic Terminology

- Feasibility Problems

$$\begin{aligned} &\text{find} && x \\ &\text{s.t.} && f_i(x) \leq 0, && i = 1, \ldots, m \\ & && h_i(x) = 0, && i = 1, \ldots, p \end{aligned}$$

  - Determine whether constraints are consistent

- Maximization Problems

$$\begin{aligned} &\max && f_0(x) \\ &\text{s.t.} && f_i(x) \leq 0, && i = 1, \ldots, m \\ & && h_i(x) = 0, && i = 1, \ldots, p \end{aligned}$$

  - It can be solved by minimizing $-f_0$
  - Optimal Value $p^\star$

$$p^\star = \sup \{f_0(x) | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$$

Basic Terminology

□ Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & h_i(x) = 0, \qquad i = 1, \ldots, p \end{aligned}$$

□ Box constraints

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & l_i \leq x_i \leq u_i, \qquad i = 1, \ldots, n \end{aligned}$$

□ Reformulation

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & l_i - x_i \leq 0, \qquad i = 1, \ldots, n \\ & x_i - u_i \leq 0, \qquad i = 1, \ldots, n \end{aligned}$$

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Equivalent Problems

- Two Equivalent Problems
  - If from a solution of one, a solution of the other is readily found, and vice versa
- A Simple Example

$$\begin{aligned} \min \quad & \tilde{f}(x) = \alpha_0 f_0(x) \\ \text{s.t.} \quad & \tilde{f}_i(x) = \alpha_i f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & \tilde{h}_i(x) = \beta_i h_i(x) = 0, \qquad i = 1, \ldots, p \end{aligned}$$

  - $\alpha_i > 0, i = 0, \ldots, m$
  - $\beta_i \neq 0, i = 1, \ldots, p$
  - Equivalent to the problem (1)

Change of Variables

- $\phi: \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and $\phi(\operatorname{dom}\phi) \supseteq \mathcal{D}$, and define

$$\tilde{f}_i(z) = f_i(\phi(z)), \qquad i = 0, \ldots, m$$
$$\tilde{h}_i(z) = h_i(\phi(z)), \qquad i = 1, \ldots, p$$

- An Equivalent Problem

$$\begin{aligned} \min \quad & \tilde{f}_0(z) \\ \text{s.t.} \quad & \tilde{f}_i(z) \le 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, \quad i = 1, \ldots, p \end{aligned}$$

  - If z solves it, $x = \phi(z)$ solves the problem (1)
  - If x solves (1), $z = \phi^{-1}(x)$ solves it

Transformation of Functions

- $\psi_0 : \mathbf{R} \to \mathbf{R}$ is monotone increasing
- $\psi_1, \ldots, \psi_m : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) \le 0$ if and only if $u \le 0$
- $\psi_{m+1}, \ldots, \psi_{m+p} : \mathbf{R} \to \mathbf{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$
- Define

$$\begin{aligned} \tilde{f}_i(x) &= \psi_i(f_i(x)), & i &= 0, \ldots, m \\ \tilde{h}_i(x) &= \psi_{m+i}(h_i(x)), & i &= 1, \ldots, p \end{aligned}$$

- An Equivalent Problem

$$\begin{aligned} \min \quad & \tilde{f}_0(x) \\ \text{s.t.} \quad & \tilde{f}_i(x) \le 0, \quad i = 1, \ldots, m \\ & \tilde{h}_i(x) = 0, \quad i = 1, \ldots, p \end{aligned}$$

Example

- Least-norm Problems

$$\min \quad \|Ax - b\|_2$$

  - Not differentiable at any x with $Ax - b = 0$

- Least-norm-squared Problems

$$\min \quad \|Ax - b\|_2^2 = (Ax - b)^\top (Ax - b)$$

  - Differentiable for all x

Slack Variables

- $f_i(x) \le 0$ if and only if there is an $s_i \ge 0$ that satisfies $f_i(x) + s_i = 0$
- An Equivalent Problem

$$\begin{array}{lll} \min & f_0(x) & \\ \text{s.t.} & s_i \ge 0, & i = 1, \ldots, m \\ & f_i(x) + s_i = 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

  - s_i is the slack variable associated with the inequality constraint $f_i(x) \le 0$
  - x is optimal for the problem (1) if and only if (x, s) is optimal for the above problem, where $s_i = -f_i(x)$

Eliminating Equality Constraints

- Assume $\phi: \mathbf{R}^k \to \mathbf{R}^n$ is such that x satisfies

$$h_i(x) = 0, \qquad i = 1, \dots, p$$

if and only if there is some $z \in \mathbf{R}^k$ such that

$$x = \phi(z)$$

- An Equivalent Problem

$$\begin{aligned} \min \quad & \tilde{f}_0(z) = f_0(\phi(z)) \\ \text{s.t.} \quad & \tilde{f}_i(z) = f_i(\phi(z)) \leq 0, \qquad i = 1, \dots, m \end{aligned}$$

  - If z is optimal for this problem, $x = \phi(z)$ is optimal for the problem (1)
  - If x is optimal for (1), there is at least one z which is optimal for this problem

Eliminating linear equality constraints

- Assume the equality constraints are all linear $Ax = b$, and x_0 is one solution

- Let $F \in \mathbf{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbf{R}^k\}$$

- An Equivalent Problem ($x = Fz + x_0$)

$$\begin{array}{lll} \min & f_0(Fz + x_0) & \\ \text{s.t.} & f_i(Fz + x_0) \le 0, & i = 1, \ldots, m \end{array}$$

  - $k = n - \operatorname{rank}(A)$

Linear algebra

$$\mathcal{N}(A) = \mathcal{R}(A^\top)^\perp$$

Range and nullspace

- Let $A \in \mathbf{R}^{m\times n}$, the range of A, denoted $\mathcal{R}(A)$, is the set of all vectors in $\mathbf{R}^m$ that can be written as linear combinations of the columns of A:

$$\mathcal{R}(A) = \{Ax | x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$$

- The nullspace (or kernel) of A, denoted $\mathcal{N}(A)$, is the set of all vectors x mapped into zero by A:

$$\mathcal{N}(A) = \{x | Ax = 0\} \subseteq \mathbf{R}^n$$

- if $\mathcal{V}$ is a subspace of $\mathbf{R}^n$, its orthogonal complement, denoted $\mathcal{V}^\perp$, is defined as:

$$\mathcal{V}^\perp = \{x | z^\top x = 0 \text{ for all } z \in \mathcal{V}\}$$

Introducing Equality Constraints

- Consider the problem

$$\begin{aligned} \min \quad & f_0(A_0x + b_0) \\ \text{s.t.} \quad & f_i(A_ix + b_i) \leq 0, \qquad i = 1,\ldots,m \\ & h_i(x) = 0, \qquad i = 1,\ldots,p \end{aligned}$$

  - $x \in \mathbf{R}^n, A_i \in \mathbf{R}^{k_i \times n}$ and $f_i: \mathbf{R}^{k_i} \to \mathbf{R}$

- An Equivalent Problem

$$\begin{aligned} \min \quad & f_0(y_0) \\ \text{s.t.} \quad & f_i(y_i) \leq 0, \qquad i = 1,\ldots,m \\ & y_i = A_ix + b_i, \qquad i = 0,\ldots,m \\ & h_i(x) = 0, \qquad i = 1,\ldots,p \end{aligned}$$

  - Introduce $y_i \in \mathbf{R}^{k_i}$ and $y_i = A_ix + b_i$

Optimizing over Some Variables

- Suppose $x \in \mathbf{R}^n$ is partitioned as $x = (x_1, x_2)$, with $x_1 \in \mathbf{R}^{n_1}, x_2 \in \mathbf{R}^{n_2}$ and $n_1 + n_2 = n$
- Consider the problem

$$\begin{array}{ll} \min & f_0(x_1, x_2) \\ \text{s.t.} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1 \\ & \tilde{f}_i(x_2) \leq 0, \quad i = 1, \ldots, m_2 \end{array}$$

- An Equivalent Problem

$$\begin{array}{ll} \min & \tilde{f}_0(x_1) \\ \text{s.t.} & f_i(x_1) \leq 0, \quad i = 1, \ldots, m_1 \end{array}$$

  - where

$$\tilde{f}_0(x_1) = \inf\{f_0(x_1, z) | \tilde{f}_i(z) \leq 0, i = 1, \ldots, m_2\}$$

Example

- Minimize a Quadratic Function

$$\begin{aligned} \min \quad & x_1^\top P_{11} x_1 + 2 x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \\ \text{s.t.} \quad & f_i(x_1) \leq 0, \qquad i = 1, \ldots, m \end{aligned}$$

- Minimize over x_2

$$\inf_{x_2} \left(x_1^\top P_{11} x_1 + 2 x_1^\top P_{12} x_2 + x_2^\top P_{22} x_2 \right)$$

$$= x_1^\top \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1$$

- An Equivalent Problem

$$\begin{aligned} \min \quad & x_1^\top \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \right) x_1 \\ \text{s.t.} \quad & f_i(x_1) \leq 0, \qquad i = 1, \ldots, m \end{aligned}$$

Epigraph Problem Form

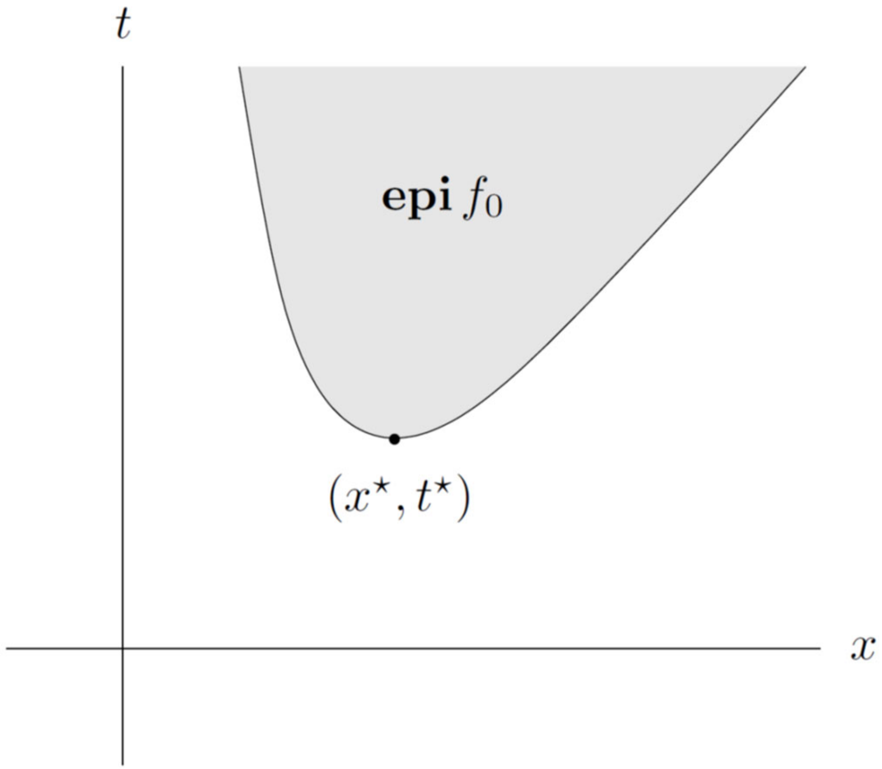
Epigraph Form

$$\begin{array}{ll} \min & t \\ \text{s.t.} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & h_i(x) = 0, \qquad i = 1, \ldots, p \end{array}$$

- Introduce a variable $t \in \mathbf{R}$
- (x, t) is optimal for this problem if and only if x is optimal for (1) and $t = f_0(x)$
- The objective function of the epigraph form problem is a linear function of x, t

Epigraph Problem Form

□ Geometric Interpretation



- Find the point in the epigraph that minimizes t

Making Constraints Implicit

- Unconstrained problem

$$\min\ F(x)$$

- $\mathrm{dom}\, F = \{x \in \mathrm{dom}\, f_0 | f_i(x) \le 0, i = 1, \ldots, m,$
  $h_i(x) = 0, i = 1, \ldots, p\}$
- $F(x) = f_0(x)$ for $x \in \mathrm{dom}\, F$
- It has not make the problem any easier
- It could make the problem more difficult, because F is probably not differentiable

Making Constraints Explicit

- A Unconstrained Problem

$$\min f(x)$$

  - where

$$f(x) = \begin{cases} x^\top x & Ax = b \\ \infty & \text{otherwise} \end{cases}$$

  - An implicit equality constraint $Ax = b$
- An Equivalent Problem

$$\begin{aligned} \min \quad & x^\top x \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

  - Objective and constraint functions are differentiable

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Problem Descriptions

- Parameter Problem Description
  - Functions have some analytical or closed form
  - Example: $f_0(x) = x^\top P x + q^\top x + r$, where $P \in \mathbf{S}^n, q \in \mathbf{R}^n$ and $r \in \mathbf{R}$
  - Give the values of the parameters
- Oracle Model (Black-box Model)
  - Can only query the objective and constraint functions by an oracle/subroutine
  - Evaluate $f(x)$ and its gradient $\nabla f(x)$
  - Know some prior information (convexity)

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Convex Optimization Problems

Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & a_i^\top x = b_i, \qquad i = 1, \ldots, p \end{aligned}$$

- The objective function must be convex
- The inequality constraint functions must be convex
- The equality constraint functions $h_i(x) = a_i^\top x - b_i$ must be affine

Convex Optimization Problems

Properties

- Feasible set of a convex optimization problem is convex

$$\bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{m} \{x \mid f_i(x) \le 0\} \cap \bigcap_{i=1}^{p} \{x \mid a_i^\top x = b_i\}$$

  - ✓ Minimize a convex function over a convex set
- ε-suboptimal set is convex
- The optimal set is convex
- If the objective is strictly convex, then the optimal set contains at most one point

Concave Maximization Problems

Standard Form

$$
\begin{aligned}
\max \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\
& a_i^\top x = b_i, \qquad i = 1, \ldots, p
\end{aligned}
$$

- It is referred as a convex optimization problem if f_0 is concave and $f_1, \ldots, f_m$ are convex
- It is readily solved by minimizing the convex objective function $-f_0$

Abstract Form Convex Optimization Problem

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1/(1 + x_2^2) \le 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{aligned}$$

  - Not a convex optimization problem
    - ✓ f_1 is not convex and h_1 is not affine
  - But the feasible set is indeed convex
  - Abstract convex optimization problem
- An Equivalent Convex Problem

$$\begin{aligned} \min \quad & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & f_1(x) = x_1 \le 0 \\ & h_1(x) = x_1 + x_2 = 0 \end{aligned}$$

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Local and Global Optima

- Any locally optimal point of a convex problem is also (globally) optimal
- Proof by Contradiction
  - x is locally optimal implies

    $$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \le R\}$$

    for some R
  - Suppose x is not globally optimal, i.e., there exists $f_0(y) < f_0(x)$ and $\|y - x\|_2 > R$
  - Define

    $$z = (1 - \theta)x + \theta y, \theta = \frac{R}{2\|y - x\|_2} \in (0,1)$$

Local and Global Optima

- By convexity of the feasible set

z is feasible

- It is easy to check

$$\|z - x\|_2 = \|\theta(y - x)\|_2 = \left\|\frac{R(y - x)}{2\|y - x\|_2}\right\|_2 = \frac{R}{2} < R$$

- Thus, $f_0(x) < f_0(z)$

$$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z - x\|_2 \le R\}$$

- By convexity of f_0

$$f_0(z) \le (1 - \theta)f_0(x) + \theta f_0(y) < f_0(x)$$

Contradiction

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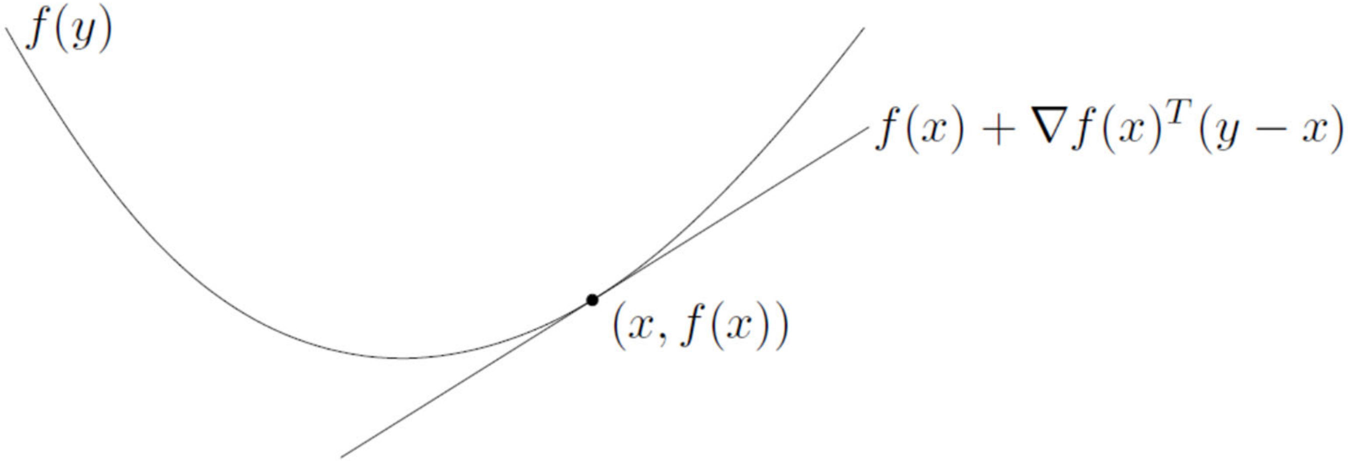
An Optimality Criterion for Differentiable f_0

- Suppose f_0 is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \mathrm{dom}\, f_0$$



An Optimality Criterion for Differentiable f_0

- Suppose f_0 is differentiable

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y - x), \forall x, y \in \mathrm{dom}\, f_0$$

- Let X denote the feasible set

$$X = \{x \mid f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$$

- x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

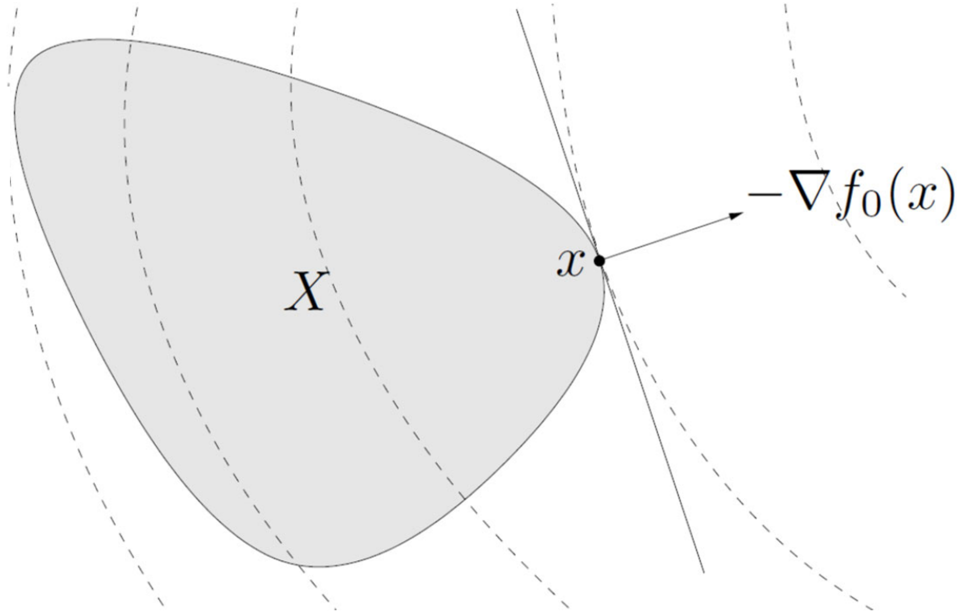
An Optimality Criterion for Differentiable f_0

- x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all } y \in X$$

- $-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x

If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$. The hyperplane $\{x | a^\top x = a^\top x_0\}$ is called a supporting hyperplane to C at x_0



Proof of Optimality Condition

- Sufficient Condition

$$\left.\begin{array}{c}\nabla f_0(x)^\top (y-x) \geq 0 \\ f_0(y) \geq f_0(x) + \nabla f_0(x)^\top (y-x)\end{array}\right\} \Rightarrow f_0(y) \geq f_0(x)$$

- Necessary Condition
  - Suppose x is optimal but

$$\exists y \in X, \nabla f_0(x)^\top (y-x) < 0$$

  - Define $z(t) = ty + (1-t)x, t \in [0,1]$

$$f_0\big(z(0)\big) = f_0(x), \qquad \frac{d}{dt} f_0\big(z(t)\big)\Big|_{t=0} = \nabla f_0(x)^\top (y-x) < 0$$

  - So, for small positive t, $f_0\big(z(t)\big) < f_0(x)$

Unconstrained Problems

- x is optimal if and only if $\nabla f_0(x) = 0$
  - Consider $y = x - t\nabla f_0(x)$ and $t > 0$
  - When t is small, y is feasible

$$\nabla f_0(x)^\top (y - x) = -t\|\nabla f_0(x)\|_2^2 \geq 0 \Leftrightarrow \nabla f_0(x) = 0$$

- Unconstrained Quadratic Optimization

$$\min \ f_0(x) = (1/2)x^\top P x + q^\top x + r, \qquad \text{where } P \in \mathbf{S}_+^n$$

  - x is optimal if and only if $\nabla f_0(x) = Px + q = 0$

1. If $q \notin \mathcal{R}(P)$, no solution, f_0 is unbound below
2. If $P \succ 0$, unique minimizer $x^\star = -P^{-1}q$
3. If P is singular, but $q \in \mathcal{R}(P)$, $X_{\text{opt}} = -P^\dagger q + \mathcal{N}(P)$

Linear Equality Constraints

- Let $F \in \mathbf{R}^{n \times k}$ be any matrix with $\mathcal{R}(F) = \mathcal{N}(A)$, then

$$\{x | Ax = b\} = \{Fz + x_0 | z \in \mathbf{R}^k\}$$

$$= \{v + x_0 | v \in \mathcal{N}(A)\}$$

- Consider $Px + q = 0$

$$X_{\text{opt}} = -P^{\dagger} q + \mathcal{N}(P)$$

Problems with Equality Constraints Only

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- x is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall\ Ay = b$$

Problems with Equality Constraints Only

- Consider the Problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & Ax = b \end{array}$$

Lagrange Multiplier Optimality Condition

$$Ax = b$$
$$\nabla f_0(x) + A^\top v = 0$$

- x is optimal if and only if

$$\left.\begin{array}{c} \nabla f_0(x)^\top (y - x) \geq 0, \forall\, Ay = b \\ \{y \mid Ay = b\} = \{x + v \mid v \in \mathcal{N}(A)\} \end{array}\right\}$$

$$\Leftrightarrow \nabla f_0(x)^\top v \geq 0, \forall\, v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x)^\top v = 0, \forall\, v \in \mathcal{N}(A)$$

$$\Leftrightarrow \nabla f_0(x) \perp \mathcal{N}(A) \Leftrightarrow \nabla f_0(x) \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$$

$$\Leftrightarrow \exists v \in \mathbf{R}^p, \nabla f_0(x) + A^\top v = 0$$

Minimization over the Nonnegative Orthant

- Consider the Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & x \succcurlyeq 0 \end{aligned}$$

- x is optimal if and only if

$$\nabla f_0(x)^\top (y - x) \geq 0, \forall y \succcurlyeq 0$$

$$\Leftrightarrow \begin{cases} \nabla f_0(x) \succcurlyeq 0 \\ -\nabla f_0(x)^\top x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \nabla f_0(x) \succcurlyeq 0 \\ \nabla f_0(x)^\top x = 0 \end{cases}$$

- The Optimality Condition

$$x \succcurlyeq 0, \qquad \nabla f_0(x) \succcurlyeq 0, \qquad x_i \big(\nabla f_0(x)\big)_i = 0, i = 1, \dots, n$$

  - The last condition is called complementarity

Outline

- Optimization Problems
  - Basic Terminology
  - Equivalent Problems
  - Problem Descriptions
- Convex Optimization
  - Standard Form
  - Local and Global Optima
  - An Optimality Criterion
  - Equivalent Convex Problems
  - Quasiconvex Optimization

Equivalent Convex Problems

- Standard Form

$$
\begin{aligned}
\min \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\
& a_i^\top x = b_i, \qquad i = 1, \ldots, p
\end{aligned}
$$

- Eliminating Equality Constraints

$$
\begin{aligned}
\min \quad & f_0(Fz + x_0) \\
\text{s.t.} \quad & f_i(Fz + x_0) \le 0, \qquad i = 1, \ldots, m
\end{aligned}
$$

  - $A = [a_1^\top; \ldots; a_p^\top]$, $b = (b_1; \ldots; b_p)$, $Ax_0 = b$, $\mathcal{R}(F) = \mathcal{N}(A)$
  - The composition of a convex function with an affine function is convex
  - At least in principle, we can restrict our attention to convex problems with no equality constraints

Equivalent Convex Problems

- Introducing Equality Constraints
  - If an objective or constraint function has the form $f_i(A_i x + b_i)$, where $A_i \in \mathbf{R}^{k_i \times n}$, we can replace it with $f_i(y_i)$ and add the constraint $y_i = A_i x + b_i$, where $y_i \in \mathbf{R}^{k_i}$
- Slack Variables
  - Introduce new constraints $f_i(x) + s_i = 0$
  - Introduce slack variables for linear inequalities preserves convexity of a problem
- Minimizing over Some Variables
  - It preserves convexity
  - $f_0(x_1, x_2)$ needs to be jointly convex in x_1 and x_2

Equivalent Convex Problems

Epigraph Problem Form

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & f_0(x) - t \le 0 \\ & f_i(x) \le 0, \qquad i = 1, \dots, m \\ & a_i^\top x = b_i, \qquad i = 1, \dots, p \end{aligned}$$

- The objective is linear (hence convex)
- The new constraint function $f_0(x) - t$ is also convex in (x, t)
- This problem is convex
- Any convex optimization problem is readily transformed to one with linear objective

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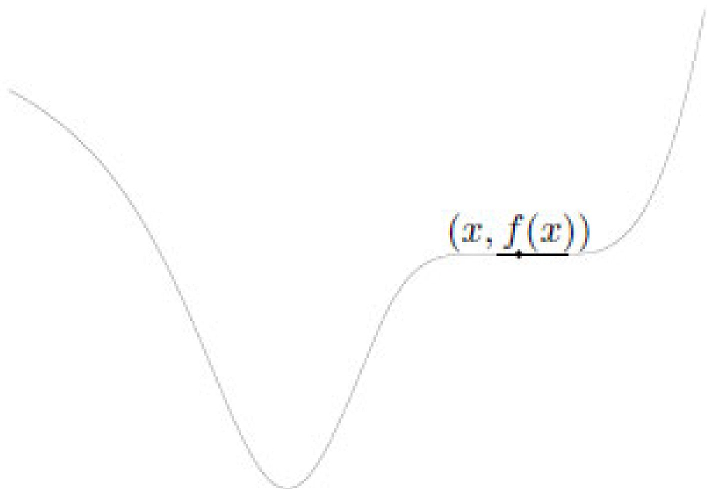
Quasiconvex Optimization

Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- f_0 is quasiconvex and $f_1, \ldots, f_m$ are convex
- Have locally optimal solutions that are not (globally) optimal



Quasiconvex Optimization

Standard Form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- f_0 is quasiconvex and $f_1, \ldots, f_m$ are convex
- Have locally optimal solutions that are not (globally) optimal

Optimality Conditions for Differentiable f_0

- Let X denote the feasible set, x is optimal if

$$x \in X, \qquad \nabla f_0(x)^\top (y - x) > 0 \text{ for all } y \in X \setminus \{x\}$$

1. Only a sufficient condition
2. Requires $\nabla f_0(x)$ to be nonzero

Representation via family of convex functions

- Represent the sublevel sets of a quasiconvex function f via inequalities of convex functions.
  - $\phi_t : \mathbf{R}^n \to \mathbf{R}$ is convex, $t \in \mathbf{R}$

$$f(x) \le t \Leftrightarrow \phi_t(x) \le 0$$

  - ϕ_t is a nonincreasing function of t
  - Examples

$$\phi_t(x) = \begin{cases} 0 & f(x) \le t \\ \infty & \text{otherwise} \end{cases}$$

$$\phi_t(x) = \operatorname{dist}(x, \{z | f(z) \le t\})$$

Quasiconvex Optimization via Convex Feasibility Problems

- Let $\phi_t: \mathbf{R}^n \to \mathbf{R}, t \in \mathbf{R}$, be a family of convex functions such that

$$f_0(x) \le t \Leftrightarrow \phi_t(x) \le 0$$

  and for each x, $\phi_s(x) \le \phi_t(x)$ whenever $s \ge t$
  - Let $p^\star$ be the optimal value of quasiconvex problem
  - Consider the feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{s.t.} & \phi_t(x) \le 0 \\ & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  - If it is feasible, $p^\star \le t$. Conversely, $p^\star \ge t$

Bisection for Quasiconvex Optimization

Algorithm

given $l \leq p^\star, u \geq p^\star$, tolerance $\epsilon > 0$

repeat

1. $t := (l + u)/2$
2. Solve the convex feasibility problem
3. **if** it is feasible, $u := t$; **else** $l := t$

until $u - l \leq \epsilon$

- The interval $[l, u]$ is guaranteed to contain $p^\star$
- The length of the interval after k iterations is $2^{-k}(u - l)$
- $\lceil \log_2((u - l)/\epsilon) \rceil$ iterations are required

Bisection for Quasiconvex Optimization

- An $\epsilon-$suboptimal Solution
  - $l \leq p^* \leq u$
  - $u - l \leq \epsilon$
  - $u - p^* \leq \epsilon$

$$\begin{array}{ll} \text{find} & x \\ \text{s.t.} & \phi_u(x) \leq 0 \\ & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  - $f_0(x) \leq u = p^* + u - p^* \leq p^* + \epsilon$

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