Optimization Methods **Fall 2021**

Homework 2

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Notice

- The submission email is: **opt4grad@163.com**.
- Please use the provided LaTeX file as a template. If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Convex functions

a) Prove that the function $f : \mathbb{R}^n_{++} \to \mathbb{R}$, defined as

$$f(x) = -\sum_{i=1}^{n} \log(x_i),$$

is strictly convex.

b) Let f be twice differentiable, with $\mathrm{dom}(f)$ convex. Prove that f is convex if and only if

$$(\nabla f(x) - \nabla f(y))^T (x - y) \geq 0,$$

for all x, y.

c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Its *perspective transform* $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is defined by

$$g(x, t) = tf(x/t),$$

with domain $\mathrm{dom}(g) = \{(x, t) \in \mathbb{R}^{n+1} : x \in \mathrm{dom}(f),\ t > 0\}$. Use the definition of convexity to prove that if f is convex, then so is its perspective transform g.

Problem 2: Concave function

Suppose $p < 1, p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

with $\mathrm{dom}\, f = \mathbb{R}_{++}$ is concave.

Problem 3: Convexity

Let $\psi : \Omega \mapsto \mathbb{R}$ be a strictly convex and continuously differentiable function. We define

$$\Delta_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle, \quad \forall x, y \in \Omega.$$

a) Prove that $\Delta_\psi(x, y) \geq 0, \forall x, y \in \Omega$ and the equality holds only when $x = y$.

b) Let L be a convex and differentiable function defined on Ω and $C \subset \Omega$ be a convex set. Let $x_0 \in \Omega - C$ and define

$$x^* = \operatorname*{arg\,min}_{x \in C} \ L(x) + \Delta_\psi(x, x_0).$$

Prove that for any $y \in C$,

$$L(y) + \Delta_\psi(y, x_0) \geq L(x^*) + \Delta_\psi(x^*, x_0) + \Delta_\psi(y, x^*). \tag{1}$$

Problem 4: Projection
For any point y, the projection onto a nonempty and closed convex set X is defined as

$$\Pi_X(y) = \operatorname*{argmin}_{x \in X} \frac{1}{2}\|x - y\|_2^2. \tag{2}$$

a) Prove that $\|\Pi_X(x) - \Pi_X(y)\|_2^2 \leq \langle \Pi_X(x) - \Pi_X(y), x - y \rangle$.

b) Prove that $\|\Pi_X(x) - \Pi_X(y)\|_2 \leq \|x - y\|_2$.

Problem 5: Convexity
Derive the conjugates of the following functions.

a) $f(x) = \ln(1 + e^{-x})$.

b) $f(x) = x^p$ on $\mathbb{R}_{++}$, where $p > 1$.