Efficient Learning on Large Data Sets

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Outline

1. Introduction: Unsupervised learning
2. Nyström Approximation
3. Supervised learning: Accelerated gradient
4. Conclusion

Unsupervised Learning

Unsupervised feature extraction / clustering / novelty detection

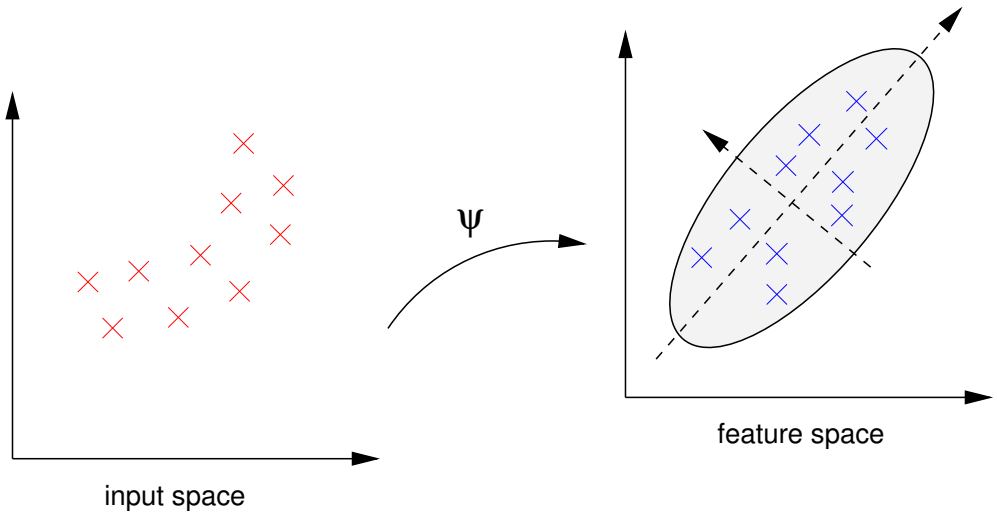
- e.g., feature extraction for optical character recognition
- e.g., network intrusion detection

Principal Component Analysis (PCA)

- $\mathbf{S}\mathbf{v} = \lambda \mathbf{v}$
- $\mathbf{S} = \frac{1}{n} \sum_{j=1}^{n} \mathbf{x}_j \mathbf{x}_j'$ (scatter matrix)

Kernel PCA (KPCA)

Perform PCA in the feature space ($\mathbf{x} \mapsto \varphi(\mathbf{x})$)



- when mapped back to the input space, eigenvectors becomes nonlinear
- eigen-decompose the kernel matrix **K**

$$\mathbf{K}\boldsymbol{\alpha} = n\lambda\boldsymbol{\alpha}$$

Image Segmentation: Normalized Cut

Normalized cut

- normalized fraction of total costs of edges between A and B to the total edge connections to all the nodes

Optimization

$$(\mathbf{D} - \mathbf{W})\mathbf{y} = \lambda \mathbf{D}\mathbf{y}$$

- $\mathbf{W}$: weight matrix; $\mathbf{D}$: degree matrix;

Eigen-decomposition

$$\mathbf{D}^{-\frac{1}{2}}\mathbf{L}\mathbf{D}^{-\frac{1}{2}}\mathbf{z} = \lambda \mathbf{z}$$

- $\mathbf{L} = \mathbf{D} - \mathbf{W}$: Laplacian matrix

Large Data Sets

Problem

Eigenvalue decomposition of the $n \times n$ matrix takes $O(n^3)$ time

- PCA, KPCA: n = number of samples
- normalized cut: n = number of pixels in the image

Nyström Algorithm

$$\int k(x,y)\phi(x)dx = \lambda\phi(y)$$

- $k(\cdot,\cdot)$: kernel function; λ: eigenvalue; $\phi(\cdot)$: eigenfunction

Approximation using a set $\mathcal{Z} = \{z_1, \ldots, z_m\}$ of landmark points

$$\int k(x,y)\phi(x)dx \simeq \frac{1}{m}\sum_{j=1}^{m} k(z_j,y)\phi(z_j) = \lambda\phi(y)$$

Take $y = z_i$

$$\frac{1}{m}\sum_{j=1}^{m} k(z_i,z_j)\phi(z_j) = \lambda\phi(z_i), \quad i = 1,2,\ldots,m$$

- small eigen-system $\tilde{\mathbf{K}}\boldsymbol{\phi} = m\lambda\boldsymbol{\phi}$
- $\tilde{\mathbf{K}} = [k(z_i,z_j)]$, $m \times m$ matrix
- $\boldsymbol{\phi} = [\phi(z_1), \ldots, \phi(z_m)]'$, m-vector

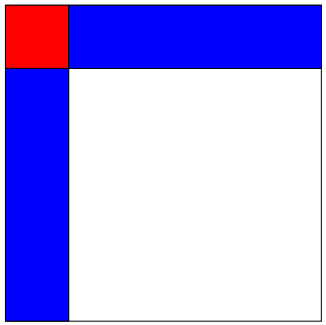
Extrapolate this "small" eigenvector to the "large" eigenvector

$$\phi(y) = \frac{1}{m\lambda}\sum_{j=1}^{m} k(z_i,y)\phi(z_i)$$

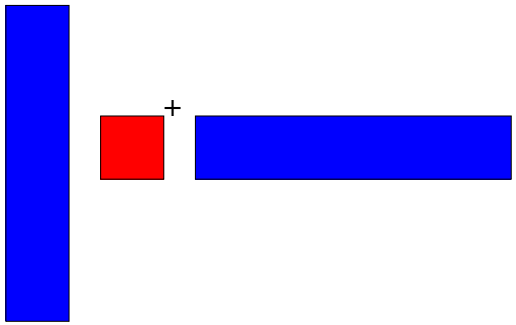
Low-Rank Approximation

Sample m landmark points → sample m columns from input matrix G

- $C = \begin{bmatrix} W \\ S \end{bmatrix} \quad G = \begin{bmatrix} W & S^T \\ S & B \end{bmatrix}$



Rank-k Nyström approximation:
$\tilde{G}_k = C W_k^+ C^T$



Time complexity: $O(nmk + m^3)$

- $m \ll n$ → much lower than the $O(n^3)$ complexity required by a direct SVD on G

How to Choose the Landmark Points?

1. random sampling (used in standard Nyström)
2. probabilistic [Drineas & Mahoney, JMLR-2005]
   - chooses the columns based on a data-dependent probability
3. greedy approach [Ouimet & Bengio, AISTATS-2005]
   - much more time-consuming
4. clustering-based
   - inexpensive; with interesting theoretical properties
   - (K. Zhang, I.W. Tsang, J.T. Kwok. *Improved Nyström low rank approximation and error analysis*. ICML-2008)

Tradeoff between Accuracy and Efficiency

- more columns sampled, more accurate is the approximation
- on very large data sets, the SVD step on W will dominate the computations and become prohibitive

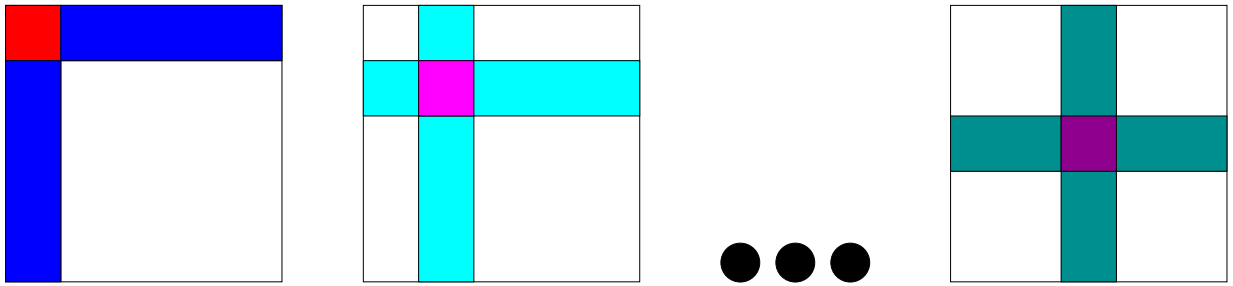
Example

Data set with several millions samples

- sampling only 1% of the columns
- W larger than $10,000 \times 10,000$

Ensemble Nyström Algorithm [Kumar et al., NIPS-2009]

Use an ensemble of Nyström approximators

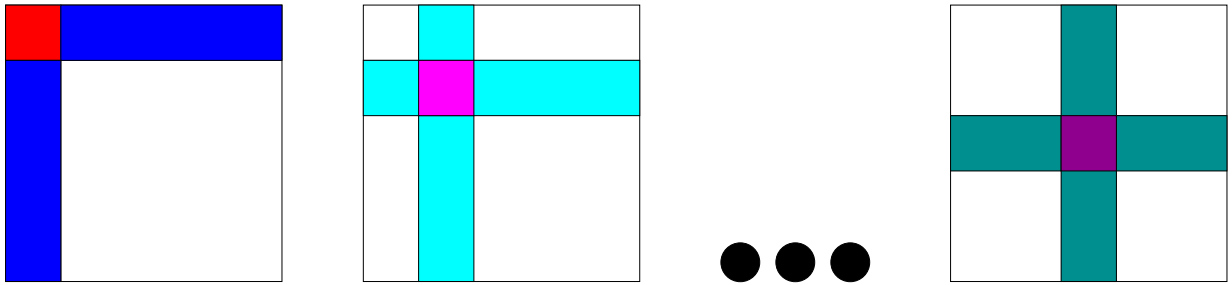


- n_e experts, each sample m columns; sample a total of mn_e columns
- each expert performs a standard Nyström approximation
  - single expert = standard Nyström
  - obtain rank-k approximations $\tilde{G}_{1,k}, \tilde{G}_{2,k}, \ldots, \tilde{G}_{n_e,k}$
- resultant approximations are linearly combined

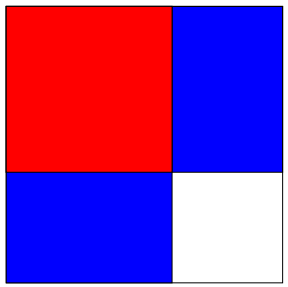
$$\tilde{G}^{ens} = \sum_{i=1}^{n_e} \mu_i \tilde{G}_{i,k}$$

  - μ_i's: mixture weights: uniform / heuristics / trained

Ensemble Nyström Algorithm...



1. more columns are sampled → more accurate approximation



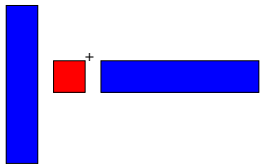
2. replace the expensive SVD by n_e SVDs, each on a small $m \times m$ matrix

Time complexity

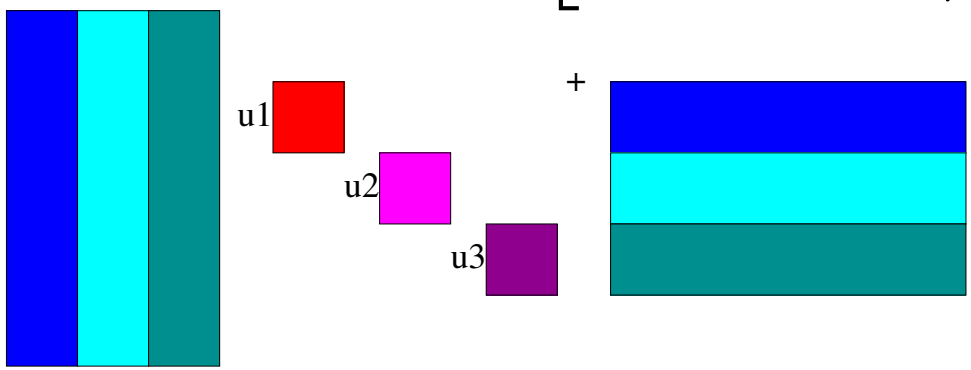
- $O(n_e nmk + n_e m^3 + C_\mu)$, where C_μ: cost of computing the mixture weights
- (roughly) n_e times that of standard Nyström

Motivation: Ensemble Nyström Algorithm

- $\tilde{G}^{ens} = \sum_{i=1}^{n_e} \mu_i \tilde{G}_{i,k}$



$$\sum_{i=1}^{n_e} \mu_i \tilde{G}_{i,k} = \sum_{i=1}^{n_e} \mu_i C_i W_{i,k}^{+} C_i^T = C \begin{bmatrix} \mu_1 W_{1,k}^{+} & & \\ & \ddots & \\ & & \mu_{n_e} W_{n_e,k}^{+} \end{bmatrix} C^T$$



- approximate $W^{+} \in \mathbb{R}^{n_e m \times n_e m}$ by a block diagonal matrix
- inverse of block diagonal matrix is block diagonal
- no matter how sophisticated the mixture weights μ_i's are estimated, this block diagonal approximation is rarely valid

Idea

1. high efficiency of the Nyström algorithm
2. sample more columns (like the ensemble Nyström algorithm)
3. produce a SVD approximation that is more accurate but still efficient → randomized algorithm [Halko et al., TR 2009]

Randomized Algorithm [Halko et al., 2009]

- W: $n \times n$ symmetric matrix; k: rank
- p: over-sampling parameter (typically, $p = 5$)
- q: parameter of the power method (accelerate decay of eigenvalues, typically, $q = 1$ or 2)

1: $\Omega \leftarrow$ a $n \times (k + p)$ standard Gaussian random matrix.
2: $Z \leftarrow W\Omega$, $Y \leftarrow W^{q-1}Z$.
3: Find an orthonormal matrix Q (e.g., by QR decomposition) such that $Y = QQ^T Y$. ▷ an approximate, low-dimensional basis for the range of W
4: Solve $B(Q^T\Omega) = Q^T Z$.
5: Perform SVD on B to obtain $V\Lambda V^T = B$.
6: $U \leftarrow QV$.

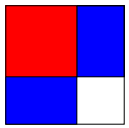
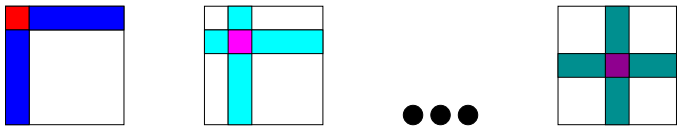
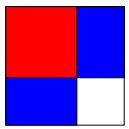
- only needs to perform SVD on B: small $k \times k$ matrix
- time complexity: $O(n^2 k + k^3)$

Proposed Algorithm

1: $C \leftarrow m$ columns of G sampled uniformly at random without replacement. ▷ m can be large
2: $W \leftarrow m \times m$ matrix
3: $[\tilde{U}, \Lambda] \leftarrow \text{randsvd}(W, k, p, q)$
4: $U \leftarrow C\tilde{U}\Lambda^{+}$. ▷ standard Nyström extension
5: $\hat{G} \leftarrow U\Lambda U^{T}$.

Is it Efficient?

Recall that $n \gg m \gg k$

Nyström 	$O(nmk + m^3)$
ensemble Nyström 	$O(nmk + n_e k^3 + C_\mu)$
randomized SVD	$O(n^2 k + k^3)$
proposed method 	$O(nmk + m^2 k + k^3) = O(nmk + k^3)$

Is it Accurate?

Rank-k standard Nyström approximation $\hat{G}$

- m randomly sampled columns

$$\mathbb{E}\|G - \hat{G}\|_2 \leq \|G - G_k\|_2 + \frac{2n}{\sqrt{m}}(\max_i G_{ii})$$

- G_k: best rank-k approximation

Proposed method

$$\mathbb{E}\|G - \hat{G}\|_2 \leq \zeta^{1/q}\|G - G_k\|_2 + (1 + \zeta^{1/q})\frac{n}{\sqrt{m}}(\max_i G_{ii})$$

- ζ: constant depending on k, p, m
- $\zeta^{1/q}$ close to 1 → becomes $\|G - G_k\|_2 + \frac{2n}{\sqrt{m}}(\max_i G_{ii})$, same as that for standard Nyström using m columns

Proposed method is as accurate as standard Nyström

More Error Analysis Results

Spectral norm

- sample columns probabilistically [Drineas & Mahoney, JMLR-2005]

$$\mathbb{E}\|G - \hat{G}\|_2 \leq \zeta^{1/q}\|G - G_k\|_2 + (1 + \zeta^{1/q})\frac{1}{\sqrt{m}}\,\mathrm{tr}(G)$$

Frobenius norm

- sample columns uniformly at random without replacement

$$\mathbb{E}\|G - \hat{G}\|_F \leq 2\zeta_F\|G - G_k\|_F + \left(1 + \frac{4\zeta_F}{\sqrt{m}}\right) n(\max_i G_{ii})$$

- sample columns probabilistically

$$\mathbb{E}\|G - \hat{G}\|_F \leq 2\zeta_F\|G - G_k\|_F + \left(1 + \frac{4\zeta_F}{\sqrt{m}}\right) \mathrm{tr}(G)$$

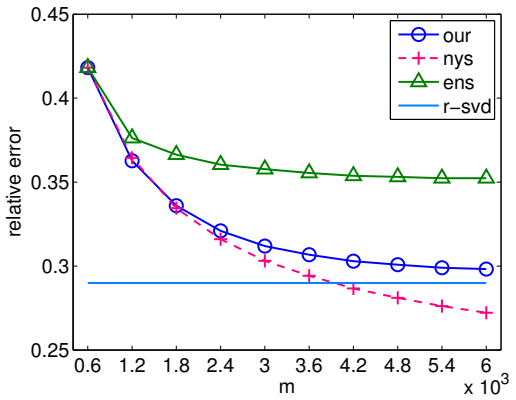
Different Numbers of Columns

- data sets: rcv1, mnist, covtype

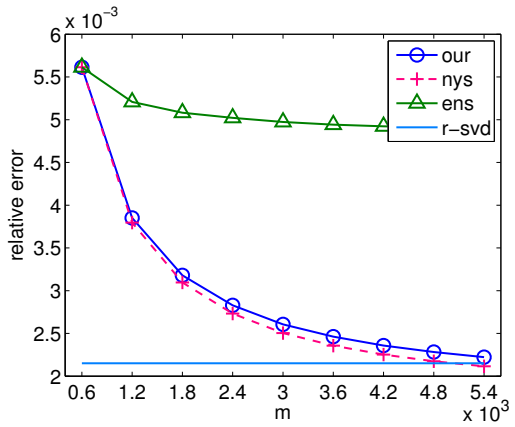
DATA	#SAMPLES	DIM
RCV1	23,149	47,236
MNIST	60,000	784
COVTYPE	581,012	54

- $k = 600$, and gradually increase the number of sampled columns (m)
- all Nyström-based methods have access to the same number of m columns
  - for the ensemble Nyström, $n_e = m/k$
- randomized algorithm: $p = 5, q = 2$

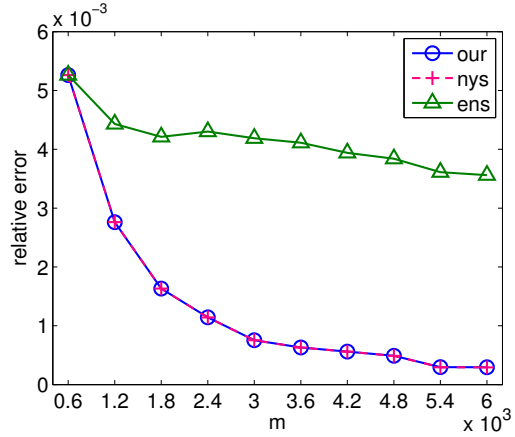
Approximation Error and CPU Time



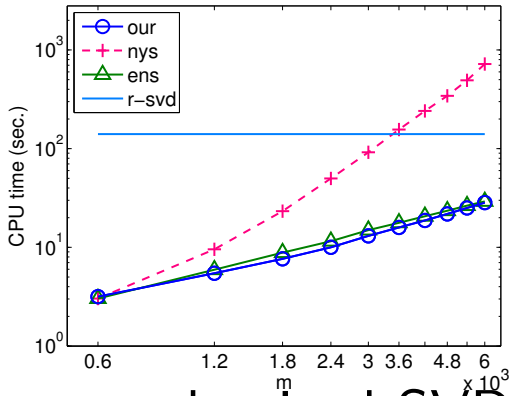
rcv1

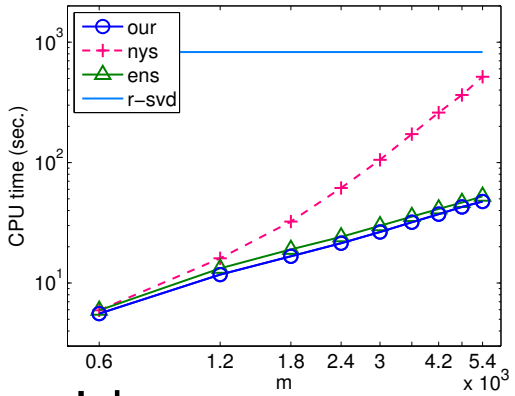


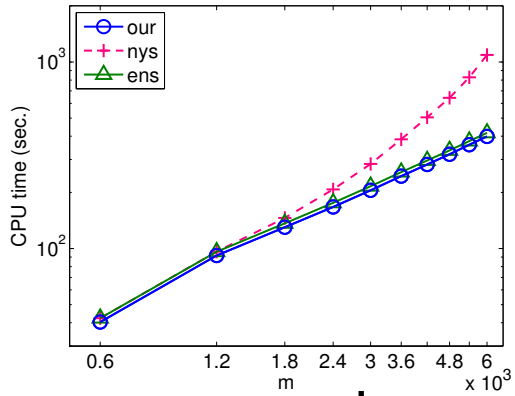
mnist



covtype

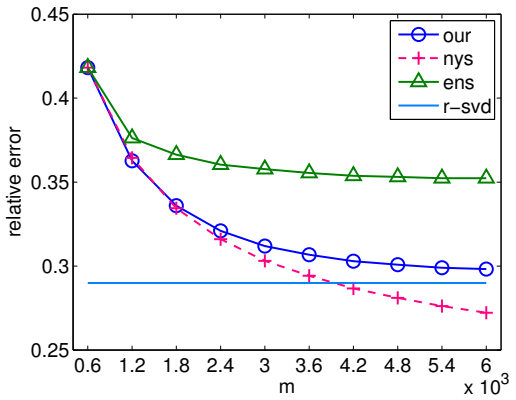




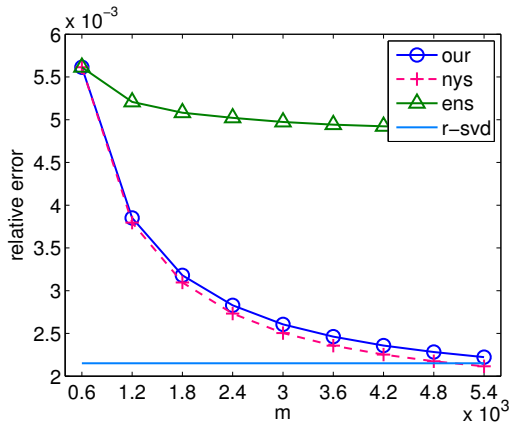


- randomized SVD algorithm: most accurate, most expensive
- standard Nyström
  - as accurate as randomized SVD (when m is large enough)
  - quickly becomes computationally infeasible

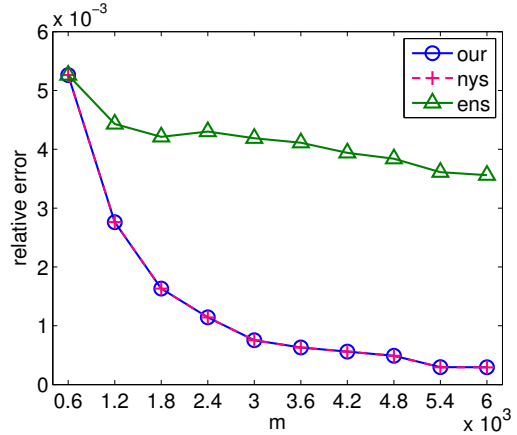
Approximation Error and CPU Time



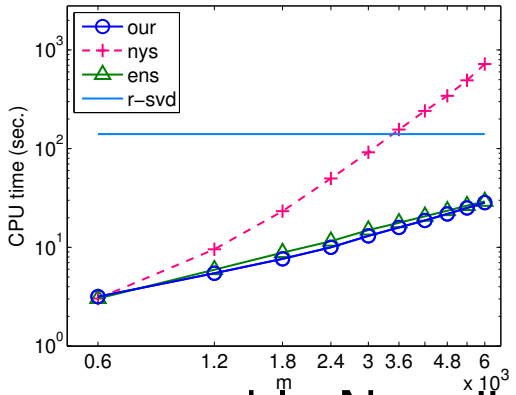
rcv1

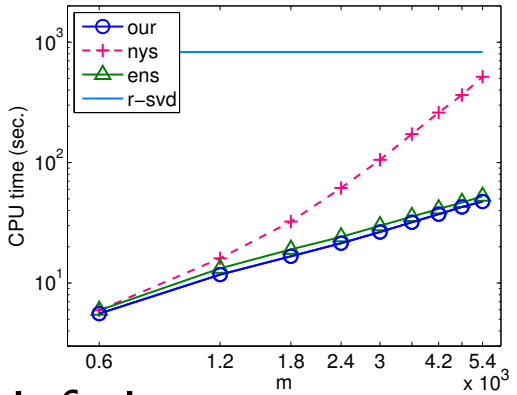


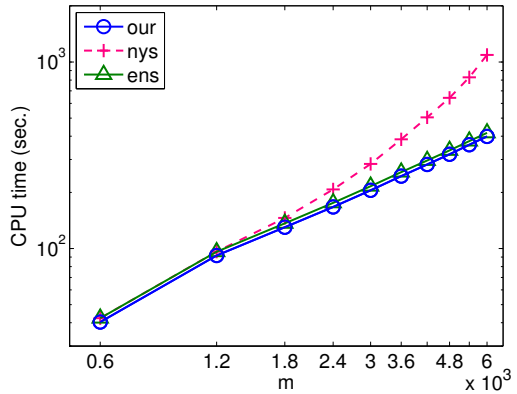
mnist



covtype



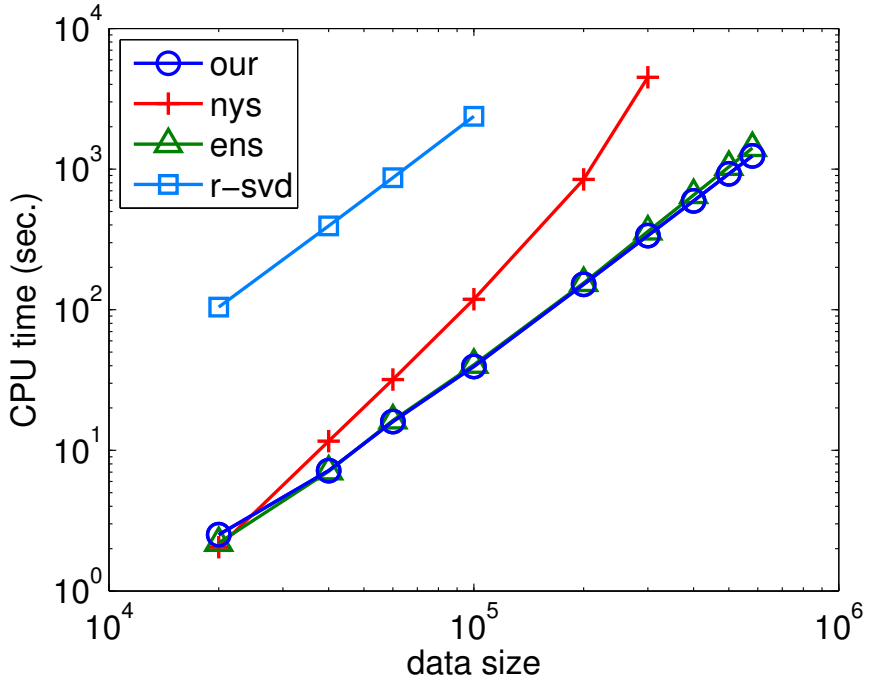




- ensemble Nyström: inferior accuracy
- proposed method
  - almost as accurate as standard Nyström
  - CPU time comparable or even smaller than ensemble Nyström

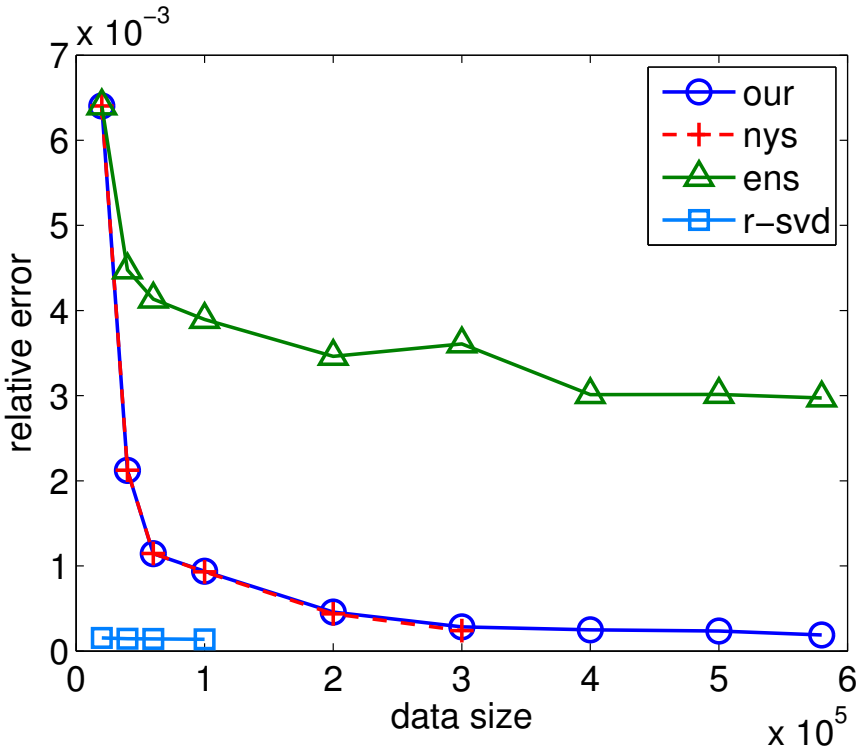
Scaling Behavior (Time)

- "covtype"; rank $k = 600$; #columns $m = 0.03n$
- log-log plot



- standard Nyström method scales cubically with n
- others (including ours) scale quadratically (note: m also scales linearly with n)

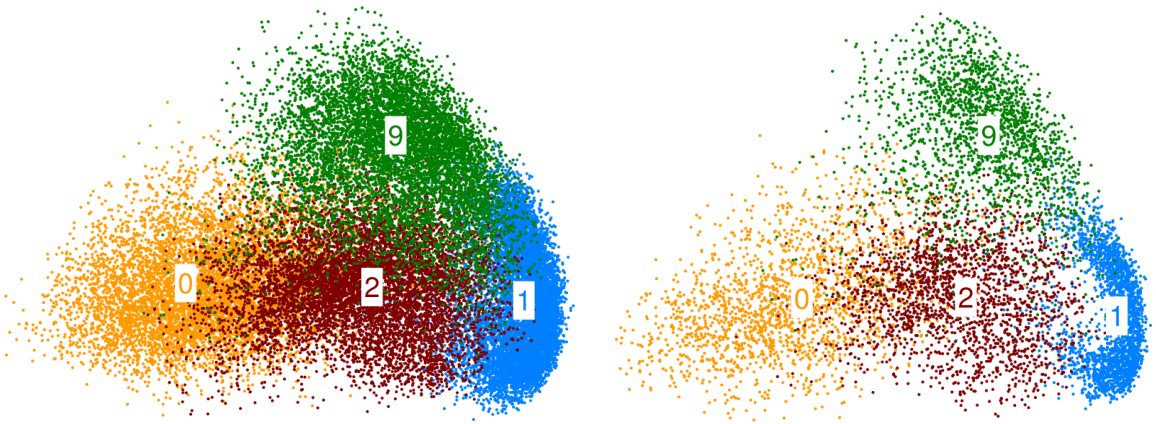
Scaling Behavior (Accuracy)



- proposed method is as accurate as the standard Nyström that performs a large SVD

Spectral Embedding

- Laplacian eigenmap
- digits 0, 1, 2 and 9 of MNIST: about 3.3M samples
  - neither standard SVD nor Nyström can be run on the whole set
  - for comparison, standard SVD on a random subset of 8,000 samples
- data projected onto the 2d space



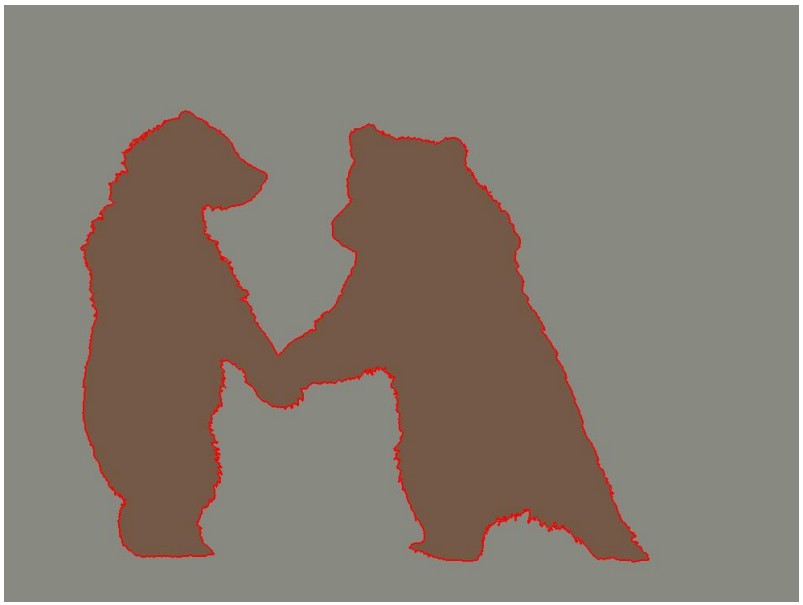
- embedding of the proposed method is obtained within an hour on a PC

Spectral Clustering

- eigen-decompose $\mathbf{D}^{-\frac{1}{2}}\mathbf{W}\mathbf{D}^{-\frac{1}{2}}$
- MNIST data
- compare with
  1. standard spectral clustering [Fowlkes *et al.*, PAMI-2004]
  2. parallel spectral clustering (PSC) [Chen *et al.*, PAMI-2010]
     - single machine
  3. k-means-based spectral clustering (KASP) [Yan *et al.*, KDD-2009] (implemented in R)

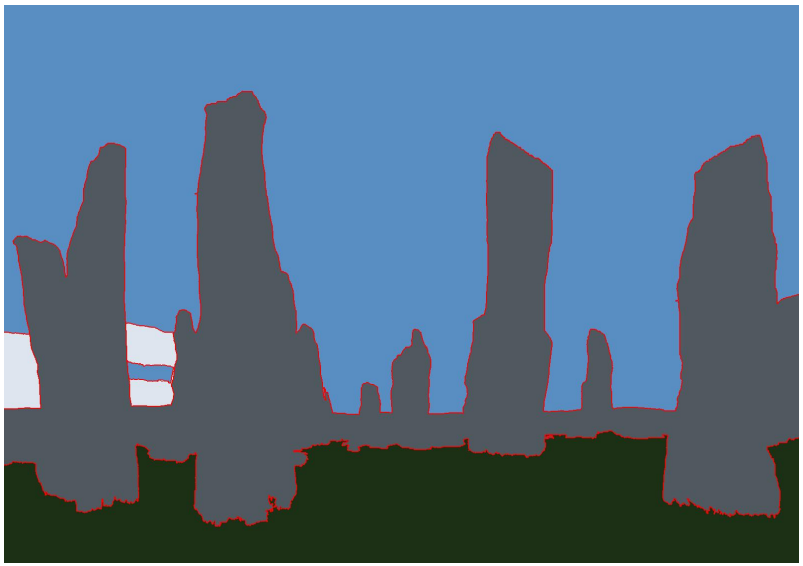
size	method	accuracy (%)	time (sec)
35,735	standard	81.37 ± 0.15	289.4
	PSC	80.47 ± 0.13	41.2
	KASP	59.08 ± 6.25	40 min
	ours	82.55 ± 1.97	5.4
4,130,460	standard	-	-
	PSC (using 10^6 samples)	78.68 ± 0.18	391.8
	KASP	-	-
	ours	77.91 ± 0.78	407.0

Image Segmentation Example



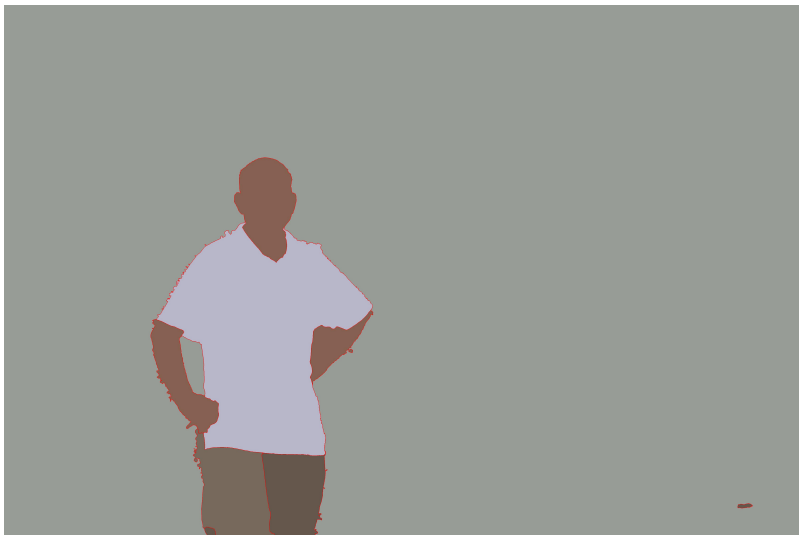
- 989×742 (733,838 pixels)
- 4 sec (excluding the time for generating the affinity matrix)
- Xeon 5440 2.83Ghz CPU, matlab 2009b, 16GB memory

Image Segmentation Example...



- 1600×1122 (1,795,200 pixels)
- 8 sec

Image Segmentation Example...



- 4752×3168 (15,054,336 pixels)
- 30 sec

Graphics Processors (GPU)

- popularly used in entertainment, high-performance computing, etc
- many-core high-performance parallel computing

NVIDIA Tesla C1060:

- 240 streaming processor cores
- peak single-precision (SP) performance: 933 GFLOPS
- peak double-precision (DP) performance: 78 GFLOPS

Intel Core i7-980X CPU

- 6 cores
- peak SP: 158.4 GFLOPS
- peak DP: 79.2 GFLOPS

Proposed Algorithm on GPU

1: $C \leftarrow m$ columns of G sampled uniformly at random without replacement. ▷ m can be large

2: $W \leftarrow m \times m$ matrix

3: $[\tilde{U}, \Lambda] \leftarrow \mathrm{randsvd}(W, k, p, q)$

4: $U \leftarrow C\tilde{U}\Lambda^{+}$. ▷ standard Nyström extension

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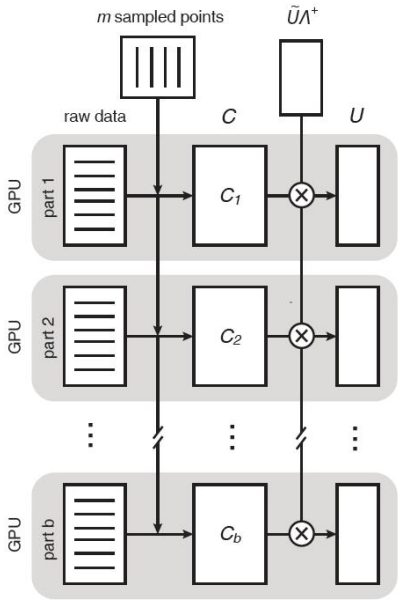
- matrix-matrix multiplication



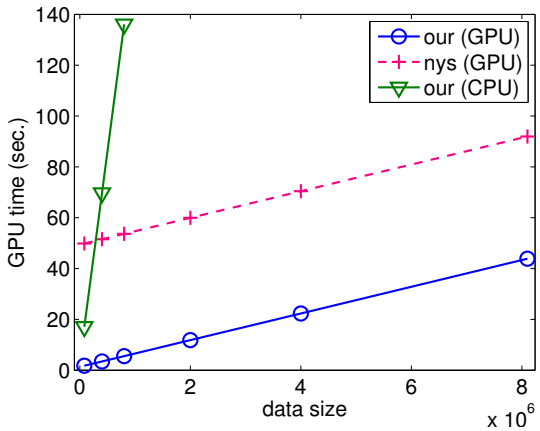
GPU Experiments

Machine

- two Intel Xeon X5560 2.8GHz CPUs, 32G RAM
- four NVIDIA Tesla C1060 GPU cards

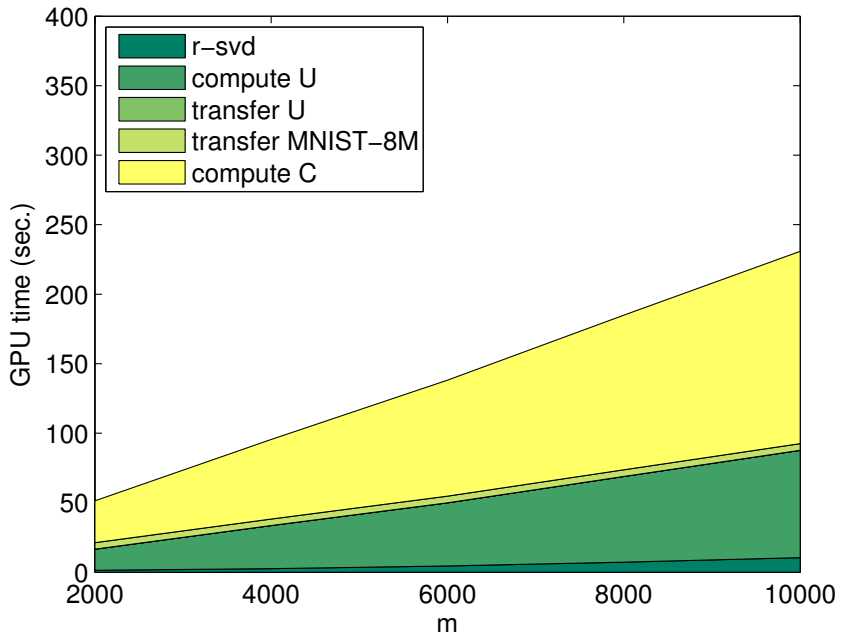
MNIST-8M data

- fixed number of sampling columns ($m = 6000$)



n	CPU (sec)	GPU (sec)	speedup
8×10^4	19	4.9	4x
4×10^5	71	6.8	10x
8×10^4	137	9.0	15x
2×10^6	332	15.9	20x
4×10^6	657	27.2	24x
8.1×10^6	1310	50.0	26x

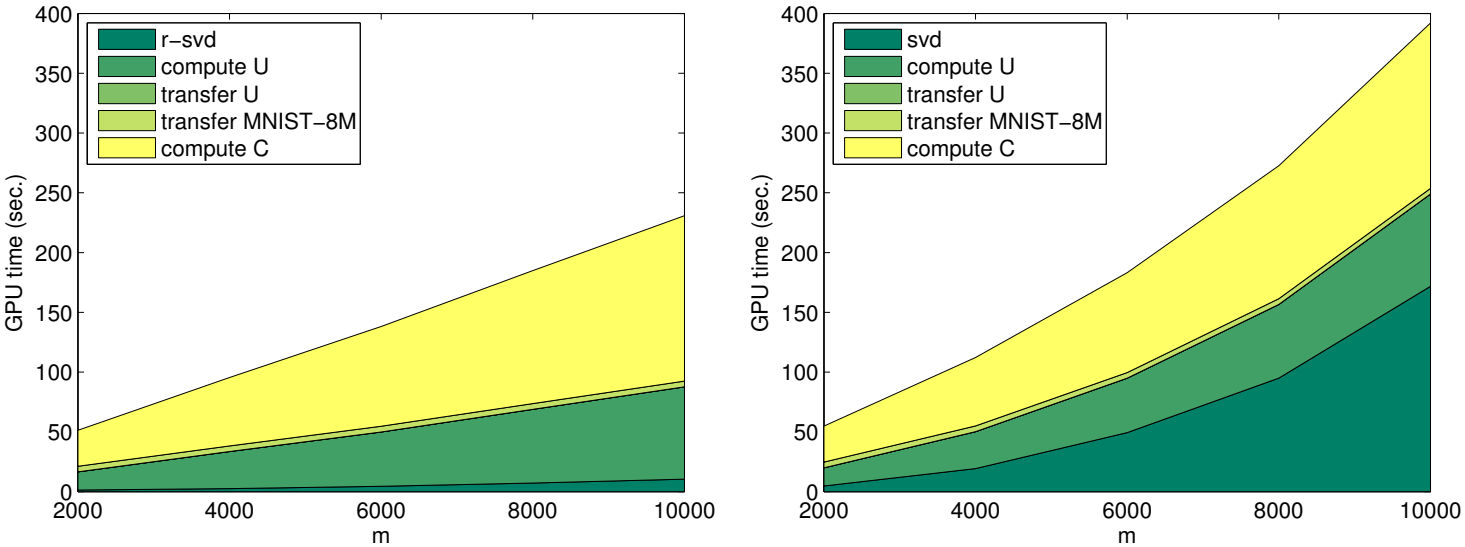
Breakdown of Time



- minimal data transfer
- time spent on data transfer: about 1.5 seconds

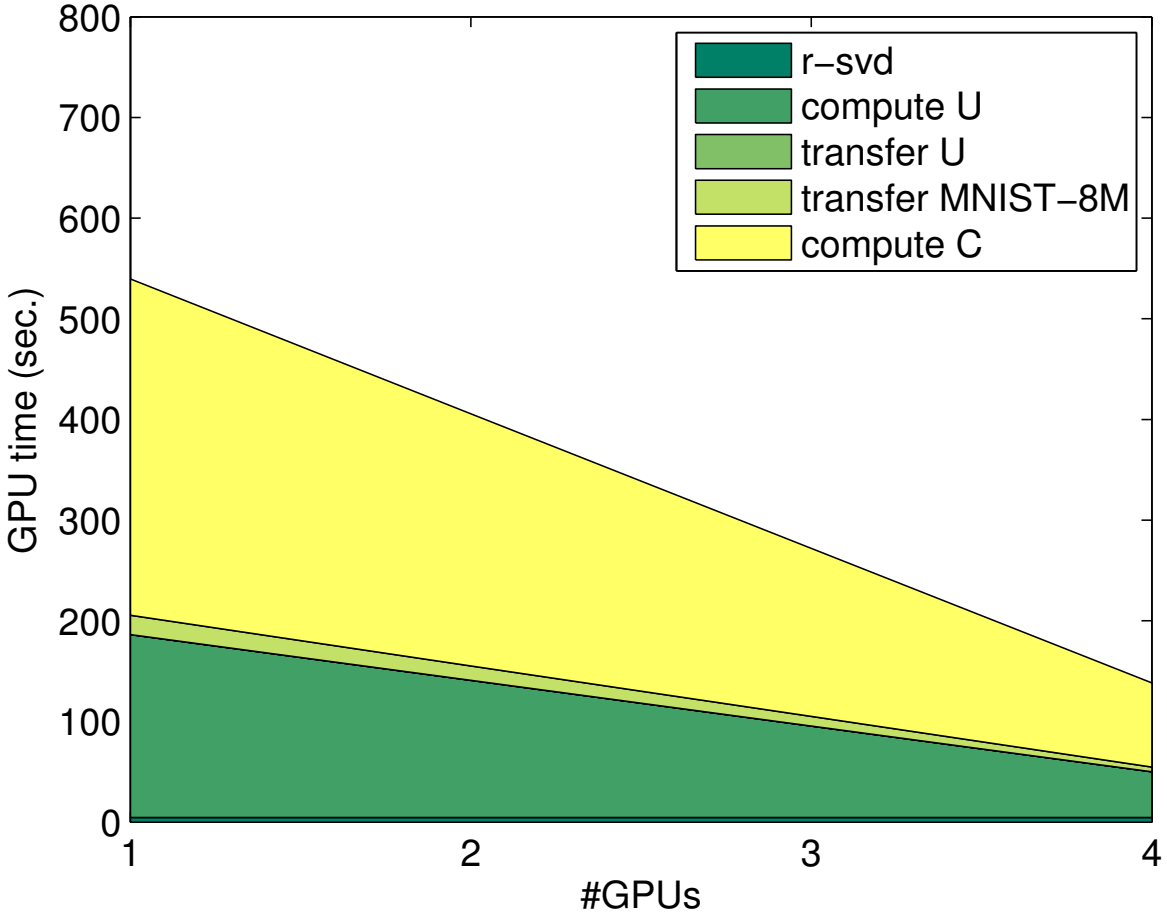
Proposed Method vs Standard Nyström

- standard Nyström can also benefit by running on the GPU
- $k = 600$
- number of sampled columns m varied from 2K to 10K



- standard Nyström: time is dominated by the SVD decomposition step

Varying the Number of GPU Cards



- speedup linear in number of GPU cards

Supervised Learning: Regularized Risk Minimization

minimize *loss* + *regularizer*

Example

- hinge loss $+\|w\|_2^2$ ➔ SVM
- square loss$+\|w\|_2^2$ ➔ ridge regression
- square loss $+\|w\|_1$ ➔ lasso

$$\min_w \mathbb{E}_{XY}[\ell(w; X, Y)] + \lambda\Omega(w)$$

Replace the expectation by its empirical average on a training sample $\{(x_1, y_1), \ldots, (x_m, y_m)\}$

$$\min_w \frac{1}{m}\sum_{i=1}^m \ell(w; x_i, y_i) + \lambda\Omega(w)$$

Typically, both $\ell(\cdot,\cdot)$ and $\Omega(\cdot)$ are convex ➔ convex optimization

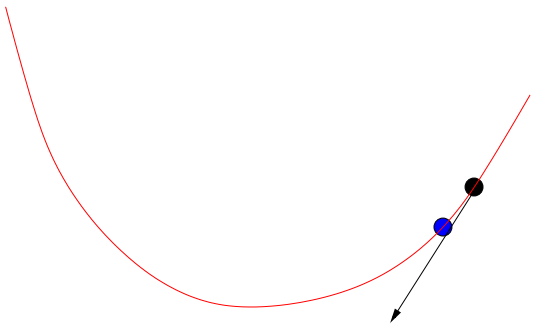
Limitations with Existing Solvers

Example (SVM)

- need to solve a large quadratic program (QP)
- very large data set ➡ very large QP

Back to Basics

Gradient descent $\min_w \ \frac{1}{m}\sum_{i=1}^{m} \ell(w; x_i, y_i) + \lambda\Omega(w)$



LOOP

1. find descent direction
2. choose stepsize
3. descent

Subgradient

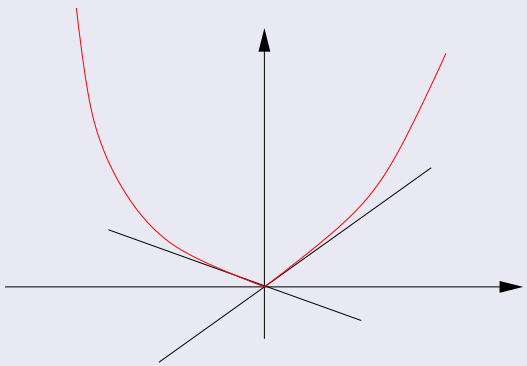
$$\frac{1}{m}\sum_{i=1}^{m}\ell(w;x_i,y_i)+\lambda\Omega(w)$$

Problem

ℓ and/or Ω may be non-smooth (e.g., hinge loss, ℓ_1 regularizer)

Subgradient

- extend gradient to non-smooth functions
- g is a subgradient of f at x iff $f(y) \geq f(x) + g'(y - x)$



Computing the Gradient

$$\nabla_w \left(\frac{1}{m} \sum_{i=1}^{m} \ell(w; x_i, y_i) + \lambda \Omega(w) \right)$$

Another Problem

Computing the gradient using all the training samples may still be costly

Estimates the gradient from a small data subset (mini-batch)

Stochastic Gradient Descent (SGD)

Example

- Pegasos [Shalev-Shwartz et al, ICML-2007]
- FOBOS [Duchi and Singer, NIPS-2009]
- SGD-QN [Bordes et al, JMLR-2009]

Advantages

- easy to implement
- low per-iteration complexity ➡ good scalability

Disadvantage

- uses first-order information
- slow convergence rate ➡ may require a large number of iterations

Accelerated Gradient Methods

First developed by Nesterov in 1983

- deterministic algorithm for smooth optimization

Extension to composite optimization [Nesterov MP-2007]

- objective has both smooth and non-smooth components

Extension to stochastic composite optimization [Lan, TR 2009]

- setting of learning parameters relies on quantities (such as number of iterations, variance of the stochastic subgradient) that are difficult to estimate in practice
- does not consider strong convexity
- cannot produce sparse solutions

Accelerated Gradient Method for Stochastic Learning

$$\min_x \phi(x) \equiv \mathbb{E}[F(x,\xi)] + \psi(x)$$

- ξ: random component
- $f(x) \equiv \mathbb{E}[F(x,\xi)]$: convex and differentiable
  - $G(x_t,\xi_t)$: stochastic gradient of $F(x,\xi_t)$
- $\psi(x)$: convex but possibly non-smooth

Stochastic Accelerated GradiEnt (SAGE)

Input: Sequences $\{L_t\}$ and $\{\alpha_t\}$.

for $t = 0$ to N **do**

$x_t = (1-\alpha_t)y_{t-1} + \alpha_t z_{t-1}$.

$y_t = \arg\min_x \left\{\langle G(x_t,\xi_t), x - x_t\rangle + \frac{L_t}{2}\|x - x_t\|^2 + \psi(x)\right\}$.

$z_t = z_{t-1} - (L_t\alpha_t + \mu)^{-1}[L_t(x_t - y_t) + \mu(z_{t-1} - x_t)]$.

end for

Output y_N.

Special Case: $\alpha_t \equiv 0$

$$x_t = y_{t-1}$$
$$y_t = \arg\min_x \left\{ \langle G(x_t, \xi_t), x - x_t \rangle + \frac{L_t}{2}\|x - x_t\|^2 + \psi(x) \right\}$$

Generalized gradient update

$$x_{t+1} = \arg\min_x \left(f(x_t) + \langle \nabla f(x), x - x_t \rangle + \frac{1}{2\lambda}\|x - x_t\|^2 + \psi(x) \right)$$

- f: smooth; ψ: nonsmooth

$\psi(x) \equiv 0$

$$\arg\min_x \left(f(x_t) + \langle \nabla f(x), x - x_t \rangle + \frac{1}{2\lambda}\|x - x_t\|^2 \right) = x_t - \lambda \nabla f(x_t)$$

- standard gradient descent

In general, $\alpha_t \neq 0$

$$x_t = (1 - \alpha_t) y_{t-1} + \alpha_t z_{t-1}$$
$$y_t = \arg\min_x \left\{ \langle G(x_t, \xi_t), x - x_t \rangle + \frac{L_t}{2}\|x - x_t\|^2 + \psi(x) \right\}.$$
$$z_t = z_{t-1} - (L_t \alpha_t + \mu)^{-1} [L_t(x_t - y_t) + \mu(z_{t-1} - x_t)] \text{ (history)}$$

Efficient Computation of y_t

Can often be efficiently computed with various smooth and non-smooth regularizers

- $\ell_1, \ell_2, \ell_2^2, \ell_\infty$, and matrix norms [Duchi and Singer 2009]

Example ($\psi(x) = \|x\|_1$)

$$\begin{aligned} & \arg\min_x \left(\langle \nabla f(x), x - x_t \rangle + \frac{1}{2\lambda} \|x - x_t\|_2^2 + \|x\|_1 \right) \\ = \quad & S_t(x_t - \lambda \nabla f(x_t)) \end{aligned}$$

where

$$[S_t(y)]_k = \begin{cases} y_k - \lambda & y_k \geq \lambda \\ 0 & -\lambda \leq y_k \leq \lambda \\ y_k + \lambda & y_k \leq -\lambda \end{cases}$$

(soft thresholding)

Convergence Analysis

$$\min_x \phi(x) \equiv \mathbb{E}[F(x,\xi)] + \psi(x)$$

$\phi(x)$ is convex

Set $L_t = b(t+1)^{\frac{3}{2}} + L, \alpha_t = \frac{2}{t+2}$, where $b > 0$ is a constant.

$$\mathbb{E}[\phi(y_N)] - \phi(x^*) \leq \frac{3D^2L}{N^2} + \left(3D^2b + \frac{5\sigma^2}{3b}\right)\frac{1}{\sqrt{N}}.$$

- gradient of $f(x)$: L-Lipschitz

$\phi(x)$ is μ-strongly convex

Set $\lambda_0 = 1$. For $t \geq 1$, set $L_t = L + \mu\lambda_{t-1}^{-1}$ and

$\alpha_t = \sqrt{\lambda_{t-1} + \frac{\lambda_{t-1}^2}{4}} - \frac{\lambda_{t-1}}{2}$, where $\lambda_t \equiv \Pi_{k=1}^t(1-\alpha_t)$.

$$\mathbb{E}[\phi(y_N)] - \phi(x^*) \leq \frac{2(L+\mu)D^2}{N^2} + \frac{6\sigma^2}{N\mu}.$$

Error Bounds

The error bounds consist of two terms

$$\mathbb{E}[\phi(y_N)] - \phi(x^*) \leq \frac{3D^2L}{N^2} + \left(3D^2b + \frac{5\sigma^2}{3b}\right)\frac{1}{\sqrt{N}}$$

$$\mathbb{E}[\phi(y_N)] - \phi(x^*) \leq \frac{2(L+\mu)D^2}{N^2} + \frac{6\sigma^2}{N\mu}$$

- faster term: related to the smooth component
  - $\mathcal{O}(\frac{1}{N^2})$: optimal convergence rate for smooth optimization
- slower term: related to the stochastic / non-smooth component
  - $\mathcal{O}(\frac{1}{\sqrt{N}})$: optimal convergence rate for (stochastic) non-smooth optimization

SAGE uses the structure of the problem and accelerates the convergence of the smooth component

Remarks

1. Unlike previous algorithms, setting of L_t and α_t does not require knowledge of σ and the number of iterations
2. With a sparsity-promoting $\psi(x)$, SAGE can produce a sparse solution
   - y_t reduces to a soft thresholding step
   - some other algorithms: output is a combination of two variables ➡ adding two vectors is unlikely to produce a sparse vector

Accelerated Gradient Method for Online Learning

$$\min_x \sum_{t=1}^{N} \phi_t(x) \equiv \sum_{t=1}^{N} (f_t(x) + \psi(x))$$

SAGE-based Online Learning Algorithm

Inputs: Sequences $\{L_t\}$ and $\{\alpha_t\}$, where $L_t > L$ and $0 < \alpha_t < 1$.

Initialize: $z_1 = y_1$.

loop

$x_t = (1 - \alpha_t) y_{t-1} + \alpha_t z_{t-1}$.

Output

$y_t = \arg\min_x \left\{ \langle \nabla f_{t-1}(x_t), x - x_t \rangle + \frac{L_t}{2} \|x - x_t\|^2 + \psi(x) \right\}$.

$z_t = z_{t-1} - \alpha_t (L_t + \mu\alpha_t)^{-1} [L_t(x_t - y_t) + \mu(z_{t-1} - x_t)]$.

end loop

- Regret bounds can be obtained

Experiments

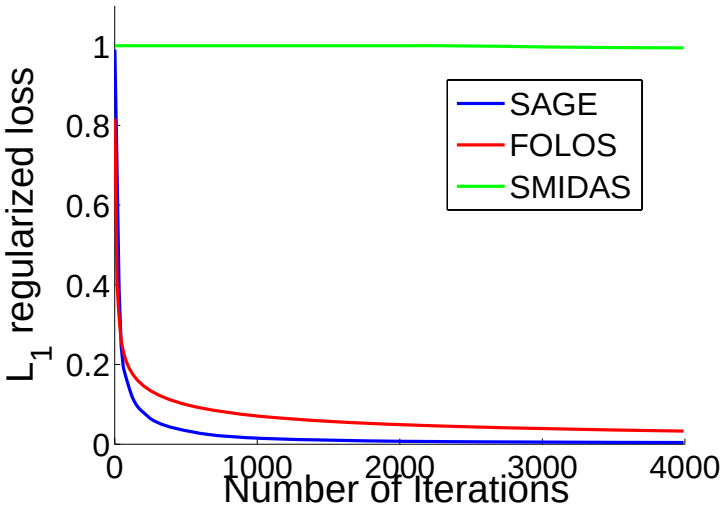
Stochastic optimization of $\min_w \mathbb{E}_{XY}[\ell(w; X, Y)] + \lambda\Omega(w)$

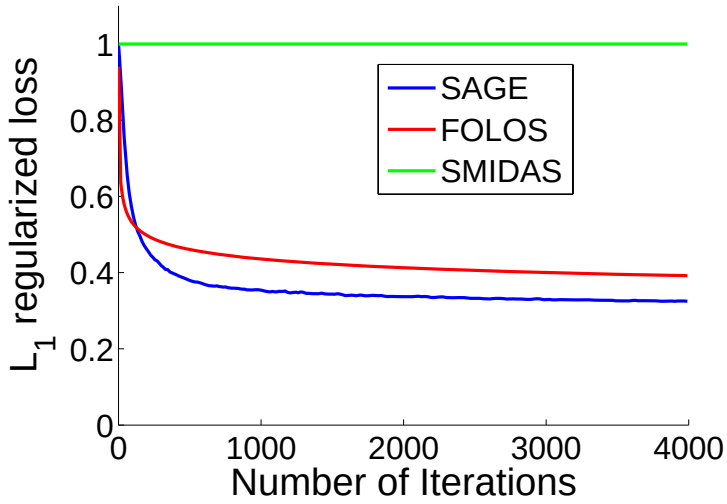
- ℓ: square loss; Ω: ℓ_1 regularizer
  - generalized gradient update can be efficiently computed by soft thresholding

data set	#features	#instances
pcmac	7,511	1,946
RCV1	47,236	193,844

  - pcmac: subset of the 20-newsgroup data set
  - RCV1: Reuters RCV1
- Compare with
  1. FOBOS [Duchi and Singer, NIPS-2009]
  2. SMIDAS [Shalev-Shwartz and Tewari, ICML-2009]
  3. SCD [Shalev-Shwartz and Tewari, ICML-2009]
- Subgradient is computed from mini-batch

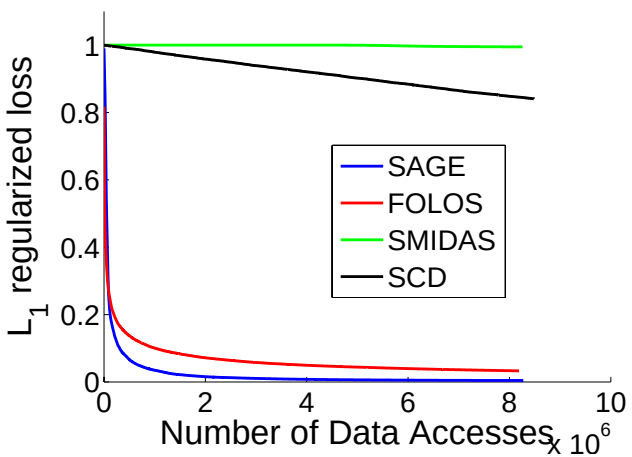
Convergence (Number of Iterations)

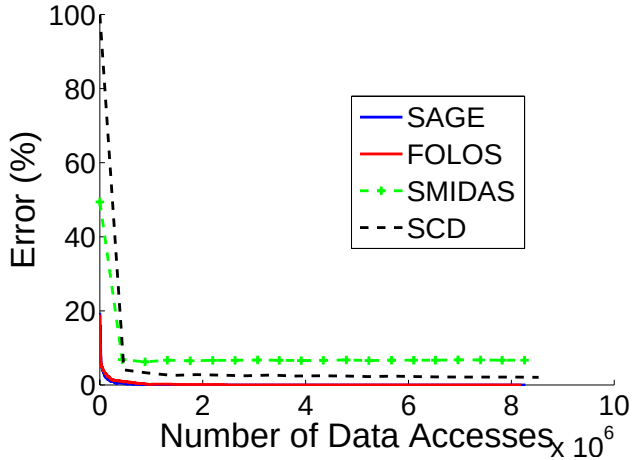


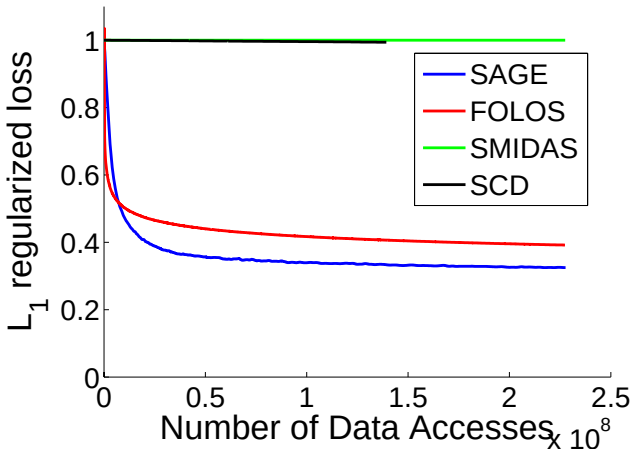


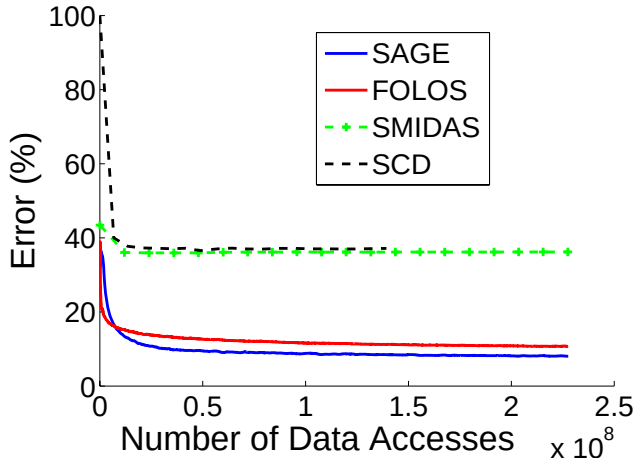
- SAGE requires much fewer iterations for convergence than the others

Convergence (Number of Data Access Operations)









- SAGE is still the fastest
  - the most expensive step is the generalized gradient update
  - per-iteration complexity is comparable with others

Conclusion: Approximation of Large Eigen-systems

Nyström method with randomized SVD

- samples a large column subset from the input matrix
- performs approximate SVD on the inner submatrix by using randomized low-rank matrix approximation algorithm
- as accurate as standard Nyström method that directly performs a large SVD on the inner submatrix
- time complexity is only as low as the ensemble Nyström method
- can be used for spectral clustering on very large images
- further speed up with GPU possible

(*Making large-scale Nyström approximation possible*. M. Li, J.T. Kwok, B. Lu. *ICML 2010*)

Conclusion: Accelerated Gradient Algorithm (SAGE)

Scalable stochastic convex composite optimization solver

- enjoys the computational simplicity and scalability of traditional subgradient methods
- convergence rate: utilizes the problem structure and accelerates the convergence of the smooth component
- per-iteration cost: same as standard subgradient methods
- empirically, SAGE outperforms recent subgradient methods

(*Accelerated gradient methods for stochastic optimization and online learning*. C. Hu, J.T. Kwok, W. Pan. *NIPS 2009*)