

Convex Sets

Lijun Zhang

zlj@nju.edu.cn

<http://cs.nju.edu.cn/zlj>





Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary



Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

Line

□ Lines

$$y = \theta x_1 + (1 - \theta)x_2$$

$$y = x_2 + \theta(x_1 - x_2)$$

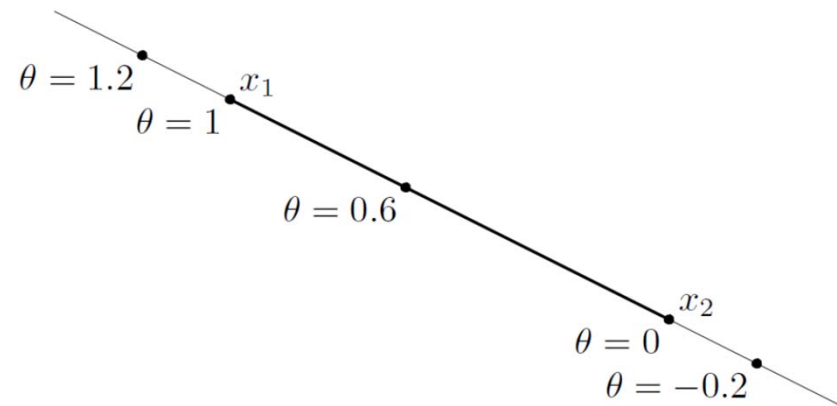
- $\theta \in \mathbf{R}$

- $x_1 \neq x_2$

□ Line segments

- $\theta \in [0,1]$

- $x_1 \neq x_2$





Affine Sets (1)

□ Definition

- $C \subseteq \mathbf{R}^n$ is affine, if

$$\theta x_1 + (1 - \theta)x_2 \in C$$

for any $x_1, x_2 \in C$ and $\theta \in \mathbf{R}$

□ Generalized form

- Affine Combination

$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$$

- $\theta_1 + \theta_2 + \cdots + \theta_k = 1$



Affine Sets (2)

□ Subspace

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

- $C \subseteq \mathbf{R}^n$ is an affine set, $x_0 \in C$
- Subspace is closed under sums and scalar multiplication

$$\alpha v_1 + \beta v_2 \in V, \quad \forall v_1, v_2 \in V$$

- C can be expressed as a subspace plus an offset $x_0 \in C$

$$C = V + x_0$$

- Dimension of C : dimension of V



Affine Sets (3)

- Solution set of linear equations is affine

$$C = \{x | Ax = b\}$$

- Suppose $x_1, x_2 \in C$

$$\begin{aligned} A(\theta x_1 + (1 - \theta)x_2) &= \theta Ax_1 + (1 - \theta)Ax_2 \\ &= \theta b + (1 - \theta)b \\ &= b \end{aligned}$$

- Every affine set can be expressed as the solution set of a system of linear equations.



Affine Sets (4)

□ Affine hull of set C

$$\text{aff } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \dots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

- Affine hull is the smallest affine set that contains C

□ Affine dimension

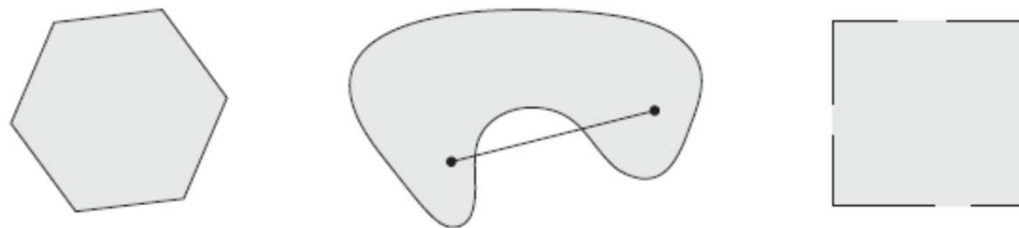
- Affine dimension of a set C as the dimension of its affine hull $\text{aff } C$
- Consider the unit circle $B = \{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\}$, $\text{aff } B$ is \mathbf{R}^2 . So affine dimension is 2

Convex Sets (1)

□ Convex sets

- A set \mathcal{C} is convex if for any $x_1, x_2 \in \mathcal{C}$, any $\theta \in [0,1]$, we have

$$\theta x_1 + (1 - \theta)x_2 \in \mathcal{C}$$



□ Generalized form

- Convex combination

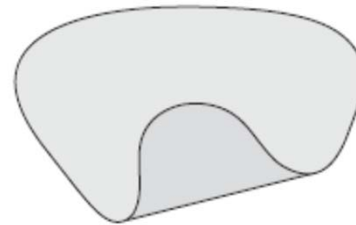
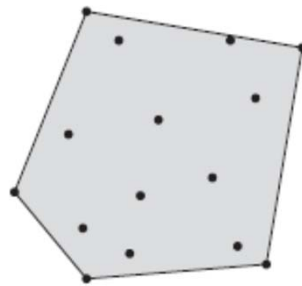
$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in \mathcal{C}$$

$$\theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \cdots, k$$

Convex Sets (2)

□ Convex hull

$$\text{conv } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \dots, k\}$$



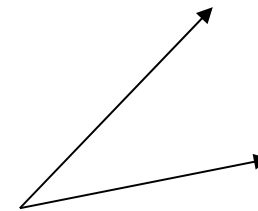
□ Infinite sums, integrals

Cone (1)

□ Cone

- Cone is a set that

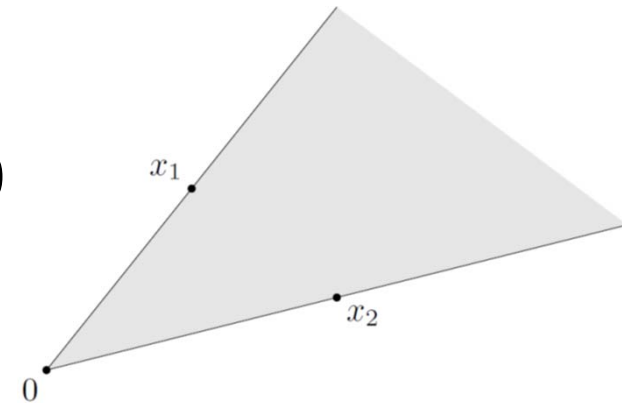
$$x \in C, \theta \geq 0 \Rightarrow \theta x \in C$$



□ Convex cone

- For any $x_1, x_2 \in C, \theta_1, \theta_2 \geq 0$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$



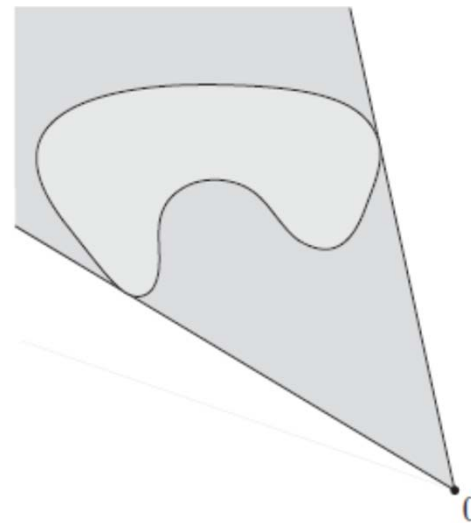
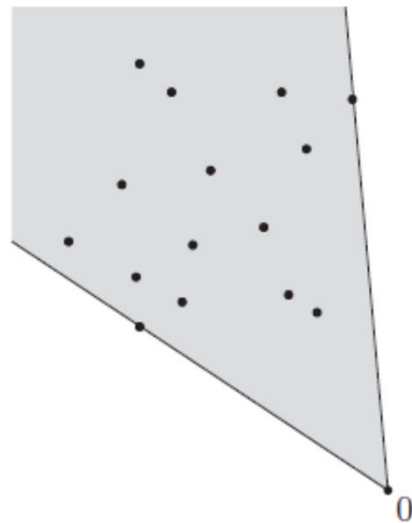
□ Conic combination

- $\theta_1 x_1 + \cdots + \theta_k x_k, \theta_i \geq 0, i = 1, \cdots, k$

Cone (2)

□ Conic hull

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_i \geq 0, i = 1, \dots, k\}$$





Some Examples

- The empty set \emptyset , any single point $\{x_0\}$, and the whole space \mathbf{R}^n are affine (hence, convex) subsets of \mathbf{R}^n
- Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point).
- A ray, which has the form $\{x_0 + \theta v \mid \theta \geq 0\}$, where $v \neq 0$, is convex, but not affine. It is a convex cone if its base x_0 is 0.
- Any subspace is affine, and a convex cone (hence convex).



Hyperplanes

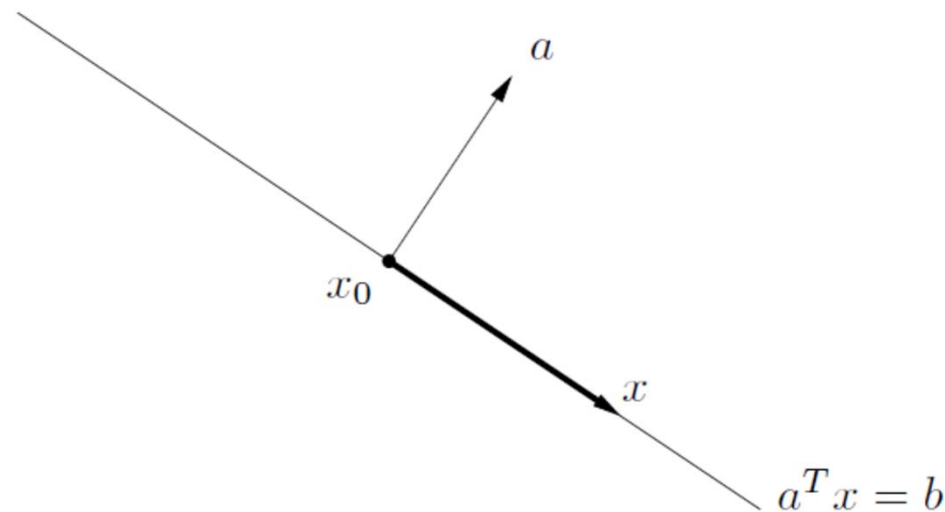
$$\{x | a^T x = b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in \mathbf{R}$

□ Other Forms

$$\{x | a^T (x - x_0) = 0\}$$

- x_0 is any point such that $a^T x_0 = b$





Hyperplanes

$$\{x | a^T x = b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in \mathbf{R}$

□ Other Forms

$$\{x | a^T (x - x_0) = 0\}$$

- x_0 is any point such that $a^T x_0 = b$

$$\{x | a^T (x - x_0) = 0\} = x_0 + a^\perp$$

- $a^\perp = \{v | a^T v = 0\}$



Halfspaces

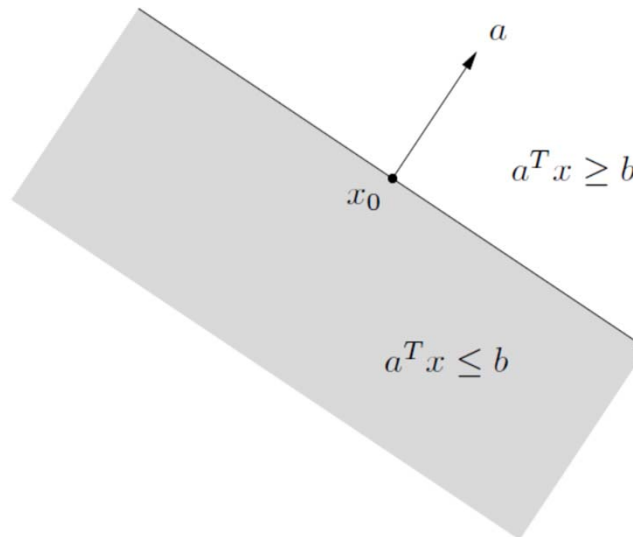
$$\{x | a^T x \leq b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in \mathbf{R}$

□ Other Forms

$$\{x | a^T (x - x_0) \leq 0\}$$

- x_0 is any point such that $a^T x_0 = b$



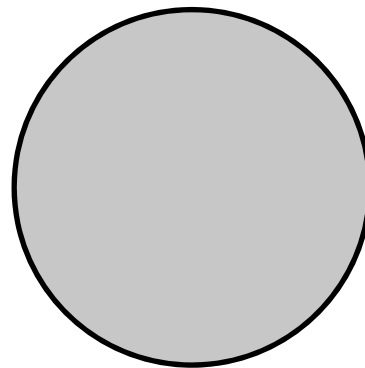
- Convex
- Not affine

Balls

□ Definition

$$\begin{aligned} B(x_c, r) &= \{x \mid \|x - x_c\|_2 \leq r\} \\ &= \{x \mid (x - x_c)^\top (x - x_c) \leq r^2\} \\ &= \{x_c + ru \mid \|u\|_2 \leq 1\} \end{aligned}$$

- $r > 0$, and $\|\cdot\|_2$ denotes the Euclidean norm
- Convex

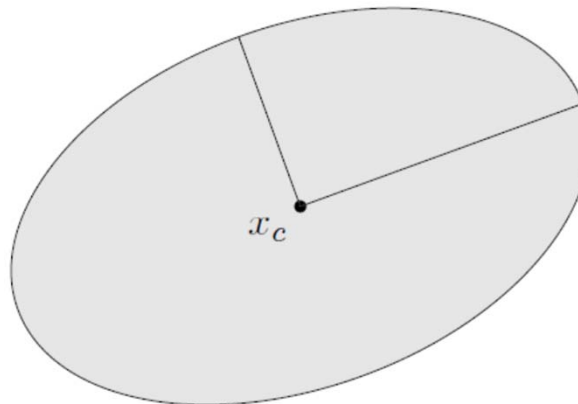


Ellipsoids

□ Definition

$$\begin{aligned}\mathcal{E} &= \{x | (x - x_c)^\top P^{-1} (x - x_c) \leq 1\} \\ &= \{x_c + Au | \|u\|_2 \leq 1\}\end{aligned}$$

- $P \in \mathbf{S}_{++}^n$ determines how far the ellipsoid extends in every direction from x_c ;
- Lengths of semi-axes are $\sqrt{\lambda_i}$
- Convex





Norm Balls and Norm Cones

□ Norm balls

$$\mathcal{C} = \{x \mid \|x - x_c\| \leq r\}$$

- $\|\cdot\|$ is any norm on \mathbf{R}^n , x_c is the center

□ Norm cones

$$\mathcal{C} = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbf{R}^{n+1}$$

- Second-order Cone

$$\begin{aligned}\mathcal{C} &= \{(x, t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \leq t\} \\ &= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq 0, t \geq 0 \right\}\end{aligned}$$



Norm Balls and Norm Cones

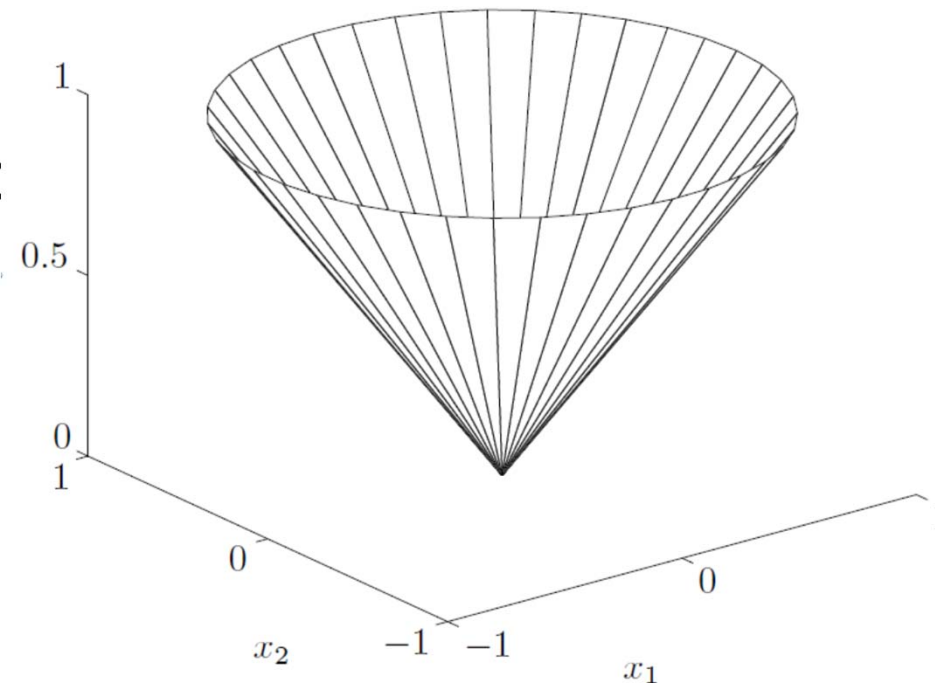
□ Norm balls

$$\mathcal{C} = \{x \mid \|x - x_c\| \leq r\}$$

■ $\|\cdot\|$ is any norm on \mathbf{R}^n , x_c is the center

□ Norm cones

■ Sec





Polyhedra (1)

□ Polyhedron

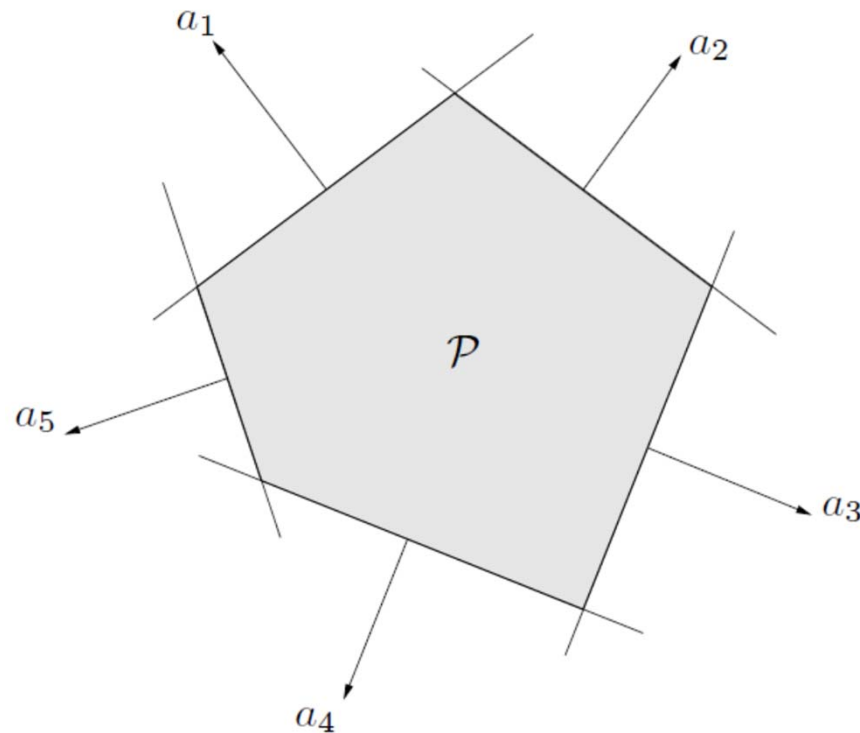
$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$

- Solution set of a finite number of linear equalities and inequalities
- Intersection of a finite number of halfspaces and hyperplanes
- Affine sets (e.g., subspaces, hyperplanes, lines), rays, line segments, and halfspaces are all polyhedra

Polyhedra (2)

□ Polyhedron

$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$





Polyhedra (2)

□ Polyhedron

$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \dots, m, c_j^\top x = d_j, j = 1, \dots, p\}$$

■ Matrix Form

$$\mathcal{P} = \{x \mid Ax \preccurlyeq b, Cx = d\}$$

$$A = \begin{bmatrix} a_1^\top \\ \vdots \\ a_m^\top \end{bmatrix}, \quad C = \begin{bmatrix} c_1^\top \\ \vdots \\ c_p^\top \end{bmatrix}$$

$u \preccurlyeq v$ means $u_i \leq v_i$ for all i



Polyhedron?

Simplexes

□ An important family of **polyhedra**

$$C = \text{conv}\{v_0, \dots, v_k\} = \{\theta_0 v_0 + \dots + \theta_k v_k \mid \theta \geq 0, 1^\top \theta = 1\}$$

■ $k + 1$ points v_0, \dots, v_k are affinely independent

■ The affine dimension of this simplex is k

□ 1-dimensional simplex: line segment

□ 2-dimensional simplex: triangle

□ Unit simplex: $x \geq 0, 1^\top x \leq 1$

■ n -dimensional

□ Probability simplex: $x \geq 0, 1^\top x = 1$

■ $(n - 1)$ -dimensional



The positive semidefinite cone

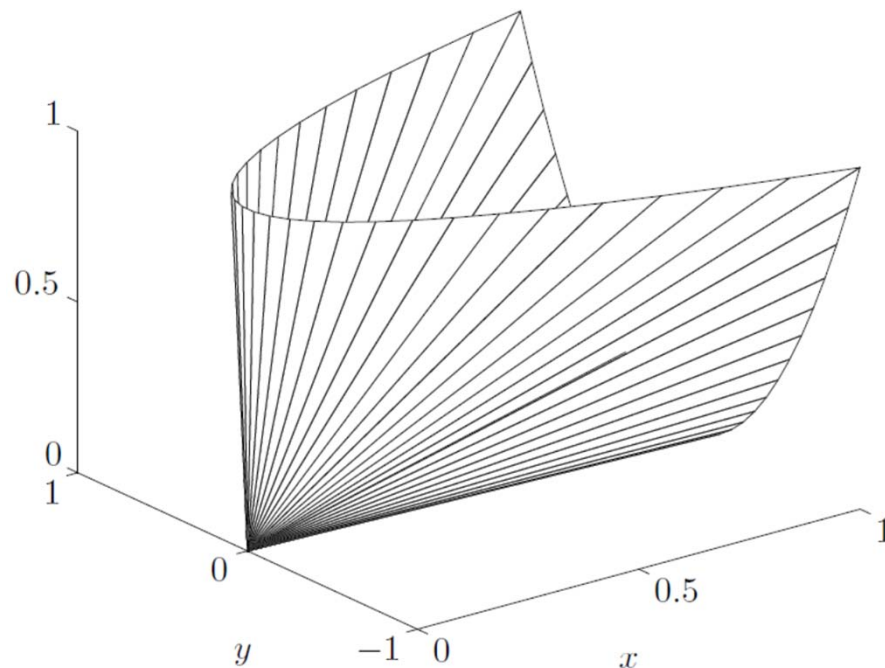
- $\mathbf{S}^n = \{X \in \mathbf{R}^{n \times n} | X = X^T\}$ is the set of symmetric $n \times n$ matrices
 - Vector space with dimension $n(n + 1)/2$
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n | X \succcurlyeq 0\}$ is the set of symmetric positive semidefinite matrices
 - Convex cone
- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n | X \succ 0\}$ is the set of symmetric positive definite



The positive semidefinite cone

□ PSD Cone in \mathbf{S}^2

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2 \iff x \geq 0, z \geq 0, xz \geq y^2$$





Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary



Intersection

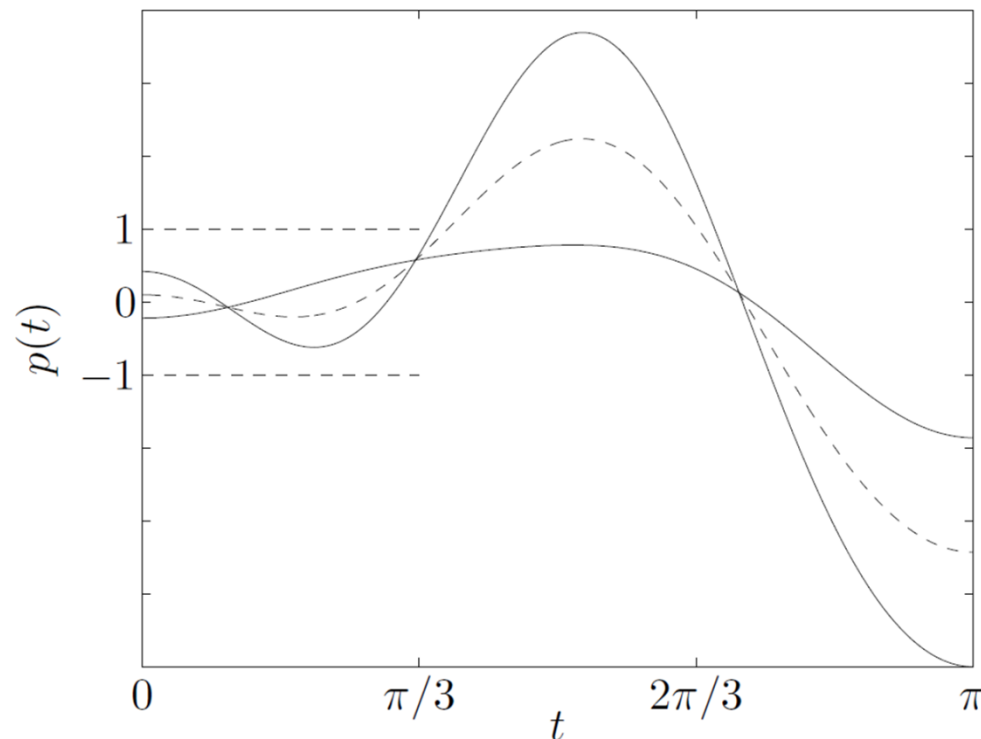
- If S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex
 - A polyhedron is the intersection of halfspaces and hyperplanes
- if S_α is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex
 - Positive semidefinite cone

$$\mathbf{S}_+^n = \bigcap_{z \neq 0} \{X \in \mathbf{S}^n \mid z^\top X z \geq 0\}$$

A Complicated Example (1)

$$S = \left\{ x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\}$$

■ $p(t) = \sum_{k=1}^m x_k \cos kt$





A Complicated Example (2)

$$S = \left\{ x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\}$$

■ $p(t) = \sum_{k=1}^m x_k \cos kt$

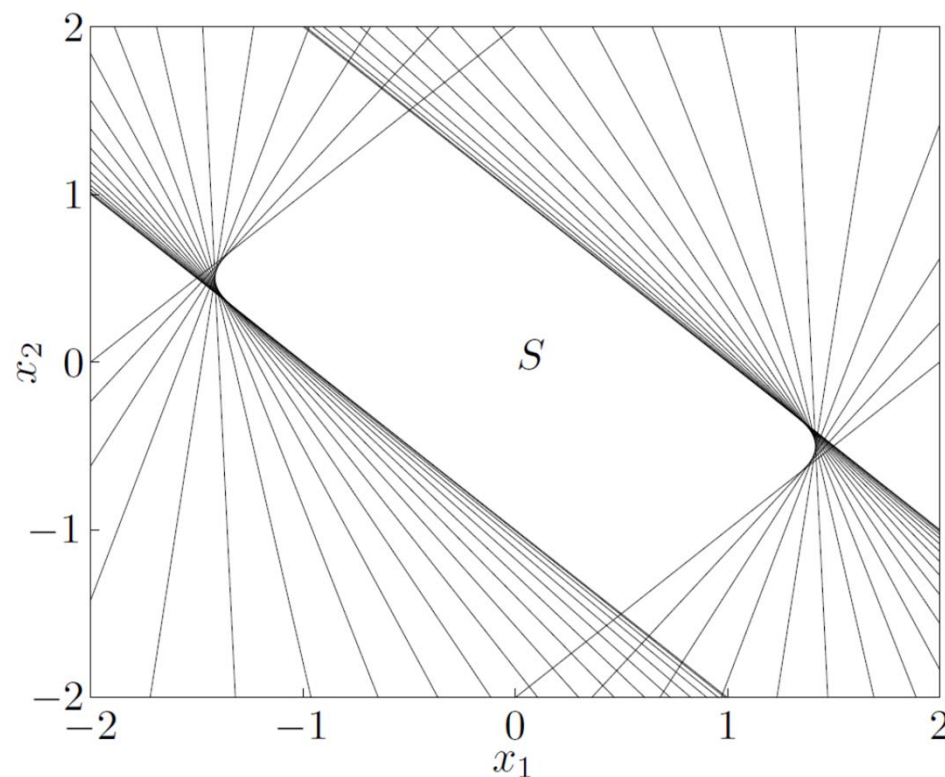
$$S = \bigcap_{|t| \leq \pi/3} S_t$$

■ $S_t = \{x \mid -1 \leq (\cos t, \dots, \cos mt)^\top x \leq 1\}$



A Complicated Example (3)

$$S = \bigcap_{|t| \leq \pi/3} S_t = \bigcap_{|t| \leq \pi/3} \{x \mid -1 \leq (\cos t, \dots, \cos mt)^\top x \leq 1\}$$





Affine Functions

□ Affine function $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$

$$f(x) = Ax + b, A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$$

□ $S \subseteq \mathbf{R}^n$ is convex

□ Then, the **image** of S under f

$$f(S) = \{f(x) \mid x \in S\}$$

and the **inverse image** of S under f

$$f^{-1}(S) = \{x \mid f(x) \in S\}$$

are convex



Examples (1)

□ Scaling

$$\alpha S = \{\alpha x \mid x \in S\}$$

□ Translation

$$S + a = \{x + a \mid x \in S\}$$

□ Projection of a convex set onto some of its coordinates

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

■ $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex



Examples (2)

□ Sum of two sets

$$S_1 + S_2 = \{x + y | x \in S_1, y \in S_2\}$$

- Cartesian product: $S_1 \times S_2 = \{(x_1, x_2) | x_1 \in S_1, x_2 \in S_2\}$
- Linear function: $f(x_1, x_2) = x_1 + x_2$

□ Partial sum of $S_1, S_2 \in \mathbf{R}^n \times \mathbf{R}^m$

$$S = \{(x, y_1 + y_2) | (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

- $m = 0$, intersection of S_1 and S_2
- $n = 0$, set addition



Examples (3)

□ Polyhedron

$$\{x | Ax \preceq b, Cx = d\} = \{x | f(x) \in \mathbf{R}_+^m \times \{0\}\}$$

■ $f(x) = (b - Ax, d - Cx)$

□ Linear Matrix Inequality

$$A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$$

■ The solution set $\{x | A(x) \preceq B\}$

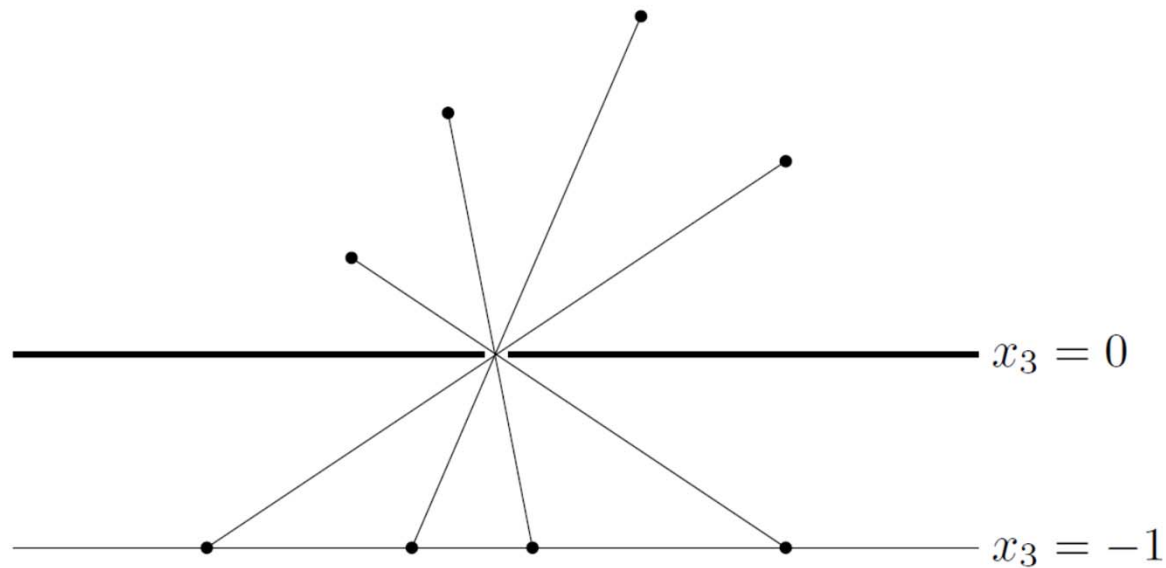
$$\{x | A(x) \preceq B\} = \{x | B - A(x) \in \mathbf{S}_+^m\}$$



Perspective Functions (1)

□ Perspective function $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$

$$P(z, t) = \frac{z}{t}, \text{ dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$$



$$(x_1, x_2, x_3) \mapsto -(x_1/x_3, x_2/x_3, 1)$$



Perspective Functions (2)

□ Perspective function $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$

$$P(z, t) = \frac{z}{t}, \text{ dom } P = \mathbf{R}^n \times \mathbf{R}_{++}$$

□ If $C \subseteq \text{dom } P$ is convex, then its image

$$P(C) = \{P(x) | x \in C\}$$

is convex

□ If $C \subseteq \mathbf{R}^n$ is convex, the inverse image

$$P^{-1}(C) = \left\{ (x, t) \in \mathbf{R}^{n+1} \mid \frac{x}{t} \in C, t > 0 \right\}$$

is convex



Linear-fractional Functions (1)

□ Suppose $g: \mathbf{R}^n \rightarrow \mathbf{R}^{m+1}$ is affine

$$g(x) = \begin{bmatrix} A \\ c^\top \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} Ax + b \\ c^\top x + d \end{bmatrix}$$

□ The function $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ given by $P \circ g$

$$f(x) = \frac{Ax + b}{c^\top x + d}, \text{ dom } f = \{c^\top x + d > 0\}$$



Linear-fractional Functions (2)

- If C is convex and $\{c^\top x + d > 0 \text{ for } x \in C\}$, then

$$f(C) = \left\{ \frac{Ax + b}{c^\top x + d} \mid x \in C \right\}$$

is convex

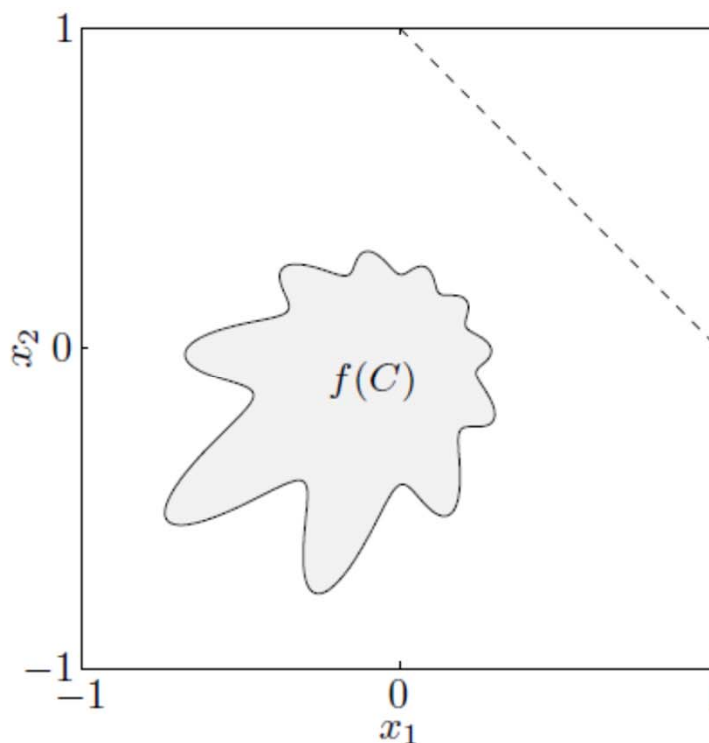
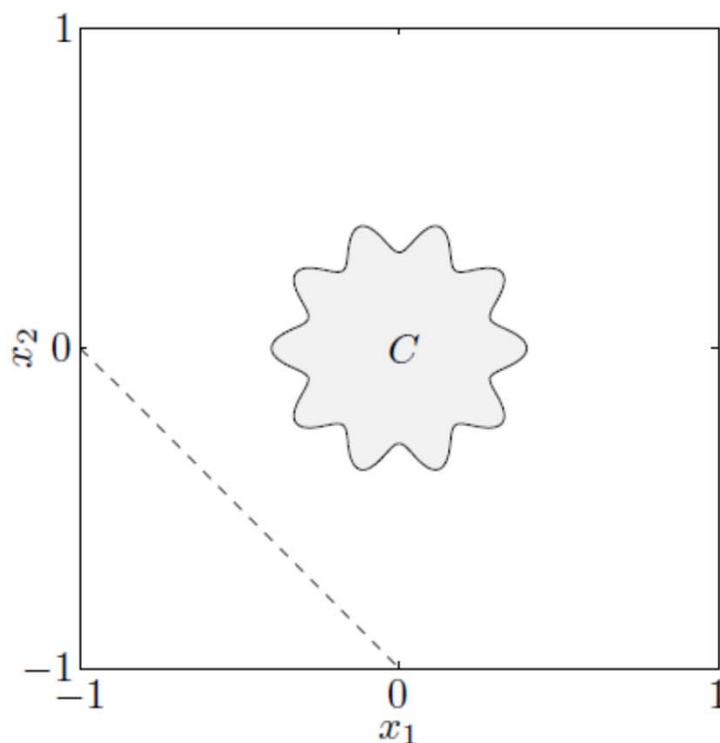
- If $C \subseteq \mathbf{R}^m$ is convex, then the inverse image

$$f^{-1}(C) = \left\{ x \mid \frac{Ax + b}{c^\top x + d} \in C \right\}$$

is convex

Example

$$f(x) = \frac{1}{x_1 + x_2 + 1} x, \text{ dom } f = \{(x_1, x_2) | x_1 + x_2 + 1 > 0\}$$





Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary



Proper Cones

- A cone $K \subseteq \mathbf{R}^n$ is called a proper cone if it satisfies the following
 - K is convex.
 - K is closed.
 - K is solid, which means it has nonempty interior.
 - K is pointed, which means that it contains no line ($x \in K, -x \in K \Rightarrow x = 0$).
- A proper cone K can be used to define a generalized inequality



Generalized Inequalities

- We associate with the proper cone K the partial ordering on \mathbf{R}^n defined by

$$x \preceq_K y \iff y - x \in K$$

- We define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \text{int } K$$



Examples

□ Nonnegative Orthant and Componentwise Inequality

- $K = \mathbf{R}_+^n$
- $x \preceq_K y$ means that $x_i \leq y_i, i = 1, \dots, n$.
- $x \prec_K y$ means that $x_i < y_i, i = 1, \dots, n$.

□ Positive Semidefinite Cone and Matrix Inequality

- $K = \mathbf{S}_+^n$
- $X \preceq_K Y$ means that $Y - X$ is PSD
- $X \prec_K Y$ means that $Y - X$ is positive definite



Properties of Generalized Inequalities

- \preceq_K is preserved under addition: If $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
- \preceq_K is transitive: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
- \preceq_K is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$ then $\alpha x \preceq_K \alpha y$.
- \preceq_K is reflexive: $x \preceq_K x$.
- \preceq_K is antisymmetric: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- \preceq_K is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \dots$, $x_i \rightarrow x$ and $y_i \rightarrow y$ as $i \rightarrow \infty$, then $x \preceq_K y$.

Properties of Strict Generalized Inequalities



- If $x \prec_K y$ then $x \preccurlyeq_K y$.
- If $x \prec_K y$ and $u \preccurlyeq_K v$ then $x + u \prec_K y + v$.
- If $x \prec_K y$ and $\alpha > 0$ then $\alpha x \prec_K \alpha y$.
- $x \not\prec_K x$.
- If $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.



Minimum and Minimal Elements

- $x \in S$ is the **minimum** element
 - If for every $y \in S$, we have $x \preceq_K y$
 - $S \subseteq x + K$
 - Minimum element is unique, if exists

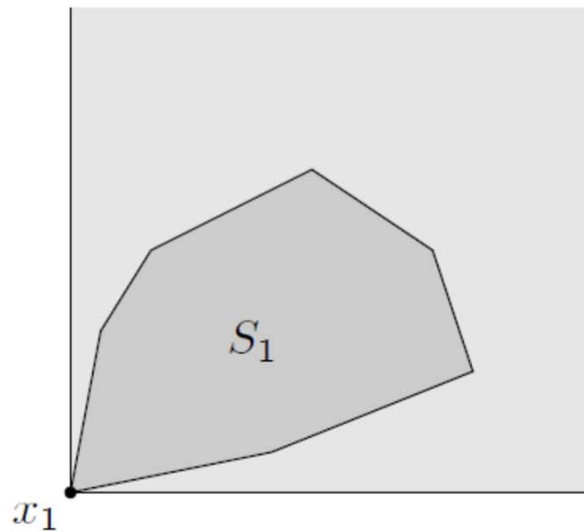
- $x \in S$ is a **minimal** element
 - if $y \in S$, $y \preceq_K x$ only if $y = x$
 - $(x - K) \cap S = \{x\}$
 - May have different minimal elements

- **Maximum, Maximal**

Example

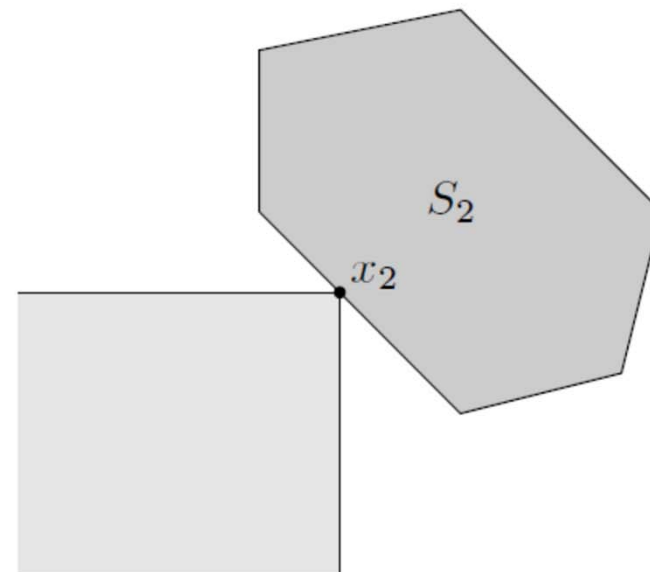
□ The Cone \mathbf{R}_+^2

- $x \preccurlyeq y$ means y is above and to the right of x



$$S \subseteq x + K$$

$$(x - K) \cap S = \{x\}$$





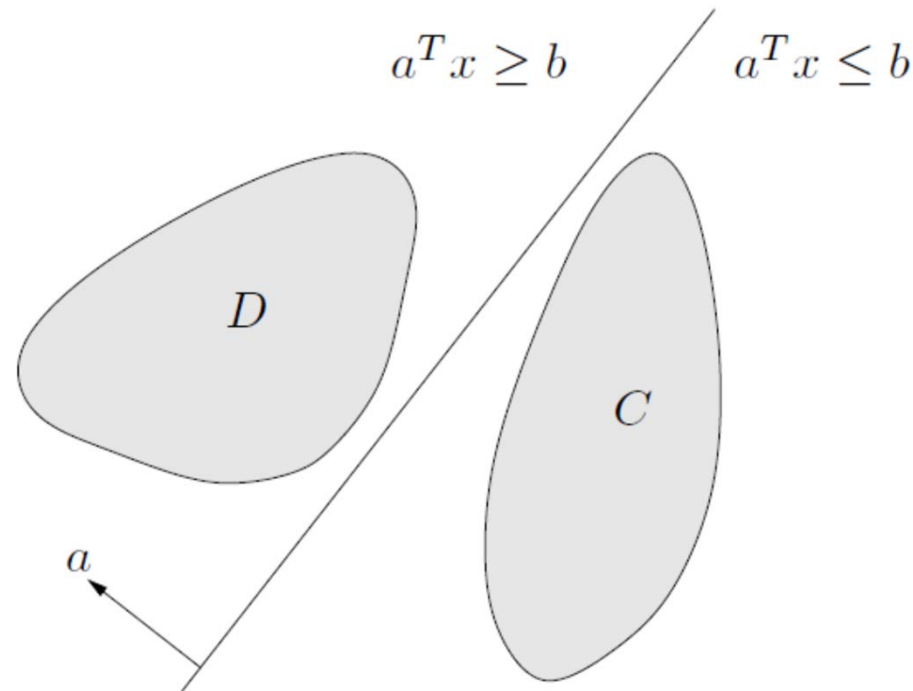
Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

Separating Hyperplane Theorem



- Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then, there exist $a \neq 0$ and b such that



Separating Hyperplane Theorem

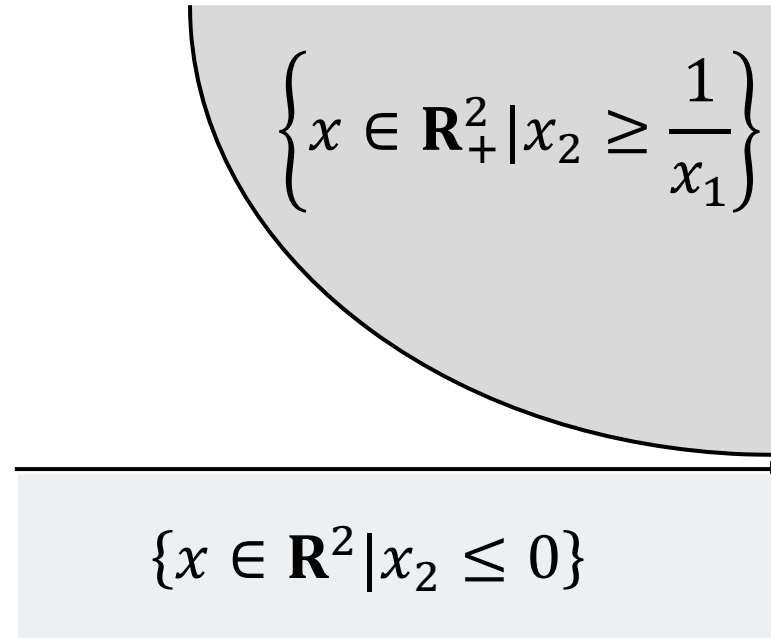


- Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then, there exist $a \neq 0$ and b such that $a^\top x \leq b$ for all $x \in C$ and $a^\top x \geq b$ for all $x \in D$.
- $\{x | a^\top x = b\}$ is called a **separating** hyperplane for the sets C and D .



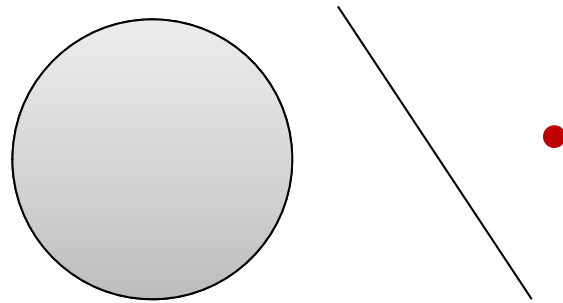
Strict Separation

- $a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$
- May not be possible in general



Strict Separation

- $a^T x < b$ for all $x \in C$ and $a^T x > b$ for all $x \in D$
- May not be possible in general
- A Point and a Closed Convex Set

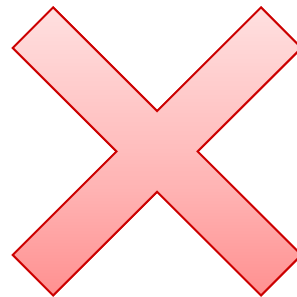


- A closed convex set is the intersection of all halfspaces that contain it

Converse separating hyperplane theorems



- Suppose C and D are convex sets,
and there exists an
affine function f that is nonpositive
on C and nonnegative on D . Then C
and D are disjoint.



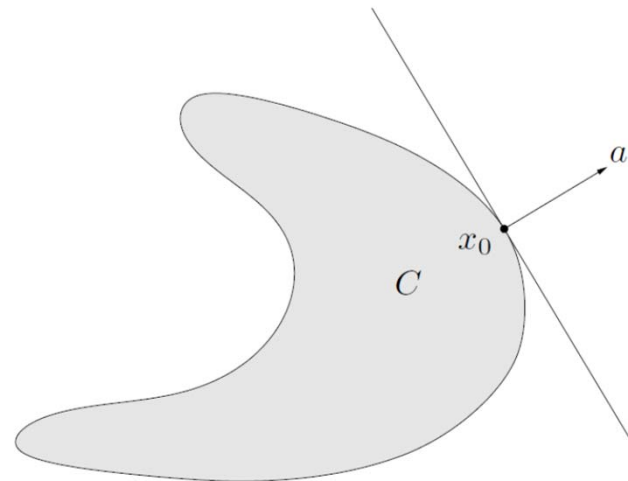
Converse separating hyperplane theorems



- Suppose C and D are convex sets, with C open, and there exists an affine function f that is nonpositive on C and nonnegative on D . Then C and D are disjoint.
- Any two convex sets C and D , at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Supporting Hyperplanes

- Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\text{bd } C$, i.e.,
$$x_0 \in \text{bd } C = \text{cl } C \setminus \text{int } C$$
- If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$. The hyperplane $\{x | a^\top x = a^\top x_0\}$ is called a **supporting** hyperplane to C at the point x_0





Supporting Hyperplanes

- Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\text{bd } C$, i.e.,

$$x_0 \in \text{bd } C = \text{cl } C \setminus \text{int } C$$

- If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$. The hyperplane $\{x | a^\top x = a^\top x_0\}$ is called a **supporting** hyperplane to C at the point x_0

- The point x_0 and the set C are separated by the hyperplane $\{x | a^\top x = a^\top x_0\}$

- The hyperplane $\{x | a^\top x = a^\top x_0\}$ is tangent to C at x_0 , and the halfspace $\{x | a^\top x \leq a^\top x_0\}$ contains C



Two Theorems

□ Supporting Hyperplane Theorem

- For any nonempty convex set C , and any $x_0 \in \text{bd } C$, there exists a supporting hyperplane to C at x_0 .

□ Converse Theorem

- If a set is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex.



Outline

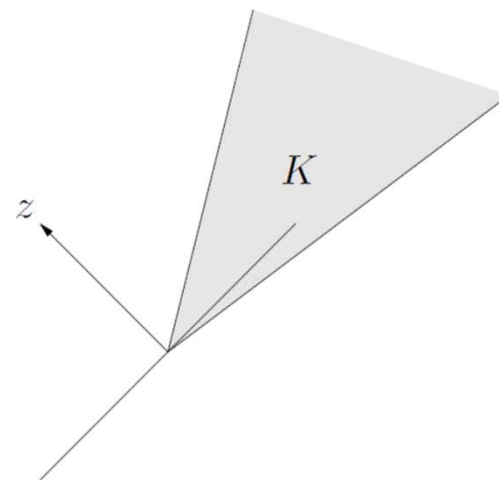
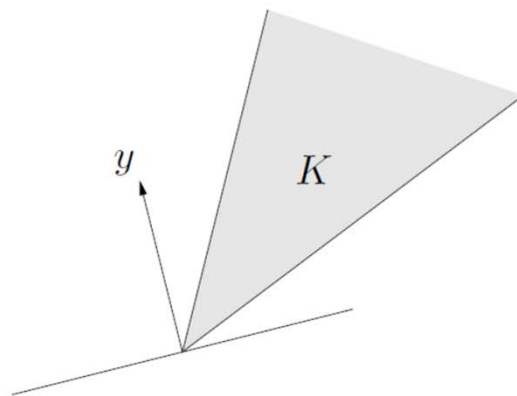
- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

Dual Cone

□ Dual Cone of a Given Cone K

$$K^* = \{y | x^\top y \geq 0 \text{ for all } x \in K\}$$

- K^* is convex, even when K is not
- $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin





Examples

□ Subspace

- The dual cone of a subspace $V \in \mathbf{R}^n$

$$V^\perp = \{y | v^\top y = 0 \text{ for all } v \in V\}$$

□ Nonnegative Orthant

- The cone \mathbf{R}_+^n is its own dual

$$x^\top y \geq 0 \text{ for all } x \succcurlyeq 0 \iff y \succcurlyeq 0$$

□ Positive Semidefinite Cone

- \mathbf{S}_+^n is self-dual

$$\text{tr}(XY) \geq 0 \text{ for all } X \succcurlyeq 0 \iff Y \succcurlyeq 0$$



Properties of Dual Cone

- K^* is closed and convex.
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$
- If K has nonempty interior, then K^* is pointed.
- If the closure of K is pointed then K^* has nonempty interior.
- K^{**} is the closure of the convex hull of K . (Hence if K is convex and closed, $K^{**} = K$.)



Dual Generalized Inequalities

- Suppose that the convex cone K is proper, so it induces a generalized inequality \preceq_K .
- Its dual cone K^* is also proper. We refer to the generalized inequality \preceq_{K^*} as the dual of the generalized inequality \preceq_K .



Dual Generalized Inequalities

- Suppose that the convex cone K is proper, so it induces a generalized inequality \preceq_K .
- Its dual cone K^* is also proper. We refer to the generalized inequality \preceq_{K^*} as the dual of the generalized inequality \preceq_K .
- $x \preceq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$

$$\left. \begin{array}{l} x \preceq_K y \Rightarrow y - x \in K \\ \lambda \succcurlyeq_{K^*} 0 \Rightarrow \lambda \in K^* \\ K^* = \{y | x^\top y \geq 0 \text{ for all } x \in K\} \end{array} \right\} \lambda^\top (y - x) \geq 0$$



Dual Generalized Inequalities

- Suppose that the convex cone K is proper, so it induces a generalized inequality \preceq_K .
- Its dual cone K^* is also proper. We refer to the generalized inequality \preceq_{K^*} as the dual of the generalized inequality \preceq_K .
 - $x \preceq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succeq_{K^*} 0$
 - $x \prec_K y$ if and only if $\lambda^\top x < \lambda^\top y$ for all $\lambda \succeq_{K^*} 0, \lambda \neq 0$
 - These properties hold if the generalized inequality and its dual are swapped



Dual Characterization of Minimum Element

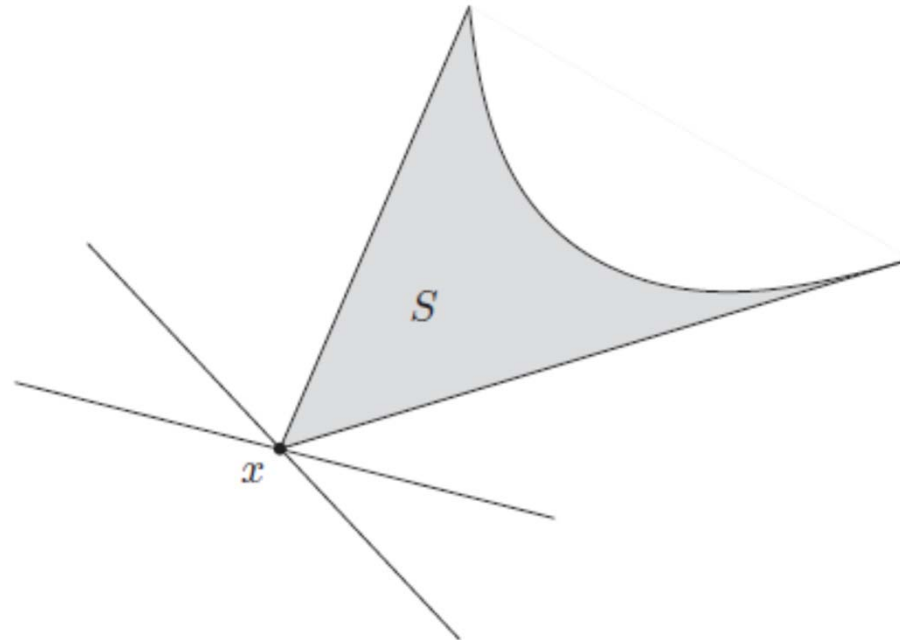
- x is the minimum element of S , with respect to the generalized inequality \preceq_K , **if and only if** for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^\top z$ over $z \in S$.
- That means, for **any** $\lambda \succ_{K^*} 0$, the hyperplane $\{z \mid \lambda^\top (z - x) = 0\}$ is a strict supporting hyperplane to S at x .

$$\lambda^\top z \geq \lambda^\top x \Leftrightarrow -\lambda^\top z \leq -\lambda^\top x, \quad \forall z \in S$$

Dual Characterization of Minimum Element



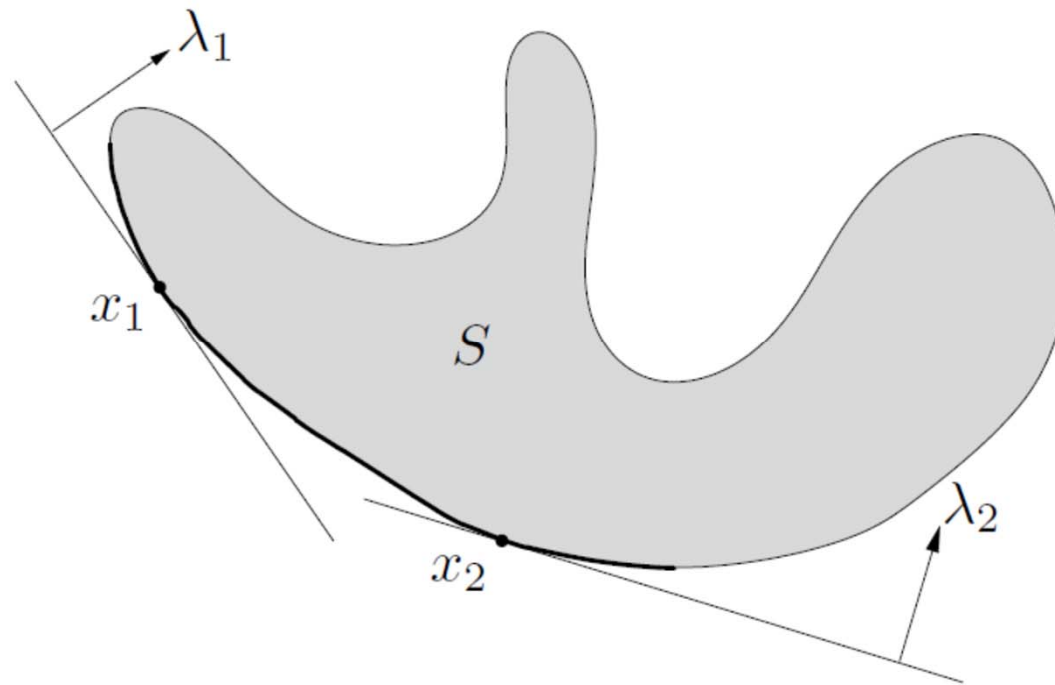
- x is the minimum element of S , with respect to the generalized inequality \preceq_K , **if and only if** for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^\top z$ over $z \in S$.



Dual Characterization of Minimal Elements (1)



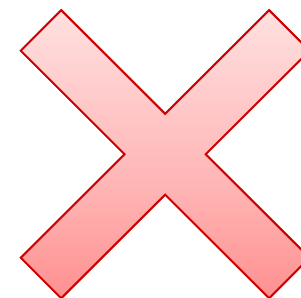
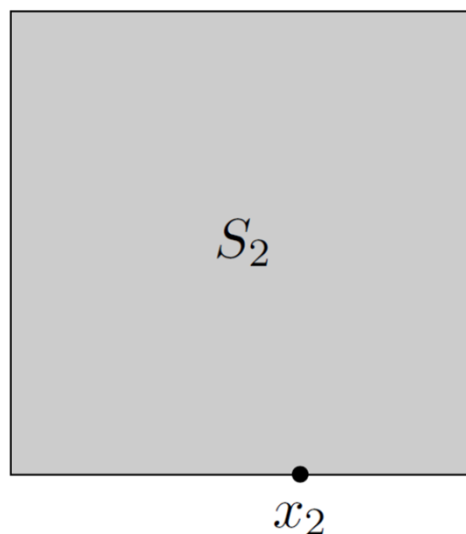
- If $\lambda \succ_{K^*} 0$, and x minimizes $\lambda^\top z$ over $z \in S$, then x is minimal.



Dual Characterization of Minimal Elements (1)



- Any minimizer of $\lambda^T z$ over $z \in S$, with $\lambda \succ_{K^*} 0$, is minimal.

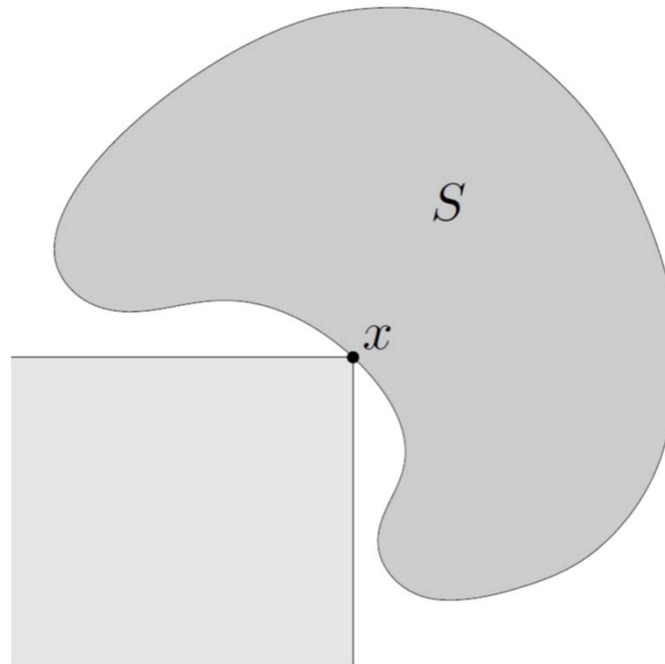
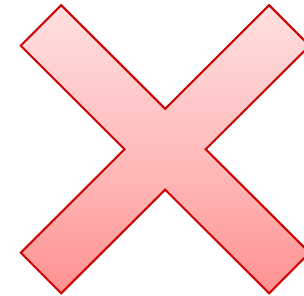


x_2 minimizes $\lambda^T z$ over $z \in S_2$ for $\lambda = (0,1) \succ 0$

Dual Characterization of Minimal Elements (2)



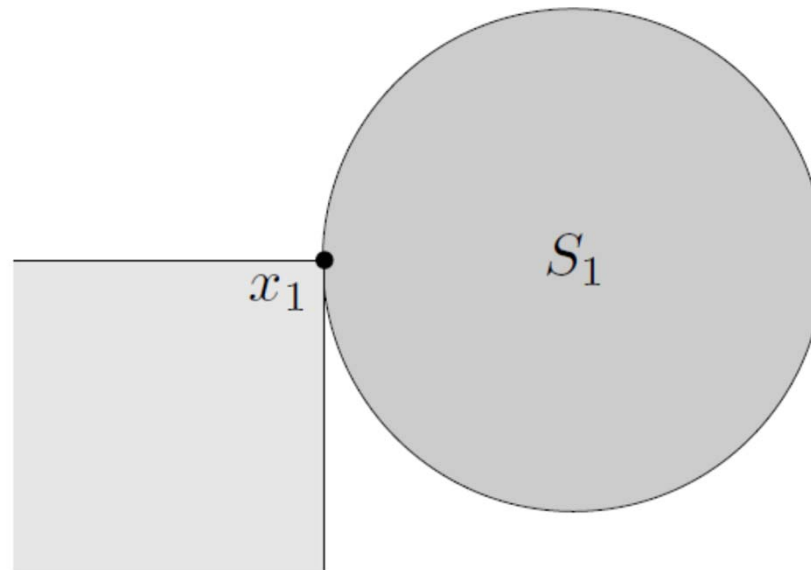
- If x is minimal, then x minimizes $\lambda^T z$ over $z \in S$ with $\lambda \succ_{K^*} 0$.



Dual Characterization of Minimal Elements (2)



- If S is convex, for any minimal element x there exists a nonzero $\lambda \succcurlyeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over $z \in S$.

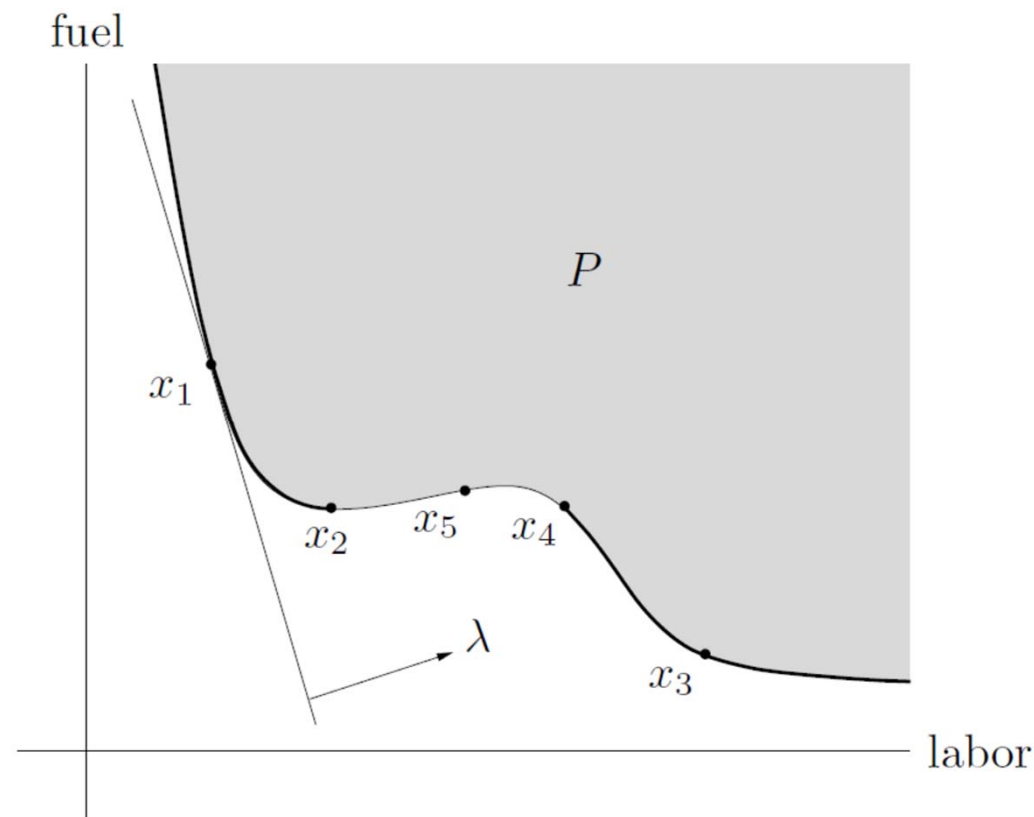


x_1 minimizes $\lambda^T z$ over $z \in S_1$ for $\lambda = (1,0) \succcurlyeq 0$

Pareto Optimal Production Frontier

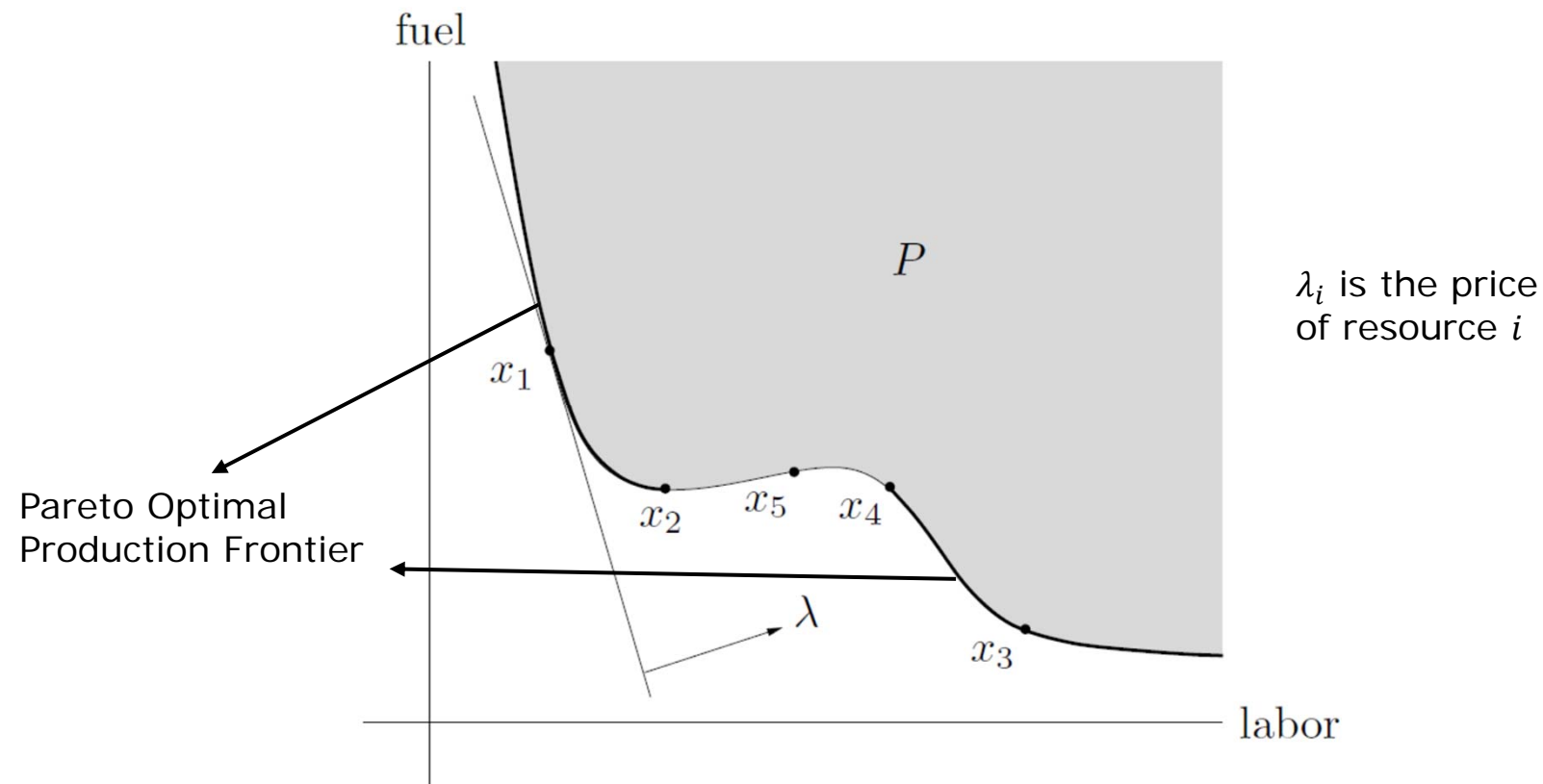


- A product which requires n sources
- A resource vector $x \in \mathbf{R}^n$





- A product which requires n sources
- A resource vector $x \in \mathbf{R}^n$





Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary



Summary

- Affine and convex
- Operations that preserve convexity
- Generalized Inequalities
- Separating and supporting hyperplanes
 - Theorems
- Dual cones and generalized inequalities