# Introduction

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# Outline

- Mathematical Optimization
- Least-squares
- Linear Programming
- Convex Optimization
- Nonlinear Optimization
- Summary

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# Mathematical Optimization (1)

- Optimization Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \ldots, m \end{aligned}$$

  - Optimization Variable: $x = (x_1, \ldots, x_n)$
  - Objective Function: $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$
  - Constraint Functions: $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$
- $x^\star$ is called optimal or a solution
  - $f_i(x^\star) \le b_i$, $i = 1, \ldots, m$
  - For any $z$ with $f_i(z) \le b_i$, we have $f_0(z) \ge f_0(x^*)$

# Mathematical Optimization (2)

- Linear Program

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)$$

  - for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$

- Nonlinear Program
  - If the optimization problem is not linear

- Convex Optimization Problem

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  - for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

# Applications

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \ldots, m \end{aligned}$$

- Abstraction
  - $x$ represents the choice made
  - $f_i(x) \le b_i$ represent firm requirements that limit the possible choices
  - $f_0(x)$ represents the cost of choosing $x$
  - A solution corresponds to a choice that has minimum cost, among all choices that meet the requirements

# Portfolio Optimization

- Variables
  - $x_i$ represents the investment in the $i$-th asset
  - $x \in \mathbf{R}^n$ describes the overall portfolio allocation across the set of asset
- Constraints
  - A limit on the budget the requirement
  - Investments are nonnegative
  - A minimum acceptable value of expected return for the whole portfolio
- Objective
  - Minimize the variance of the portfolio return

# Device Sizing

- Variables
  - $x \in \mathbf{R}^n$ describes the widths and lengths of the devices
- Constraints
  - Limits on the device sizes
  - Timing requirements
  - A limit on the total area of the circuit
- Objective
  - Minimize the total power consumed by the circuit

# Data Fitting

- Variables
  - $x \in \mathbf{R}^n$ describes parameters in the model
- Constraints
  - Prior information
  - Required limits on the parameters (such as nonnegativity)
- Objective
  - Minimize the prediction error between the observed data and the values predicted by the model

# Solving Optimization Problems

- General Optimization Problem
  - Very difficult to solve
  - Constraints can be very complicated, the number of variables can be very large
  - Methods involve some compromise, e.g., computation time, or suboptimal solution
- Exceptions
  - Least-squares problems
  - Linear programming problems
  - Convex optimization problems

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# Least-squares Problems (1)

## The Problem

$$\min \quad \|Ax - b\|_2^2 = \sum_{i=1}^{k} \left(a_i^\top x - b_i\right)^2$$

- $A \in \mathbf{R}^{k \times n}$, $a_i^\top$ is the $i$-th row of $A$, $b \in \mathbf{R}^k$
- $x \in \mathbf{R}^n$ is the optimization variable

How to solve it?

# Least-squares Problems (1)

- The Problem

$$\min \quad \|Ax-b\|_2^2 = \sum_{i=1}^{k} \left(a_i^\top x - b_i\right)^2$$

  - $A \in \mathbf{R}^{k\times n}$, $a_i^\top$ is the $i$-th row of $A$, $b \in \mathbf{R}^k$
  - $x \in \mathbf{R}^n$ is the optimization variable

- Setting the gradient to be 0

$$\begin{aligned} & 2A^\top(Ax-b) = 0 \\ \Longrightarrow \quad & A^\top A x = A^\top b \\ \Longrightarrow \quad & x = (A^\top A)^{-1} A^\top b \end{aligned}$$

# Least-squares Problems (2)

- A Set of Linear Equations

$$A^\top A x = A^\top b$$

- Solving least-squares problems
  - Reliable and efficient algorithms and software
  - Computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
  - A mature technology
  - Challenging for extremely large problems

# Using Least-squares

- Easy to Recognize
- Weighted least-squares

$$\sum_{i=1}^{k} w_i \left(a_i^\top x - b_i\right)^2$$

  - Different importance

# Using Least-squares

- Easy to Recognize
- Weighted least-squares

$$\sum_{i=1}^{k} w_i (a_i^\top x - b_i)^2 = \sum_{i=1}^{k} (\sqrt{w_i} a_i^\top x - \sqrt{w_i} b_i)^2$$

  - Different importance
- Regularization

$$\sum_{i=1}^{k} (a_i^\top x - b_i)^2 + \rho \sum_{i=1}^{n} x_i^2$$

  - More stable

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# Linear Programming

## The Problem

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \le b_i, \qquad i = 1, \ldots, m \end{aligned}$$

- $c, a_1, \ldots, a_m \in \mathbf{R}^n$, $b_1, \ldots, b_m \in \mathbf{R}$

## Solving Linear Programs

- No analytical formula for solution
- Reliable and efficient algorithms and software
- Computation time proportional to $n^2 m$ if $m \ge n$; less with structure
- A mature technology
- Challenging for extremely large problems

# Using Linear Programming

- Not as easy to recognize
- Chebyshev Approximation Problem

$$\min \quad \max_{i=1,\ldots,k} \left| a_i^\top x - b_i \right|$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & t = \max_{i=1,\ldots,k} \left| a_i^\top x - b_i \right| \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & t \geq \left| a_i^\top x - b_i \right|, i = 1, \ldots, k \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & -t \leq a_i^\top x - b_i \leq t, i = 1, \ldots, k \end{array}$$

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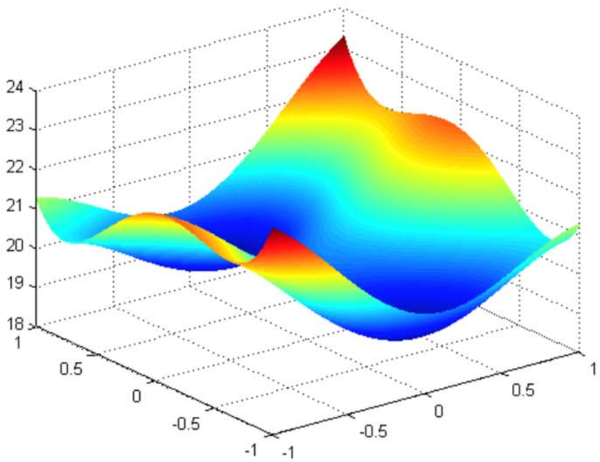
# Convex Optimization

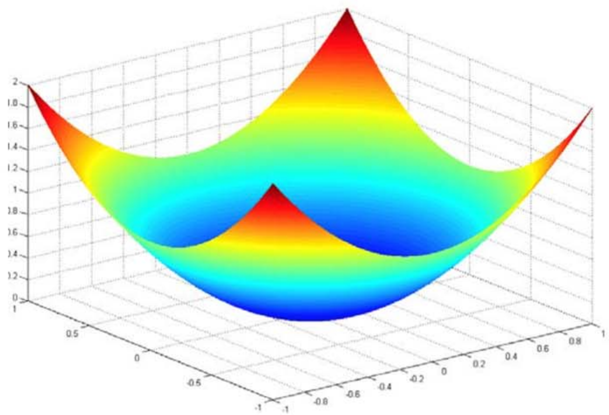
## □ Why Convexity?

“ The great watershed in optimization isn’t between linearity and nonlinearity, but convexity and nonconvexity.”

— R. Rockafellar, SIAM Review 1993




**Non-Convex Optimization**



**Convex Optimization**

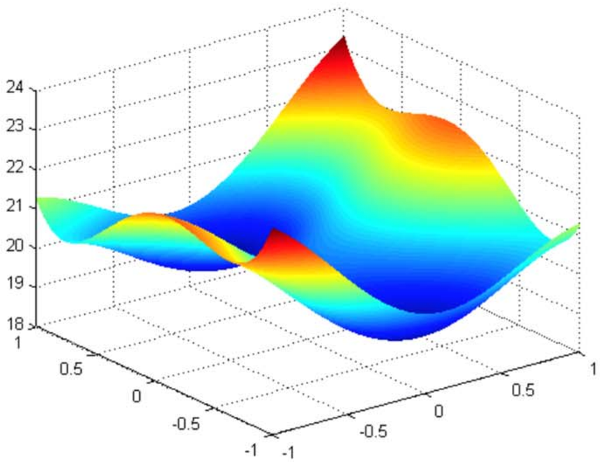
# Convex Optimization

## ☐ Why Convexity?

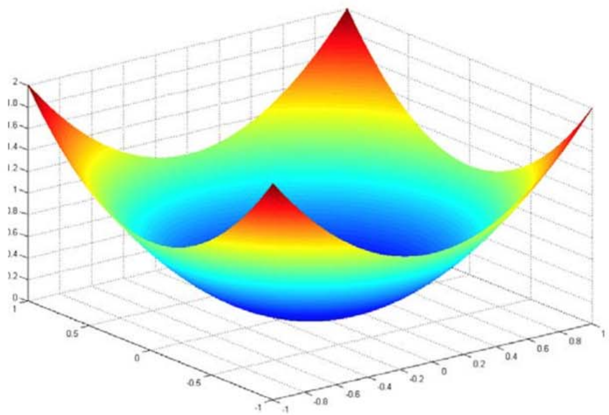
" The great watershed in optimizati… nd nonlinearity, but convexity and no…

— R. … 93

Local minimizers are also global minimizers.




**Non-Convex Optimization**



**Convex Optimization**

# Convex Optimization Problems (1)

## The Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \ldots, m \end{aligned}$$

- Functions $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$ are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1,\ \alpha \ge 0,\ \beta \ge 0$

- Least-squares and linear programs as special cases

# Convex Optimization Problems (2)

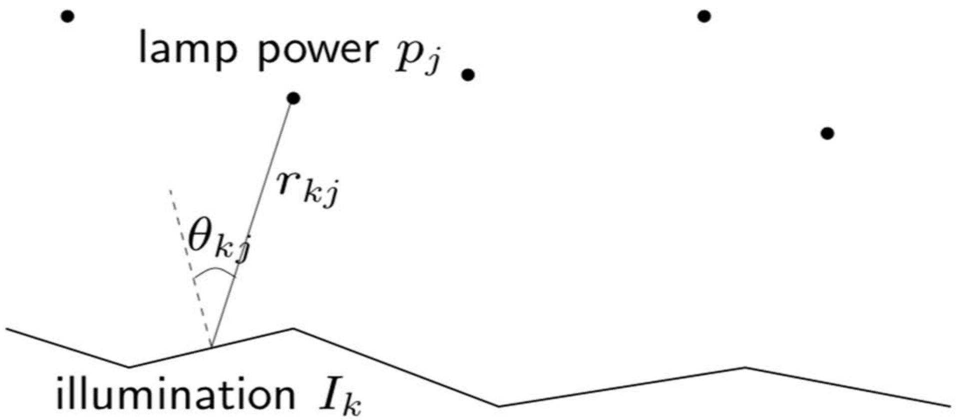
- Solving Convex Optimization Problems
  - No analytical solution
  - Reliable and efficient algorithms (e.g., interior-point methods)
  - Computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$
    - ✓ $F$ is cost of evaluating the first and second derivatives of $f_i'$s
  - Almost a technology

# Using Convex Optimization

- Often difficult to recognize
- Many tricks for transforming problems into convex form
- Surprisingly many problems can be solved via convex optimization

# An Example (1)

- $m$ lamps illuminating $n$ patches




  - Intensity $I_k$ at patch $k$ depends linearly on lamp powers $p_j$

$$I_k = \sum_{j=1}^{m} a_{kj} p_j, \qquad a_{kj} = r_{kj}^{-2} \max\{\cos\theta_{kj}, 0\}$$

# An Example (2)

- Achieve desired illumination $I_{\text{des}}$ with bounded lamp powers

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

How to solve it?

# An Example (3)

1. Use uniform power: $p_j = p$, vary $p$
2. Use least-squares

$$\min \sum_{i=1}^{k} (I_k - I_{\text{des}})^2 = \sum_{i=1}^{k} \left( \sum_{j=1}^{m} a_{kj} p_j - I_{\text{des}} \right)^2$$

- Round $p_j$ if $p_j > p_{\max}$ or $p_j < 0$

3. Use weighted least-squares

$$\min \sum_{i=1}^{k} (I_k - I_{\text{des}})^2 + \sum_{j=1}^{m} w_j \left( p_j - \frac{p_{\max}}{2} \right)^2$$

- Adjust weights $w_j$ until $0 \le p_j \le p_{\max}$

# An Example (4)

4. Use linear programming

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |I_k - I_{\text{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

5. Use convex optimization

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & \max_{k=1,\ldots,n} \left|\log \frac{I_k}{I_{\text{des}}}\right| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

# An Example (5)

$$\Leftrightarrow \quad \begin{array}{ll} \min & \max_{k=1,\ldots,n} \max\left(\log\frac{I_k}{I_{\text{des}}}, -\log\frac{I_k}{I_{\text{des}}}\right) \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1,\ldots,m \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \min & \max_{k=1,\ldots,n} \max\left(\log\frac{I_k}{I_{\text{des}}}, \log\frac{I_{\text{des}}}{I_k}\right) \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1,\ldots,m \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \min & \max_{k=1,\ldots,n} \max\left(\frac{I_k}{I_{\text{des}}}, \frac{I_{\text{des}}}{I_k}\right) \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1,\ldots,m \end{array}$$

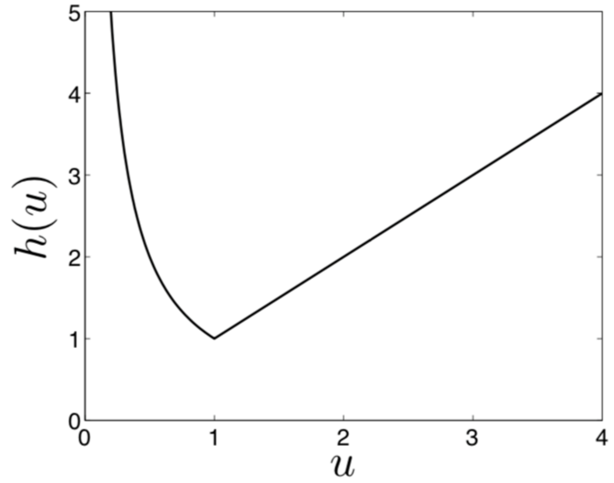
# An Example (5)

$$\Longleftrightarrow \quad \min \quad \max_{k=1,\dots,n} h\left(\frac{I_k}{I_{\text{des}}}\right)$$

$$\text{s.t.} \quad 0 \le p_j \le p_{\max}, j = 1, \dots, m$$

$$I_k = \sum_{j=1}^{m} a_{kj} p_j$$

- $h(u) = \max\left(u, \frac{1}{u}\right)$



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# Nonlinear Optimization

- An optimization problem when the objective or constraint functions are not linear, but not known to be convex
- Sadly, there are no effective methods for solving the general nonlinear programming problem
  - Could be NP-hard
- We need compromise

# Local Optimization Methods

- Find a point that minimizes $f_0$ among feasible points near it
  - The compromise is to give up seeking the optimal $x$
- Fast, can handle large problems
  - Differentiability
- Require initial guess, Provide no information about distance to (global) optimum, Sensitive to parameter values
- Local optimization methods are more art than technology

# Comparisons

| | Problem Formulation | Solving the Problem |
|---|---|---|
| Local Optimization Methods for Nonlinear Programming | Straightforward | **Art** |
| Convex Optimization | **Art** | Standard |

# Global Optimization (1)

- Find the global solution
  - The compromise is efficiency
- Worst-case complexity grows exponentially with problem size
- Applications
  - A small number of variables, where computing time is not critical
  - The value of finding the true global solution is very high

# Global Optimization (2)

- Worst-case Analysis of a high value or safety-critical system
  - Variables represent uncertain parameters
  - Objective function is a utility function
  - Constraints represent prior knowledge
  - If the worst-case value is acceptable, we can certify the system as safe or reliable
- Local optimization methods can be tried
  - If finding values that yield unacceptable performance, then the system is not reliable
  - But it cannot certify the system as reliable

# Role of Convex Optimization in Nonconvex Problems

- Initialization for local optimization
  - An approximate, but convex, formulation
- Convex heuristics for nonconvex optimization
  - Sparse solutions (compressive sensing)
- Bounds for global optimization
  - Relaxation
  - Lagrangian relaxation

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- Mathematical Optimization
- Least-squares
  - Closed-form Solution
- Linear Programming
  - Efficient algorithms
- Convex Optimization
  - Efficient algorithms, Modeling is an art
- Nonlinear Optimization
  - Compromises, Optimization is an art