Online Influence Maximization under Linear Threshold Model

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Motivation

- Online influence maximization (OIM)
  - A sequential decision-making problem
  - Selects a set of users and provides them with free products
  - Receives feedback from the information diffusion process
- Common influence propagation models: independent cascade (IC) model, linear threshold (LT) model
- Existing works mainly focus on the IC model with edge-level feedback
  - The IC model assumes the information spreads through each edge independently.
  - The edge-level feedback means that the learner could observe the influence status of each edge once its start node is influenced.
- The LT model characterizes the herd behavior that often occurs in real information diffusion process, that with more active in-neighbors, a user becomes much more likely to be influenced.

Setting

- Graph $G = (V, E)$: V is the set of users (nodes) and E is the set of relationships (edges) between users
  - Each edge $e_{u,v} \in E$ is associated with a weight $w(e_{u,v})$ representing the influence ability of u on v
  - Let $n = |V|, m = |E|, D$ to be node number, edge number and the diameter respectively, where the diameter of the graph is defined as the maximum (directed) distance between the pair of nodes in any connected component.
- The diffusion process starting from seed set S
  - Each node is assigned with a threshold θ_v, which is independently uniformly drawn from $[0, 1]$ and characterizes the susceptibility level of v.
  - Let S_τ be the set of activated nodes by the end of time τ.
    - ∗ At time $\tau = 0$, only nodes in the seed set are activated: $S_0 = S$.
    - ∗ At time $\tau + 1$ with $\tau \geq 0$, for any node $v \notin S_\tau$ that has not been activated yet, it will be activated if the aggregated influence of its active in-neighbors exceeds its threshold: $\sum_{u \in N(v) \cap S_\tau} w(e_{u,v}) \geq \theta_v$.
    - ∗ Such diffusion process will last at most D time steps.
  - The size of the influenced nodes: $r(S, w, \theta) = |S_D|$
  - Let $r(S, w) = \mathbb{E}[r(S, w, \theta)]$ be the *influence spread* of seed set S where the expectation is taken over all random variables θ_v's.
- The (offline) IM problem
  - Aims at finding the seed set S with the size at most K under weight vector w to maximize the influence spread, $\max_{S:|S| \leq K} r(S, w)$.
  - This problem is NP-hard under the LT model but can be approximately solved.
  - Let Opt_w be the maximum influence spread under weight vector w
- The online IM (OIM) problem, in each round t:
  - The learner chooses a set of nodes S_t with limited size K
  - The learner observes (full) node-level feedback $S_{t,0}, S_{t,1}, \ldots, S_{t,D}$, where $S_{t,i}$ represents the set of active nodes by time step $i \in [D]$ in the diffusion process in this round.
  - The learner updates its knowledge on unknown weights using the observed feedback, which helps the seed set selection in the next round.
- The goal of the learner is to minimize the expected cumulative η-scaled regret

$$R(T) = \mathbb{E}\left[\sum_{t=1}^{T} R_t\right] = \mathbb{E}\left[\eta \cdot T \cdot \mathrm{Opt}_w - \sum_{t=1}^{T} r(S_t, w)\right].$$

Algorithm

- $\tau_1(v) := \min_\tau \{\tau = 0, \ldots, D : N(v) \cap S_\tau \neq \emptyset\}$ is the earliest time step when node v has active neighbors, set $\tau_1(v) = D + 1$ if node v has no active in-neighbor until the diffusion ends.
- $\tau_2(v) := \left\{\tau = 0, \ldots, D : \chi(E_{\tau-2}(v))^\top w_v < \theta_v \leq \chi(E_{\tau-1}(v))^\top w_v\right\}$ is the time step that node v is influenced, set $\tau_2(v) = D + 1$ if node v is finally not influenced after the information diffusion ends.
- $E_\tau(v) := \{e_{u,v} : u \in N(v) \cap S_\tau\}$ is the set of incoming edges associated with active in-neighbors of v at time step τ.

`LT-LinUCB` **Algorithm:**

1. **Input:** Graph $G = (V, E)$; seed set cardinality K; exploration parameter $\rho_{t,v} > 0$ for any t, v; offline oracle `PairOracle`
2. **Initialize:** $M_{0,v} \leftarrow I \in \mathbb{R}^{|N(v)| \times |N(v)|}$, $b_{0,v} \leftarrow 0 \in \mathbb{R}^{|N(v)| \times 1}$, $\hat{w}_{0,v} \leftarrow 0 \in \mathbb{R}^{|N(v)| \times 1}$ for any node $v \in V$
3. **for** $t = 1, 2, 3, \ldots$
4. Compute the confidence ellipsoid $\mathcal{C}_{t,v} = \left\{w'_v \in [0,1]^{|N(v)| \times 1} : \left\|w'_v - \hat{w}_{t-1,v}\right\|_{M_{t-1,v}} \leq \rho_{t,v}\right\}$ for any node $v \in V$
5. Compute the pair (S_t, w_t) by `PairOracle` with confidence set $\mathcal{C}_t = \{\mathcal{C}_{t,v}\}_{v \in V}$ and seed set cardinality K
6. Select the seed set S_t and observe the feedback
7. //Update
8. **for** node $v \in V$
9. Initialize $A_{t,v} \leftarrow 0 \in \mathbb{R}^{|N(v)| \times 1}$, $y_{t,v} \leftarrow 0 \in \mathbb{R}$
10. Uniformly randomly choose $\tau \in \{\tau' : \tau_{t,1}(v) \leq \tau' \leq \tau_{t,2}(v) - 1\}$
11. **if** v is influenced and $\tau = \tau_{t,2}(v) - 1$
12. $A_{t,v} = \chi\left(E_{t,\tau}(v)\right), \quad y_{t,v} = 1$
13. **else if** $\tau = \tau_1(v), \ldots, \tau_2(v) - 2$ or $\tau = \tau_2(v) - 1$ but v is not influenced
14. $A_{t,v} = \chi\left(E_{t,\tau}(v)\right), \quad y_{t,v} = 0$
15. $M_{t,v} \leftarrow M_{t-1,v} + A_{t,v}A_{t,v}^\top, \quad b_{t,v} \leftarrow b_{t-1,v} + y_{t,v}A_{t,v}, \quad \hat{w}_{t,v} = M_{t,v}^{-1} b_{t,v}$

Analysis

- For the seed set S, define the set of all nodes related to a node v, $V_{S,v}$, to be the set of nodes that are on any path from S to v in graph G.
- For seed set S and node $u \in V \setminus S$, define $N_{S,u} := \sum_{v \in V \setminus S} \mathbb{1}\{u \in V_{S,v}\} \leq n - K$ to be the number of nodes that u is relevant to.
- Then for the vector $N_S = (N_{S,u})_{u \in V}$, define the upper bound of its L^2-norm over all feasible seed sets

$$\gamma(G) := \max_{S \in \mathcal{A}} \sqrt{\sum_{u \in V} N_{S,u}^2} \leq (n - K)\sqrt{n} = O(n^{3/2}),$$

which is a constant related to the graph.

Theorem 1 (GOM bounded smoothness)**.** *For any two weight vectors $w, w' \in [0,1]^m$ with $\sum_{u \in N(v)} w(e_{u,v}) \leq 1$, the difference of their influence spread for any seed set S can be bounded as*

$$\left|r(S, w') - r(S, w)\right| \leq \mathbb{E}\left[\sum_{v \in V \setminus S} \sum_{u \in V_{S,v}} \sum_{\tau = \tau_1(u)}^{\tau_2(u) - 1} \left|\sum_{e \in E_\tau(u)} (w'(e) - w(e))\right|\right]$$

where the definitions of $\tau_1(u), \tau_2(u)$ and $E_\tau(u)$ are all under weight vector w, and the expectation is taken over the randomness of the thresholds on nodes.

Theorem 2 (Upper Bound)**.** *Suppose the* `LT-LinUCB` *runs with an (α, β)-approximation* `PairOracle` *and parameter $\rho_{t,v} = \rho_t = \sqrt{n \log(1 + tn) + 2\log\frac{1}{\delta}} + \sqrt{n}$ for any node $v \in V$. Then the $\alpha\beta$-scaled regret satisfies*

$$R(T) \leq 4\rho_T \gamma(G) D \sqrt{mnT \log(1 + T)/\log(1 + n)} + n\delta \cdot T(n - k).$$

When $\delta = 1/(n\sqrt{T})$, $R(T) \leq C \cdot \gamma(G)\, Dn\sqrt{mT} \log(T)$ for some universal constant C.

Conclusions

- Formulate the problem of OIM under LT model with node-level feedback and design how to distill effective information from observations.
- Prove a novel GOM bounded smoothness property for the spread function.
- Propose `LT-LinUCB` algorithm with rigorous theoretical analysis and show a competitive regret bound of $O(\mathrm{poly}(m)\sqrt{T}\ln(T))$.
- Design `OIM-ETC` algorithm with theoretical analysis on its distribution-dependent and distribution-independent regret bounds.
  - Efficient, applies to both LT and IC models, and has less requirements on feedback and offline computation.
- Future work: The OIM problem under IC model with node-level feedback; Applying Thompson sampling to influence maximization

References

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