

Duality (I)

Lijun Zhang

zlj@nju.edu.cn

<http://cs.nju.edu.cn/zlj>





Outline

□ The Lagrange Dual Function

- The Lagrange Dual Function
- Lower Bound on Optimal Value
- The Lagrange Dual Function and Conjugate Functions

□ The Lagrange Dual Problem

- Making Dual Constraints Explicit
- Weak Duality
- Strong Duality and Slater's Constraint Qualification



Optimization Problems

□ Standard Form

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array} \quad (1)$$

- Domain is nonempty

$$\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- Denote the optimal value by p^*
- We do **not** assume the problem is convex



The Lagrangian

□ The Lagrangian $L: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x)$$

- $\text{dom } L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$
- λ_i : the Lagrange multiplier associated with the i -th inequality constraint $f_i(x) \leq 0$
- v_i : the Lagrange multiplier associated with the i -th equality constraint $h_i(x) = 0$
- Vectors λ and v : **dual variables** or Lagrange multiplier vectors



The Lagrange Dual Function

$$\square g: \mathbf{R}^m \times \mathbf{R}^p \mapsto \mathbf{R}$$

$$\begin{aligned} g(\lambda, v) &= \inf_{x \in \mathcal{D}} L(x, \lambda, v) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x) \right) \end{aligned}$$

- When L is unbounded below in x , $g = -\infty$
- g is **concave**
 - ✓ g is the pointwise infimum of a family of affine functions of (λ, v)
- The inf problem is **unconstrained**



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Lower Bounds on p^*

□ For any $\lambda \geq 0$ and any v

$$g(\lambda, v) \leq p^*$$

□ Proof

■ $\tilde{x}$ is a feasible point for original problem

$$f_i(\tilde{x}) \leq 0, h_i(\tilde{x}) = 0$$

■ Since $\lambda \geq 0$

$$\sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p v_i h_i(\tilde{x}) \leq 0$$

■ Therefore

$$L(\tilde{x}, \lambda, v) = f_0(\tilde{x}) + \sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p v_i h_i(\tilde{x}) \leq f_0(\tilde{x})$$



Lower Bounds on p^*

□ For any $\lambda \geq 0$ and any v

$$g(\lambda, v) \leq p^*$$

□ Proof

■ Hence

$$g(\lambda, v) = \inf_{x \in \mathcal{D}} L(x, \lambda, v) \leq L(\tilde{x}, \lambda, v) \leq f_0(\tilde{x})$$

■ Note that $g(\lambda, v) \leq f_0(\tilde{x})$ for any feasible $\tilde{x}$

□ Discussions

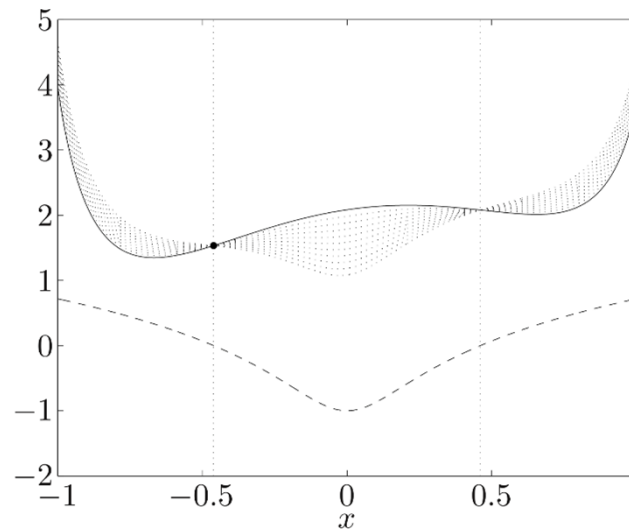
■ The lower bound is vacuous, when $g(\lambda, v) = -\infty$

■ It is nontrivial only when $\lambda \geq 0, (\lambda, v) \in \text{dom } g$

■ Dual feasible: (λ, v) with $\lambda \geq 0, (\lambda, v) \in \text{dom } g$

Example

- A Simple Problem with $x \in \mathbf{R}, m = 1, p = 0$
 - Lower bound from a dual feasible point

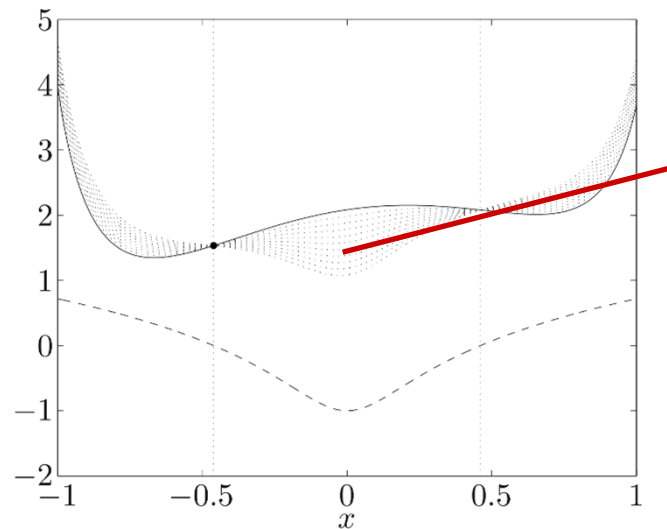


- ✓ Solid curve: objective function f_0
- ✓ Dashed curve: constraint function f_1
- ✓ Feasible set: $[-0.46, 0.46]$ (indicated by the two dotted vertical lines)



Example

- A Simple Problem with $x \in \mathbf{R}, m = 1, p = 0$
 - Lower bound from a dual feasible point

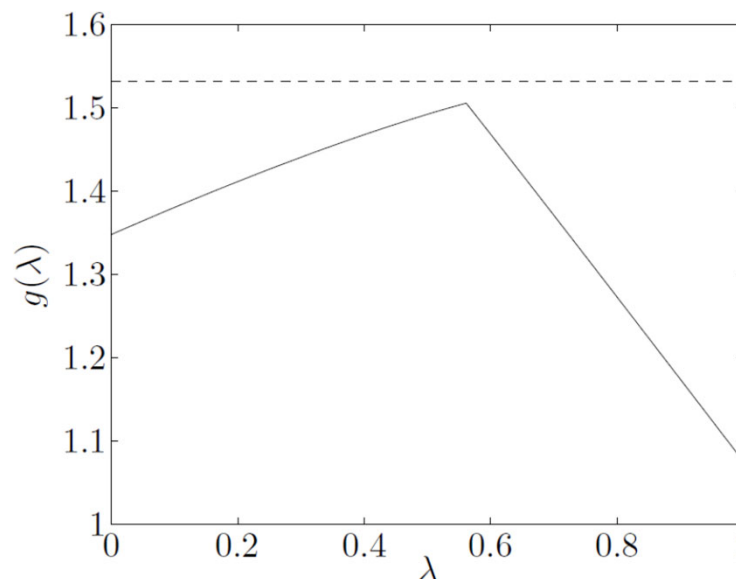


$$g(\lambda) = \inf_{x \in \mathcal{D}} L(x, \lambda) \\ \leq L(x, \lambda) \leq f_0(x)$$

- ✓ Optimal point and value: $x^* = -0.46, p^* = 1.54$
- ✓ Dotted curves: $L(x, \lambda)$ for $\lambda = 0.1, 0.2, \dots, 1.0$.
 - Each has a minimum value smaller than p^* as on the feasible set (and for $\lambda > 0$), $L(x, \lambda) \leq f_0(x)$

Example

□ The dual function g



- Neither f_0 nor f_1 is convex, but the dual function g is **concave**
- Horizontal dashed line: p^* (the optimal value of the problem)

Linear Approximation Interpretation



□ Rewrite (1) as unconstrained problem

$$\min f_0(x) + \sum_{i=1}^m I_-(f_i(x)) + \sum_{i=1}^p I_0(h_i(x)) \quad (2)$$

- $I_-: \mathbb{R} \mapsto \mathbb{R}$ is the indicator function for the nonpositive reals

$$I_-(u) = \begin{cases} 0 & u \leq 0, \\ \infty & u > 0. \end{cases}$$

- I_0 is the indicator function of $\{0\}$

Linear Approximation Interpretation



□ In the formulation (2)

- $I_-(u)$ expresses our displeasure associated with a constraint function value $u = f_i(x)$: zero if $f_i(x) \leq 0$, infinite if $f_i(x) > 0$
- $I_0(u)$ gives our displeasure for an equality constraint value $u = h_i(x)$
- Our displeasure rises from zero to infinite as $f_i(x)$ transitions from nonpositive to positive

Linear Approximation Interpretation



□ In the formulation (2)

- Suppose we replace $I_-(u)$ with linear function $\lambda_i u$, where $\lambda_i \geq 0$, and $I_0(u)$ with $v_i u$
- Objective becomes the Lagrangian $L(x, \lambda, v)$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x)$$

- Dual function value $g(\lambda, v)$ is optimal value of

$$\min f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x) \quad (3)$$

Linear Approximation Interpretation



□ In the formulation (3)

- We replace I_- and I_0 with linear or “soft” displeasure functions
- For an inequality constraint, our displeasure is zero when $f_i(x) = 0$, and is positive when $f_i(x) > 0$ (assuming $\lambda_i > 0$)
- In (2), any nonpositive value of $f_i(x)$ is acceptable
- In (3), we actually derive pleasure from constraints that have margin, i.e., from $f_i(x) < 0$

Linear Approximation Interpretation



□ Interpretation of Lower Bound

- The linear function is an underestimator of the indicator function

$$\lambda_i u \leq I_-(u)$$

$$v_i u \leq I_0(u)$$

- Lower Bound Property

$$f_0(x) + \sum_{i=1}^m I_-(f_i(x)) + \sum_{i=1}^p I_0(h_i(x)) \geq$$

$$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x)$$

Linear Approximation Interpretation



□ Interpretation of Lower Bound

- The linear function is an underestimator of the indicator function

$$\lambda_i u \leq I_-(u)$$

$$v_i u \leq I_0(u)$$

- Minimize over both sides

$$\begin{aligned} p^* &\geq g(\lambda, v) = \inf_{x \in \mathcal{D}} L(x, \lambda, v) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p v_i h_i(x) \right) \end{aligned}$$



Example

□ Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s. t.} & Ax = b \end{array}$$

- $A \in \mathbf{R}^{p \times n}$
- No inequality constraints
- p (linear) equality constraints

□ Lagrangian

$$L(x, v) = x^\top x + v^\top (Ax - b)$$

- Domain: $\mathbf{R}^n \times \mathbf{R}^p$



Example

□ Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s. t.} & Ax = b \end{array}$$

□ Dual Function

$$g(v) = \inf_x L(x, v) = \inf_x x^\top x + v^\top (Ax - b)$$

■ Optimality condition

$$\nabla_x L(x, v) = 2x + A^\top v = 0 \Rightarrow x = -(1/2)A^\top v$$



Example

□ Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s. t.} & Ax = b \end{array}$$

□ Dual Function

$$\Rightarrow g(v) = L(-(1/2)A^\top v, v) = -(1/4)v^\top AA^\top v - b^\top v$$

■ Concave Function

□ Lower Bound Property

$$-(1/4)v^\top AA^\top v - b^\top v \leq \inf \{x^\top x \mid Ax = b\}$$



Example

□ Standard Form LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

■ Inequality constraints: $f_i(x) = -x_i, i = 1, \dots, n$

□ Lagrangian

$$\begin{aligned} L(x, \lambda, v) &= c^\top x - \sum_{i=1}^n \lambda_i x_i + v^\top (Ax - b) \\ &= -b^\top v + (c + A^\top v - \lambda)^\top x \end{aligned}$$

□ Dual Function

$$\begin{aligned} g(\lambda, v) &= \inf_x L(x, \lambda, v) \\ &= -b^\top v + \inf_x (c + A^\top v - \lambda)^\top x \end{aligned}$$



Example

□ Standard Form LP

$$\begin{array}{ll}\min & c^T x \\ \text{s. t.} & Ax = b \\ & x \geq 0\end{array}$$

- Inequality constraints: $f_i(x) = -x_i, i = 1, \dots, n$

□ Dual Function

$$g(\lambda, v) = \begin{cases} -b^T v & A^T v - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

- The lower bound is nontrivial only when λ and v satisfy $\lambda \geq 0$ and $A^T v - \lambda + c = 0$



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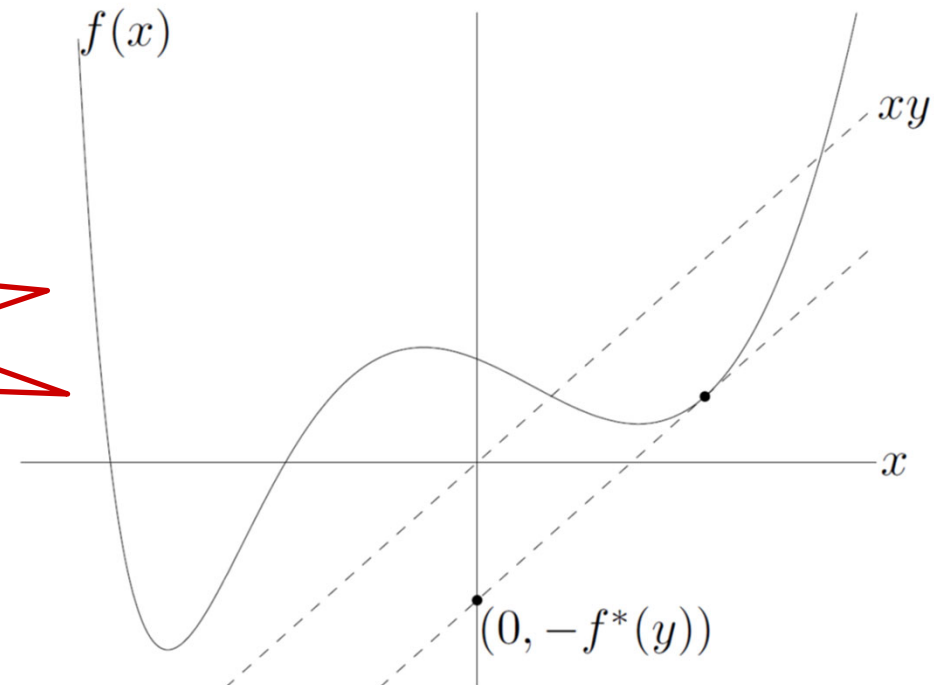
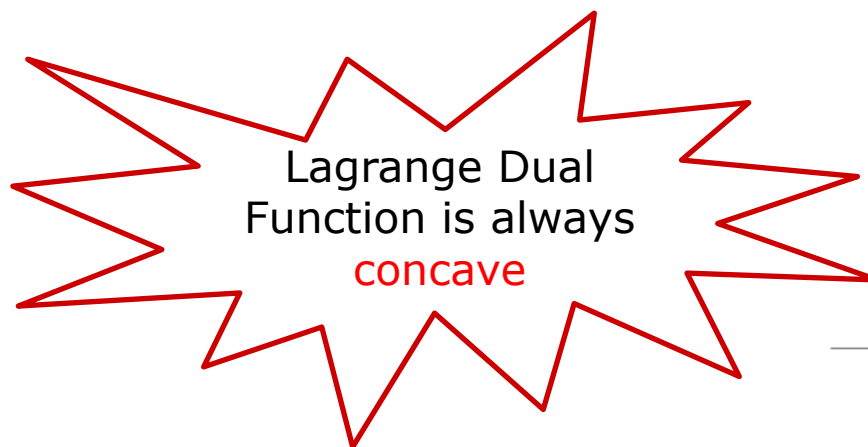
Conjugate Function

□ $f: \mathbf{R}^n \rightarrow \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^\top x - f(x))$$

■ $\text{dom } f^* = \{y | f^*(y) < \infty\}$

■ f^* is always **convex**



The Lagrange Dual Function and Conjugate Functions



□ A Simple Example

$$\begin{array}{ll} \min & f(x) \\ \text{s. t.} & x = 0 \end{array}$$

□ Lagrangian

$$L(x, v) = f(x) + v^\top x$$

□ Dual Function

$$\begin{aligned} g(v) &= \inf_x (f(x) + v^\top x) \\ &= -\sup_x ((-v)^\top x - f(x)) = -f^*(-v) \end{aligned}$$

The Lagrange Dual Function and Conjugate Functions



□ A More General Example

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & Ax \preceq b \\ & Cx = d\end{array}$$

□ Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

□ Dual Function

$$\begin{aligned}g(\lambda, \nu) &= \inf_x (f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)) \\ &= -b^\top \lambda - d^\top \nu + \inf_x (f_0(x) + (A^\top \lambda + C^\top \nu)^\top x) \\ &= -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu)\end{aligned}$$

The Lagrange Dual Function and Conjugate Functions



□ A More General Example

$$\begin{array}{ll}\min & f_0(x) \\ \text{s. t.} & Ax \preceq b \\ & Cx = d\end{array}$$

□ Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \lambda^\top (Ax - b) + \nu^\top (Cx - d)$$

□ Dual Function

$$g(\lambda, \nu) = -b^\top \lambda - d^\top \nu - f_0^*(-A^\top \lambda - C^\top \nu)$$

$$\blacksquare \text{ dom } g = \{(\lambda, \nu) \mid -A^\top \lambda - C^\top \nu \in \text{dom } f_0^*\}$$



Example

□ Equality Constrained Norm Minimization

$$\begin{array}{ll}\min & \|x\| \\ \text{s. t.} & Ax = b\end{array}$$

□ Conjugate of $f_0 = \|\cdot\|$

$$f_0^*(y) = \begin{cases} 0 & \|y\|_* \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

□ The Dual Function

$$g(v) = -b^\top v - f_0^*(-A^\top v) = \begin{cases} -b^\top v & \|A^\top v\|_* \leq 1, \\ -\infty & \text{otherwise.} \end{cases}$$



Example

□ Entropy Maximization

$$\begin{aligned} \min \quad & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \text{s. t.} \quad & Ax \preceq b \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

□ Conjugate of f_0

$$f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

□ The Dual Function

$$\begin{aligned} g(\lambda, v) &= -b^\top \lambda - v - f_0^*(-A^\top \lambda - v\mathbf{1}) \\ &= -b^\top \lambda - v - \sum_{i=1}^n e^{-a_i^\top \lambda - v - 1} \\ &= -b^\top \lambda - v - e^{-v-1} \sum_{i=1}^n e^{-a_i^\top \lambda} \end{aligned}$$



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The Lagrange Dual Problem

□ For any $\lambda \geq 0$ and any v

$$g(\lambda, v) \leq p^*$$

■ What is the best lower bound?

□ Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, v) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

□ Primal Problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \end{array} \quad (1)$$



The Lagrange Dual Problem

□ For any $\lambda \geq 0$ and any v

$$g(\lambda, v) \leq p^*$$

■ What is the best lower bound?

□ Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, v) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

- Dual feasible: (λ, v) with $\lambda \geq 0, g(\lambda, v) > -\infty$
- Dual optimal or optimal Lagrange multipliers: (λ^*, v^*)
- A convex optimization problem



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Making Dual Constraints Explicit

□ Motivation

- The $\text{dom } g = \{(\lambda, v) \mid g(\lambda, v) > -\infty\}$ may have dimension $\leq m + p$
- Identify the equality constraints that are 'hidden' or 'implicit' in g

□ Standard Form LP

$$\begin{array}{ll}\min & c^\top x \\ \text{s. t.} & Ax = b \\ & x \geq 0\end{array}$$

□ Dual Function

$$g(\lambda, v) = \begin{cases} -b^\top v & A^\top v - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$



Example

□ Lagrange Dual of Standard Form LP

■ Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

■ An Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \nu \\ \text{s.t.} \quad & A^\top \nu - \lambda + c = 0 \\ & \lambda \geq 0 \end{aligned}$$

✓ Make equality constraints explicit



Example

□ Lagrange Dual of Standard Form LP

■ Lagrange Dual Problem

$$\begin{aligned} \max \quad & g(\lambda, \nu) = \begin{cases} -b^\top \nu & A^\top \nu - \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

■ Another Equivalent Problem

$$\begin{aligned} \max \quad & -b^\top \nu \\ \text{s.t.} \quad & A^\top \nu + c \geq 0 \end{aligned}$$

✓ An LP in inequality form

Standard Form LP

Lagrange Dual
→

Inequality Form LP



Example

□ Lagrange Dual of Inequality Form LP

■ Inequality form LP (Primal Problem)

$$\begin{array}{ll}\min & c^\top x \\ \text{s. t.} & Ax \preceq b\end{array}$$

■ Lagrangian

$$L(x, \lambda) = c^\top x + \lambda^\top (Ax - b) = -b^\top \lambda + (A^\top \lambda + c)^\top x$$

■ Lagrange dual function

$$\begin{aligned}g(\lambda) &= \inf_x L(x, \lambda) = -b^\top \lambda + \inf_x (A^\top \lambda + c)^\top x \\ &= \begin{cases} -b^\top \lambda & A^\top \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases}\end{aligned}$$



Example

□ Lagrange Dual of Inequality Form LP

■ Inequality form LP (Primal Problem)

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \preceq b\end{array}$$

■ Lagrange Dual Problem

$$\begin{array}{ll}\max & g(\lambda) = \begin{cases} -b^T \lambda & A^T \lambda + c = 0, \\ -\infty & \text{otherwise.} \end{cases} \\ \text{s.t.} & \lambda \succeq 0\end{array}$$

■ An Equivalent Problem

$$\begin{array}{ll}\max & -b^T \lambda \\ \text{s.t.} & A^T \lambda + c = 0 \\ & \lambda \succeq 0\end{array}$$



Example

□ Lagrange Dual of Inequality Form LP

■ Inequality form LP (Primal Problem)

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \preceq b\end{array}$$

■ An Equivalent Problem

$$\begin{array}{ll}\max & -b^T \lambda \\ \text{s.t.} & A^T \lambda + c = 0 \\ & \lambda \succeq 0\end{array}$$

✓ An LP in standard form

Inequality Form LP

Lagrange Dual
→

Standard Form LP



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Weak Duality

□ For any $\lambda \geq 0$ and any v

$$g(\lambda, v) \leq p^*$$

■ What is the best lower bound?

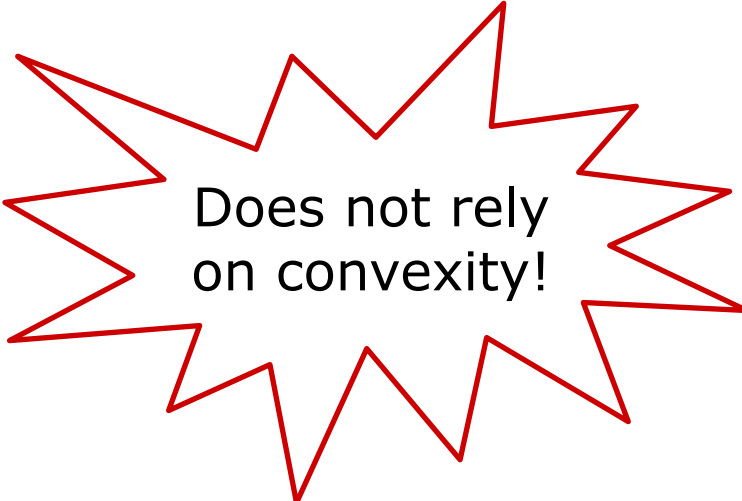
□ Lagrange Dual Problem

$$\begin{array}{ll} \max & g(\lambda, v) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

■ Optimal value d^*

□ Weak Duality

$$d^* \leq p^*$$



Does not rely
on convexity!



Weak Duality

□ Weak Duality

$$d^* \leq p^*$$

- If the primal problem is unbounded below, i.e., $p^* = -\infty$, we must have $d^* = -\infty$, i.e., the Lagrange dual problem is infeasible
- If $d^* = \infty$, we must have $p^* = \infty$

□ Optimal duality gap

$$p^* - d^*$$

- Nonnegative



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Strong Duality

□ Strong Duality

$$d^* = p^*$$

- The optimal duality gap is zero
- The best bound that can be obtained from the Lagrange dual function is tight
- In general, does not hold

□ Usually hold for convex optimization

$$\begin{array}{ll} \min & f_0(x) \\ \text{s. t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- $f_0, \dots, f_m$ are convex



Slater's Constraint Qualification

□ Constraint Qualifications

- Sufficient conditions for strong duality

□ Slater's condition

- $\exists x \in \text{relint } \mathcal{D}$ such that

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

- Such a point x is called strictly feasible

□ If Slater's condition holds and the problem is convex

- Strong duality holds
- Dual optimal value is attained when $d^* > -\infty$



Slater's Constraint Qualification

□ Constraint Qualifications

- Sufficient conditions for strong duality

□ Slater's condition (weaker form)

- If the first k constraint functions are affine
- $\exists x \in \text{relint } \mathcal{D}$ such that

$$f_i(x) \leq 0, \quad i = 1, \dots, k$$

$$f_i(x) < 0, \quad i = k + 1, \dots, m$$

$$Ax = b$$

- When constraints are all linear equalities and inequalities, and $\text{dom } f_0$ is open
 - ✓ Reduce to feasibility



Example

□ Least-squares Solution of Linear Equations

$$\begin{array}{ll} \min & x^\top x \\ \text{s. t.} & Ax = b \end{array}$$

□ Dual Problem

$$\max -(1/4)v^\top AA^\top v - b^\top v$$

□ Slater's condition

- The primal problem is feasible, i.e., $b \in \mathcal{R}(A)$

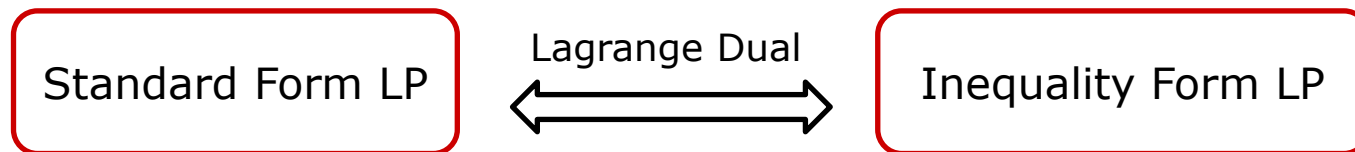
□ Strong duality always holds

- Even when $b \notin \mathcal{R}(A)$, $d^* = p^* = \infty$



Example

□ Lagrange dual of LP



□ Strong duality holds for any LP

- If the primal problem is feasible **or** the dual problem is feasible

□ Strong duality may fail

- If **both** the primal and dual problems are infeasible



Example

□ QCQP (Primal Problem)

$$\begin{aligned} \min \quad & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{s.t.} \quad & (1/2)x^\top P_i x + q_i^\top x + r_i \leq 0, \quad i = 1, \dots, m \end{aligned}$$

$$\blacksquare P_0 \in \mathbf{S}_{++}^n \text{ and } P_i \in \mathbf{S}_+^n, i = 1, \dots, m$$

□ Dual Problem

$$\begin{aligned} \max \quad & -(1/2)q(\lambda)^\top P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

$$\blacksquare P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i$$

$$\blacksquare r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i$$

□ Slater's condition

$$\blacksquare \exists x, (1/2)x^\top P_i x + q_i^\top x + r_i < 0, i = 1, \dots, m$$



Example

□ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{array}{ll}\min & x^\top A x + 2b^\top x \\ \text{s.t.} & x^\top x \leq 1\end{array}$$

■ $A \in \mathbf{S}^n, A \not\geq 0$ and $b \in \mathbf{R}^n$

■ Lagrangian

$$\begin{aligned}L(x, \lambda) &= x^\top A x + 2b^\top x + \lambda(x^\top x - 1) \\ &= x^\top (A + \lambda I)x + 2b^\top x - \lambda\end{aligned}$$

■ Dual Function

$$g(\lambda) = \begin{cases} -b^\top (A + \lambda I)^\dagger b - \lambda & A + \lambda I \geq 0, b \in \mathcal{R}(A + \lambda I) \\ -\infty & \text{otherwise} \end{cases}$$



Example

□ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

■ $A \in \mathbf{S}^n, A \not\geq 0$ and $b \in \mathbf{R}^n$

□ Dual Problem

$$\begin{aligned} \max \quad & -b^\top (A + \lambda I)^\dagger b - \lambda \\ \text{s.t.} \quad & A + \lambda I \geq 0, b \in \mathcal{R}(A + \lambda I) \end{aligned}$$

■ A convex optimization problem



Example

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Strong duality holds

□ Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n (q_i^\top b)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s. t.} \quad & \lambda \geq -\lambda_{\min}(A) \end{aligned}$$

- A convex optimization problem
- λ_i and q_i : eigenvalues and corresponding (orthonormal) eigenvectors of A



Example

□ A Nonconvex Quadratic Problem (Primal Problem)

$$\begin{aligned} \min \quad & x^\top A x + 2b^\top x \\ \text{s.t.} \quad & x^\top x \leq 1 \end{aligned}$$

■ $A \in \mathbf{S}^n, A \not\geq 0$ and $b \in \mathbf{R}^n$

Strong duality holds for any optimization problem with quadratic objective and **one** quadratic inequality constraint, provided Slater's condition holds

□ Dual Problem

$$\begin{aligned} \max \quad & -\sum_{i=1}^n (q_i^\top b)^2 / (\lambda_i + \lambda) - \lambda \\ \text{s.t.} \quad & \lambda \geq -\lambda_{\min}(A) \end{aligned}$$

■ A convex optimization problem

■ λ_i and q_i : eigenvalues and corresponding (orthonormal) eigenvectors of A



Summary

□ The Lagrange Dual Function

- The Lagrange Dual Function
- Lower Bound on Optimal Value
- The Lagrange Dual Function and Conjugate Functions

□ The Lagrange Dual Problem

- Making Dual Constraints Explicit
- Weak Duality
- Strong Duality and Slater's Constraint Qualification