

Convex Functions (II)

Lijun Zhang

zlj@nju.edu.cn

<http://cs.nju.edu.cn/zlj>





Outline

- The Conjugate Function
- Quasiconvex Functions
- Log-concave and Log-convex Functions
- Convexity with Respect to Generalized Inequalities
- Summary



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Conjugate Function

□ $f: \mathbf{R}^n \rightarrow \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^\top x - f(x))$$

■ $\text{dom } f^* = \{y \mid f^*(y) < \infty\}$

■ f^* is always convex

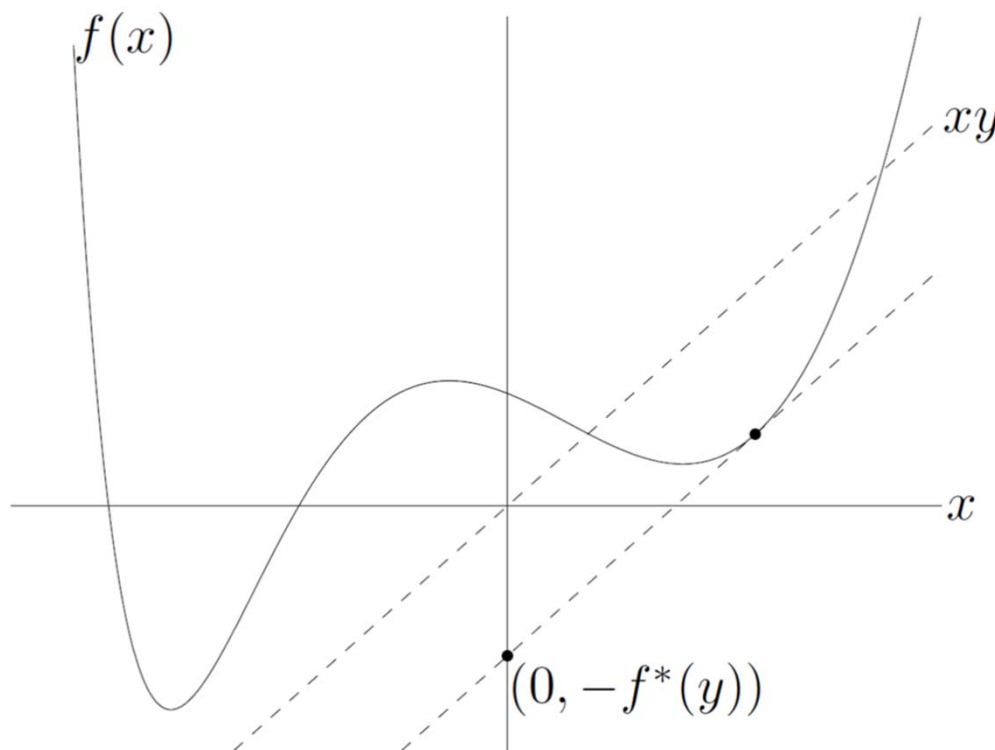




Conjugate Function

□ $f: \mathbf{R}^n \rightarrow \mathbf{R}$. Its conjugate function is

$$f^*(y) = \sup_{x \in \text{dom } f} (y^\top x - f(x))$$





Conjugate examples

□ Affine function

- $f(x) = ax + b$
- $f^*(y) = \sup_{x \in \mathbf{R}} (yx - ax - b)$
- $\text{dom } f^* = \{a\}, f^*(a) = -b$

□ Negative logarithm

- $f(x) = -\log x$
- $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (yx + \log x)$
- $\text{dom } f^* = -\mathbf{R}_{++}, f^*(y) = -\log(-y) - 1$



Conjugate examples

□ Exponential

- $f(x) = e^x$
- $f^*(y) = \sup_{x \in \mathbf{R}} (xy - e^x)$
- $\text{dom } f^* = \mathbf{R}_+, f^*(y) = y \log y - y$

□ Negative entropy

- $f(x) = x \log x$
- $f^*(y) = \sup_{x \in \mathbf{R}_+} (yx - x \log x)$
- $\text{dom } f^* = \mathbf{R}, f^*(y) = e^{y-1}$



Conjugate examples

□ Inverse

- $f(x) = 1/x, x \in \mathbf{R}_{++}$
- $f^*(y) = \sup_{x \in \mathbf{R}_{++}} (xy - 1/x)$
- $\text{dom } f^* = -\mathbf{R}_+, f^*(y) = -2(-y)^{1/2}$

□ Strictly convex quadratic function

- $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
- $f^*(y) = \sup_{x \in \mathbf{R}^n} (y^\top x - \frac{1}{2} x^\top Q x)$
- $\text{dom } f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2} y^\top Q^{-1} y$



Conjugate examples

□ Log-determinant

- $f(X) = \log \det X^{-1}, X \in \mathbf{S}_{++}^n$
- $f^*(Y) = \sup_{X \in \mathbf{S}_{++}^n} (\text{tr}(XY) + \log \det X)$
- $\text{dom } f^* = -\mathbf{S}_{++}^n, f^*(Y) = \log \det(-Y)^{-1} - n$

□ Indicator function

- $I_S(x) = 0, \text{dom } I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
- $I_S^*(y) = \sup_{x \in S} y^\top x$
- $I_S^*(y)$ is the **support function** of the set S



Conjugate examples

□ Support function of a set

- $C \subseteq \mathbf{R}^n, C \neq \emptyset$
- $S_C(x) = \sup\{x^\top y \mid y \in C\}$
- $\text{dom } S_C = \{x \mid \sup_{y \in C} x^\top y < \infty\}$

□ Indicator function

- $I_S(x) = 0, \text{dom } I_S = S, S \subseteq \mathbf{R}^n$ is not necessarily convex
- $I_S^*(y) = \sup_{x \in S} y^\top x$
- $I_S^*(y)$ is the **support function** of the set S



Conjugate examples

□ Norm

- $f(x) = \|x\|, x \in \mathbf{R}^n$, with dual norm $\|\cdot\|_*$
- $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \|x\|)$
- $\text{dom } f^* = \{y \mid \|y\|_* \leq 1\}, f^*(y) = 0$

□ Norm squared

- $f(x) = \frac{1}{2} \|x\|^2, x \in \mathbf{R}^n$, with dual norm $\|\cdot\|_*$
- $f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - \frac{1}{2} \|x\|^2)$
- $\text{dom } f^* = \mathbf{R}^n, f^*(y) = \frac{1}{2} \|y\|_*^2$



Basic properties

□ Fenchel's inequality

- $\forall x \in \text{dom } f, y \in \text{dom } f^*, f(x) + f^*(y) \geq x^\top y$
- $f^*(y) = \sup_{x \in \mathbb{R}^n} (x^\top y - f(x))$
- $f(x) = \frac{1}{2} x^\top Q x, Q \in \mathbf{S}_{++}^n$
 $\Rightarrow x^\top y \leq \frac{1}{2} x^\top Q x + \frac{1}{2} y^\top Q^{-1} y$

□ Conjugate of the conjugate

- f is convex and closed $\Rightarrow f^{**} = f$



Basic properties

□ Differentiable functions

■ f is convex and differentiable, $\text{dom } f = \mathbf{R}^n$

■
$$f^*(y) = \sup_{x \in \mathbf{R}^n} (x^\top y - f(x))$$

■
$$x^* = \operatorname{argmax}(x^\top y - f(x)) \Rightarrow \nabla f(x^*) = y$$

■
$$f^*(y) = x^{*\top} \nabla f(x^*) - f(x^*) = x^{*\top} y - f(x^*)$$

✓
$$x^* = \nabla^{-1} f(y)$$



Basic properties

□ Scaling with affine transformation

- $a > 0, b \in \mathbf{R}, g(x) = af(x) + b$

$$\Rightarrow g^*(y) = af^*\left(\frac{y}{a}\right) - b$$

- $A \in \mathbf{R}^{n \times n}$ is nonsingular, $b \in \mathbf{R}^n, g(x) =$

$$f(Ax + b) \Rightarrow g^*(y) = f^*(A^{-\top}y) -$$

$$b^{\top}A^{-\top}y, \text{dom } g^* = A^{\top} \text{dom } f^*$$

□ Sums of independent functions

- $f(u, v) = f_1(u) + f_2(v), f_1, f_2$ are convex \Rightarrow

$$f^*(w, z) = f_1^*(w) + f_2^*(z)$$



Outline

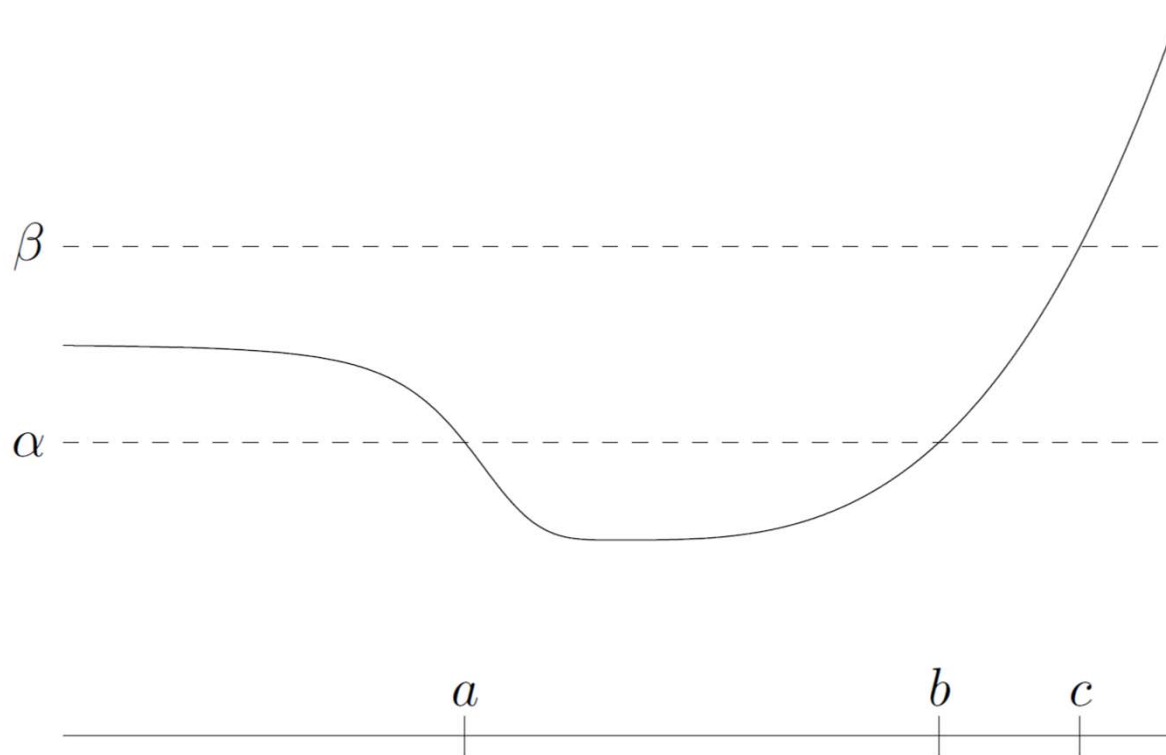
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Quasiconvex functions

□ Quasiconvex

- $f: \mathbf{R}^n \rightarrow \mathbf{R}$
- $S_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\}, \forall \alpha \in \mathbf{R}$ is convex





Quasiconvex functions

□ Quasiconvex

- $f: \mathbf{R}^n \rightarrow \mathbf{R}$
- $S_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\}, \forall \alpha \in \mathbf{R}$ is convex

□ Quasiconcave

- $-f$ is quasiconvex $\Rightarrow f$ is quasiconcave

□ Quasilinear

- f is quasiconvex and quasiconcave $\Rightarrow f$ is quasilinear



Examples

□ Some example on \mathbf{R}

- Logarithm: $\log x$ on \mathbf{R}_{++}
 - ✓ Concave, quasiconvex, quasiconcave
- Ceiling function: $\text{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$
 - ✓ quasiconvex, quasiconcave

□ Linear-fractional function

- $f(x) = \frac{a^\top x + b}{c^\top x + d}$, $\text{dom } f = \{x \mid c^\top x + d > 0\}$
- $\left\{x \mid c^\top x + d > 0, \frac{a^\top x + b}{c^\top x + d} \geq \alpha\right\}$ and $\left\{x \mid c^\top x + d > 0, \frac{a^\top x + b}{c^\top x + d} \leq \alpha\right\}$ is convex
 $\Rightarrow f$ is Quasilinear

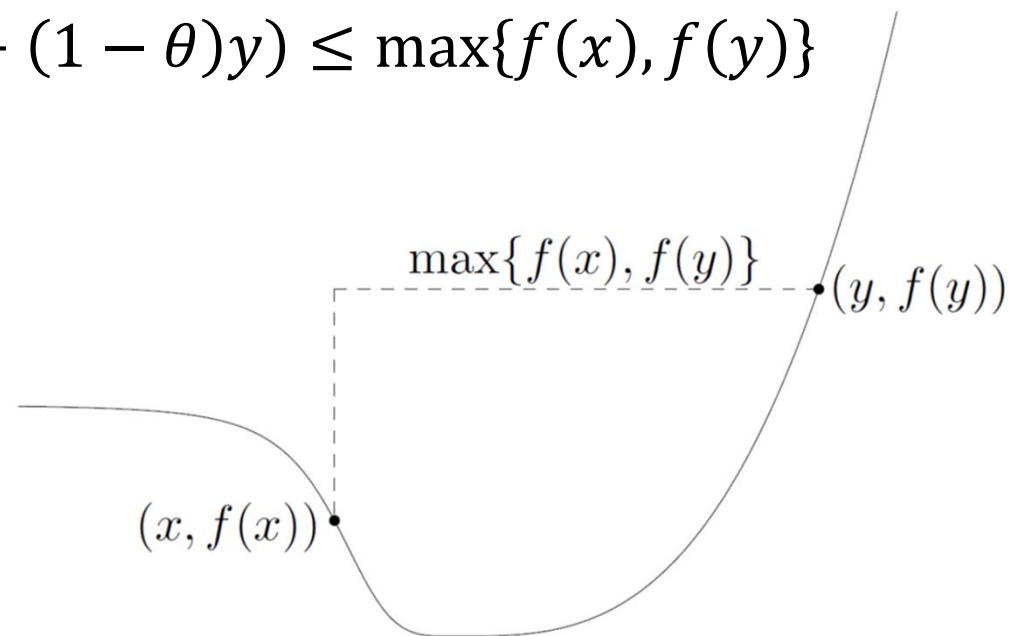


Basic properties

□ Jensen's inequality for quasiconvex functions

- f is quasiconvex \Leftrightarrow $\text{dom } f$ is convex and $\forall x, y \in \text{dom } f, 0 \leq \theta \leq 1$

$$f(\theta x + (1 - \theta)y) \leq \max\{f(x), f(y)\}$$





Basic properties

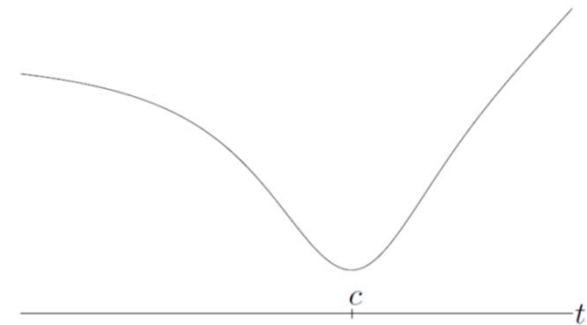
□ Condition

- f is quasiconvex \Leftrightarrow its restriction to any line intersecting its domain is quasiconvex

□ Quasiconvex functions on \mathbf{R}

- A continuous function $f: \mathbf{R} \rightarrow \mathbf{R}$ is quasiconvex \Leftrightarrow one of the following conditions holds

- ✓ f is nondecreasing
- ✓ f is nonincreasing
- ✓ $\exists c \in \text{dom } f, \forall t \in \text{dom } f, t \leq c, f$ is nonincreasing, and $t \geq c, f$ is nondecreasing



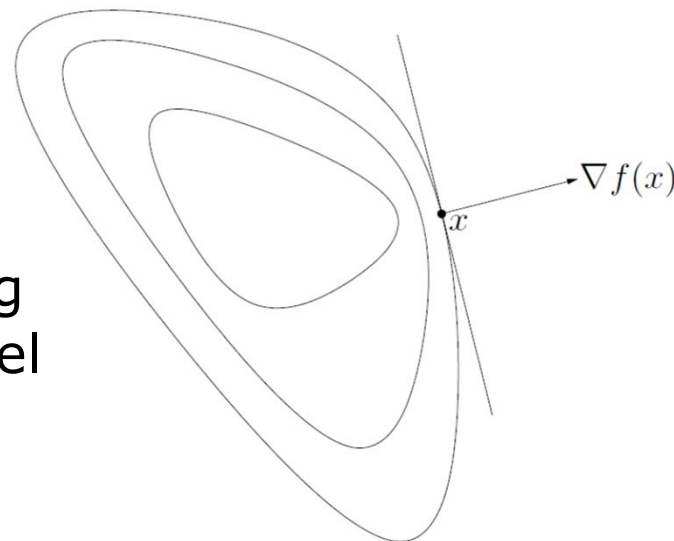
Differentiable quasiconvex functions



□ First-order conditions

- f is differentiable
- f is quasiconvex \Leftrightarrow $\text{dom } f$ is convex, $\forall x, y \in \text{dom } f, f(y) \leq f(x) \Rightarrow \nabla f(x)^\top (y - x) \leq 0$

$\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{y | f(y) \leq f(x)\}$ at x



Differentiable quasiconvex functions



□ First-order conditions

- f is differentiable
- f is quasiconvex \Leftrightarrow $\text{dom } f$ is convex, $\forall x, y \in \text{dom } f, f(y) \leq f(x) \Rightarrow \nabla f(x)^\top (y - x) \leq 0$
- It is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f .

□ Second-order conditions

- f is twice differentiable
- f is quasiconvex $\Rightarrow \forall x \in \text{dom } f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y \geq 0$

Differentiable quasiconvex functions



□ First-order conditions

- f is differentiable
- f is quasiconvex \Leftrightarrow $\text{dom } f$ is convex, $\forall x, y \in \text{dom } f, f(y) \leq f(x) \Rightarrow \nabla f(x)^\top (y - x) \leq 0$
- It is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f .

□ Second-order conditions

- f is twice differentiable
- $\forall x \in \text{dom } f, \forall y \in \mathbf{R}^n, y^\top \nabla f(x) = 0 \Rightarrow y^\top \nabla^2 f(x) y > 0 \Rightarrow f$ is quasiconvex

Operations that preserve quasiconvexity



□ Nonnegative weighted maximum

- f_i is quasiconvex, $w_i \geq 0 \Rightarrow f = \max\{w_1 f_1, \dots, w_n f_n\}$ is quasiconvex

- $g(x, y)$ is quasiconvex in x for each $y, w(y) \geq 0 \Rightarrow f(x) = \sup_{y \in C} (w(y)g(x, y))$ is quasiconvex

Operations that preserve quasiconvexity



□ Composition

- $g: \mathbf{R}^n \rightarrow \mathbf{R}$ is quasiconvex, $h: \mathbf{R} \rightarrow \mathbf{R}$ is nondecreasing $\Rightarrow f = h \circ g$ is quasiconvex
- $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f(Ax + b)$ is quasiconvex
- $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is quasiconvex $\Rightarrow g(x) = f\left(\frac{Ax+b}{c^\top x+d}\right)$ is quasiconvex, $\text{dom } g = \{x \mid c^\top x + d > 0, (Ax + b)/(c^\top x + d) \in \text{dom } f\}$

□ Minimization

- $f(x, y)$ is **quasiconvex in x and y** , C is a convex set $\Rightarrow g(x) = \inf_{y \in C} f(x, y)$ is quasiconvex



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Log-concave and log-convex functions



□ Definition

- $f: \mathbf{R}^n \rightarrow \mathbf{R}, f(x) > 0, \forall x \in \text{dom } f, \log f(x)$ is concave (convex) $\Rightarrow f$ is log-concave (convex)
- A log-convex function is convex
- A nonnegative concave function is log-concave

□ Condition

- $f: \mathbf{R}^n \rightarrow \mathbf{R}, f(x) > 0, \forall x \in \text{dom } f, f$ is log-concave $\Leftrightarrow \forall x, y \in \text{dom } f, 0 \leq \theta \leq 1$

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta}$$



Examples

- $f(x) = a^\top x + b$, $\text{dom } f = \{x | a^\top x + b > 0\}$ is log-concave
- $f(x) = x^a$, $\text{dom } f = \mathbf{R}_{++}$, $a \leq 0 \Rightarrow f$ is log-convex, $a \geq 0 \Rightarrow f$ is log-concave
- $f(x) = e^{ax}$ is log-convex and log-concave
- $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$ is log-concave
- $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$ is log-convex for $x \geq 1$
- $\det X$ and $\frac{\det X}{\text{tr } X}$ are log-concave on \mathbf{S}_{++}^n



Properties

□ Twice differentiable log-convex/concave functions

- f is twice differentiable, $\text{dom } f$ is convex
- $$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^\top$$
- f is log-convex $\Leftrightarrow f(x) \nabla^2 f(x) \succeq \nabla f(x) \nabla f(x)^\top$
- f is log-concave $\Leftrightarrow f(x) \nabla^2 f(x) \preceq \nabla f(x) \nabla f(x)^\top$



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Convexity with respect to a generalized inequality



□ K -convex

■ $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality \preceq_K

■ $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is K -convex if $\forall x, y \in \text{dom } f, 0 \leq \theta \leq 1$

$$f(\theta x + (1 - \theta)y) \preceq_K \theta f(x) + (1 - \theta)f(y)$$

■ $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is strictly K -convex if $\forall x \neq y \in \text{dom } f, 0 < \theta < 1$

$$f(\theta x + (1 - \theta)y) \prec_K \theta f(x) + (1 - \theta)f(y)$$



Examples

□ Componentwise Inequality

- $K = \mathbf{R}_+^m$
- $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is convex with respect to componentwise inequality $\Leftrightarrow \forall x, y \in \text{dom } f, 0 \leq \theta \leq 1,$
$$f(\theta x + (1 - \theta)y) \preceq \theta f(x) + (1 - \theta)f(y)$$
- Each f_i is a convex function



Examples

□ Matrix Convexity

- $f: \mathbf{R}^n \rightarrow \mathbf{S}^m$ is convex with respect to matrix inequality $\Leftrightarrow \forall x, y \in \text{dom } f, 0 \leq \theta \leq 1$
 $f(\theta x + (1 - \theta)y) \preceq \theta f(x) + (1 - \theta)f(y)$
- $f(X) = XX^T, X \in \mathbf{R}^{m \times n}$ is matrix convex
- X^p is matrix convex on \mathbf{S}_{++}^n for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$

Convexity with respect to generalized inequalities



□ Dual characterization of K -convexity

- A function f is (strictly) K -convex \Leftrightarrow For every $w \succ_{K^*} 0$, the real-valued function $w^\top f$ is (strictly) convex in the ordinary sense.

□ Differentiable K -convex functions

- A differentiable function f is K -convex \Leftrightarrow $\text{dom } f$ is convex, $\forall x, y \in \text{dom } f$,

$$f(y) \succcurlyeq_K f(x) + Df(x)(y - x)$$

- A differentiable function f is strictly K -convex \Leftrightarrow $\text{dom } f$ is convex, $\forall x, y \in \text{dom } f, x \neq y$,

$$f(y) \succ_K f(x) + Df(x)(y - x)$$

Convexity with respect to generalized inequalities



□ Composition theorem

- $g: \mathbf{R}^n \rightarrow \mathbf{R}^p$ is K -convex, $h: \mathbf{R}^p \rightarrow \mathbf{R}$ is convex, and \tilde{h} (the extended-value extension of h) is K -nondecreasing $\Rightarrow h \circ g$ is convex.

□ Example

- $g: \mathbf{R}^{m \times n} \rightarrow \mathbf{S}^n$, $g(X) = X^\top A X + B^\top X + X^\top B + C$ is convex, where $A \succcurlyeq 0$, $B \in \mathbf{R}^{m \times n}$ and $C \in \mathbf{S}^n$
- $h: \mathbf{S}^n \rightarrow \mathbf{R}$, $h(Y) = -\log \det(-Y)$ is convex and increasing on $\text{dom } h = -\mathbf{S}_{++}^n$
- $f(X) = -\log \det(-(X^\top A X + B^\top X + X^\top B + C))$ is convex on $\text{dom } f = \{X \in \mathbf{R}^{m \times n} \mid X^\top A X + B^\top X + X^\top B + C \prec 0\}$

Monotonicity with respect to a generalized inequality



□ $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality \preceq_K

■ $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is K -nondecreasing if

$$x \preceq_K y \Rightarrow f(x) \leq f(y)$$

■ $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is K -increasing if

$$x \preceq_K y, x \neq y \Rightarrow f(x) < f(y)$$



Summary

- The Conjugate Function
 - Definitions, Basic properties
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