Optimization Methods **Fall 2021**

Homework 1

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Notice

- The submission email is: **opt4grad@163.com**.
- Please use the provided LaTeX file as a template. If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Convex sets
Convex C_c sets are the sets satisfying the constraints below:

$$\theta x_1 + (1-\theta)x_2 \in C_c$$
$$\text{for all, } x_1, x_2 \in C_c, 0 \leq \theta \leq 1$$

a). Show that a set is convex if and only if its intersection with any line is convex.

b). Determine if each set below is convex.

1) $\{(x,y) \in \mathbf{R}^2_{++} | x/y \leq 1\}$

2) $\{(x,y) \in \mathbf{R}^2_{++} | x/y \geq 1\}$

3) $\{(x,y) \in \mathbf{R}^2_{++} | xy \leq 1\}$

4) $\{(x,y) \in \mathbf{R}^2_{++} | xy \geq 1\}$

5) $\{(x,y) \in \mathbf{R}^2_{++} | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$

Problem 2: Convex cone
Let K be a convex cone. The set $K^* = \{y | x^T y \geq 0, \forall x \in K\}$ is called the dual cone of K.

a) Show that K^* is a convex cone (even K is not convex).

b) Show that a dual cone of a subspace $V \subset \mathbb{R}^n$ (which is a cone) is its orthogonal complement $V^+ = \{y | y^T v = 0, \forall v \in V\}$.

c) What is the dual cone of the nonnegative orthant $(\mathbb{R}^+_n)$?

Problem 3: Hyperplane
What is the distance between two parallel hyperplanes, i.e., $\{x | a^T x = b\}$ and $\{x | a^T x = c\}$?

Problem 4: Norms
A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom} f = \mathbb{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t| f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x+y) \leq f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$

We use the notation $f(x) = \|x\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup\{z^{\mathrm{T}}x | \|x\| \leq 1\}$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of Euclidean norm (ℓ_2-*norm*) is the Euclidean norm, *i.e.*, prove that

$$\|z\|_{2*} = \sup\{z^{\mathrm{T}}x | \|x\|_2 \leq 1\} = \|z\|_2$$

(*Hint*: Use Cauchy-Schwarz inequality.)

Problem 5: Operations That Preserve Convexity

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\psi : \mathbb{R}^m \to \mathbb{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax+b}{c^\top x+d}, \psi(y) = \frac{Ey+f}{g^\top y+h}$$

with domains $\mathbf{dom}\ \phi = \left\{x \mid c^\top x + d > 0\right\}$, $\mathbf{dom}\ \psi = \left\{y \mid g^\top y + h > 0\right\}$. We associate with ϕ and ψ the matrices respectively.

$$\begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}, \begin{bmatrix} E & f \\ g^\top & h \end{bmatrix}$$

Now, consider the composition Γ of ϕ and ψ, i.e., $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}\Gamma = \{x \in \mathbf{dom}\phi \mid \phi(x) \in \mathbf{dom}\psi\}$$

Show that Γ is linear-fractional, and that the matrix associate with it is the product

$$\begin{bmatrix} E & f \\ g^\top & h \end{bmatrix} \begin{bmatrix} A & b \\ c^\top & d \end{bmatrix}$$