Convex Sets

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Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

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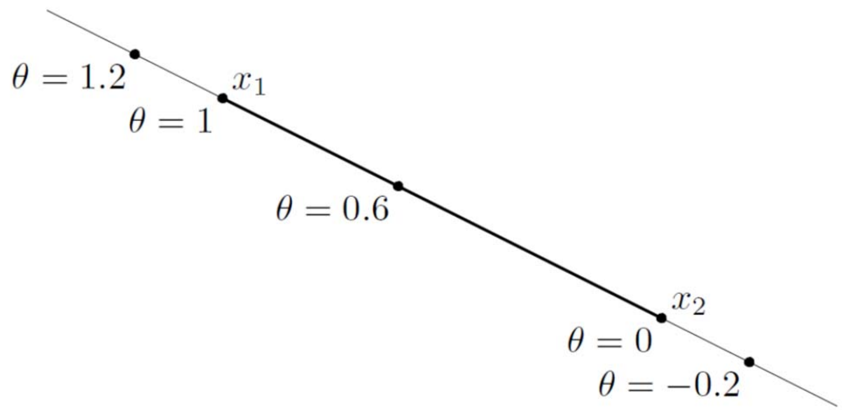
Line

- Lines

$$y = \theta x_1 + (1-\theta)x_2$$

$$y = x_2 + \theta(x_1 - x_2)$$

  - $\theta \in \mathbf{R}$
  - $x_1 \neq x_2$



- Line segments
  - $\theta \in [0,1]$
  - $x_1 \neq x_2$

Affine Sets (1)

- Definition
  - $C \subseteq \mathbf{R}^n$ is affine, if

$$\theta x_1 + (1-\theta)x_2 \in C$$

for any $x_1, x_2 \in C$ and $\theta \in \mathbf{R}$

- Generalized form
  - Affine Combination

$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$$

  - $\theta_1 + \theta_2 + \cdots + \theta_k = 1$

Affine Sets (2)

Subspace

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

- $C \subseteq \mathbf{R}^n$ is an affine set, $x_0 \in C$
- Subspace is closed under sums and scalar multiplication

$$\alpha v_1 + \beta v_2 \in V, \qquad \forall v_1, v_2 \in V$$

- C can be expressed as a subspace plus an offset $x_0 \in C$

$$C = V + x_0$$

- Dimension of C: dimension of V

Affine Sets (3)

- Solution set of linear equations is affine

$$C = \{x | Ax = b\}$$

  - Suppose $x_1, x_2 \in C$

$$\begin{aligned} A(\theta x_1 + (1-\theta)x_2) &= \theta A x_1 + (1-\theta) A x_2 \\ &= \theta b + (1-\theta) b \\ &= b \end{aligned}$$

- Every affine set can be expressed as the solution set of a system of linear equations.

Affine Sets (4)

- Affine hull of set C

$$\mathrm{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \,|x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

  - Affine hull is the smallest affine set that contains C

- Affine dimension
  - Affine dimension of a set C as the dimension of its affine hull $\mathrm{aff}\, C$
  - Consider the unit circle $B = \{x \in \mathbf{R}^2 | x_1^2 + x_2^2 = 1\}$, $\mathrm{aff}\, B$ is $\mathbf{R}^2$. So affine dimension is 2

Affine Sets (5)

- Relative interior

$$\text{relint}\, C = \{x \in C | B(x,r) \cap \text{aff}\, C \subseteq C \text{ for some } r > 0\}$$

  - $B(x,r) = \{y | \|y - x\| \leq r\}$, the ball of radius r and center x in the norm $\|\cdot\|$

- Interior Set
  - An element $x \in C \subseteq \mathbf{R}^n$ is called an interior point of C if there exists an $\epsilon > 0$ for which

$$\{y \mid \|y - x\|_2 \leq \epsilon\} \subseteq C$$

  - The set of all points interior to C is called the interior of C and is denoted $\text{int}\, C$

Affine Sets (5)

Relative interior

$$\operatorname{relint} C = \{x \in C \mid B(x,r) \cap \operatorname{aff} C \subseteq C \text{ for some } r > 0\}$$

- $B(x,r) = \{y \mid \|y - x\| \leq r\}$, the ball of radius r and center x in the norm $\|\cdot\|$

Relative boundary

$$\operatorname{cl} C \setminus \operatorname{relint} C$$

- $\operatorname{cl} C$ is the closure of C

Affine Sets (5)

- A square in (x_1, x_2)-plane in $\mathbf{R}^3$

$$C = \{x \in \mathbf{R}^3 | -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}$$

  - Interior is empty
  - Boundary is itself

$$\text{bd}\, C = \text{cl}\, C \setminus \text{int}\, C$$

Affine Sets (5)

- A square in (x_1, x_2)-plane in $\mathbf{R}^3$

$$C = \{x \in \mathbf{R}^3 | -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}$$

  - Interior is empty
  - Boundary is itself
  - Affine hull is the (x_1, x_2)-plane
  - Relative interior

$$\text{relint}\, C = \{x \in \mathbf{R}^3 | -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$$

$$\text{relint}\, C = \{x \in C | B(x, r) \cap \text{aff}\, C \subseteq C \text{ for some } r > 0\}$$

Affine Sets (5)

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$$C = \{x \in \mathbf{R}^3 | -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_3 = 0\}$$

  - Interior is empty
  - Boundary is itself
  - Affine hull is the (x_1, x_2)-plane
  - Relative interior

$$\operatorname{relint} C = \{x \in \mathbf{R}^3 | -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\}$$

  - Relative boundary

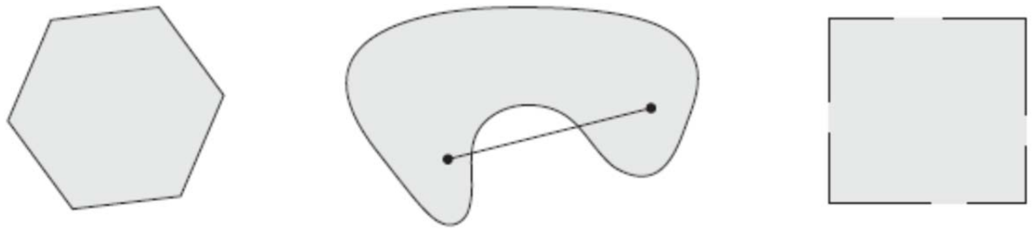
$$\{x \in \mathbf{R}^3 | \max\{|x_1|, |x_2|\} = 1, x_3 = 0\}$$

Convex Sets (1)

Convex sets

- A set C is convex if for any $x_1, x_2 \in C$, any $\theta \in [0,1]$, we have

$$\theta x_1 + (1-\theta) x_2 \in C$$



Generalized form

- Convex combination

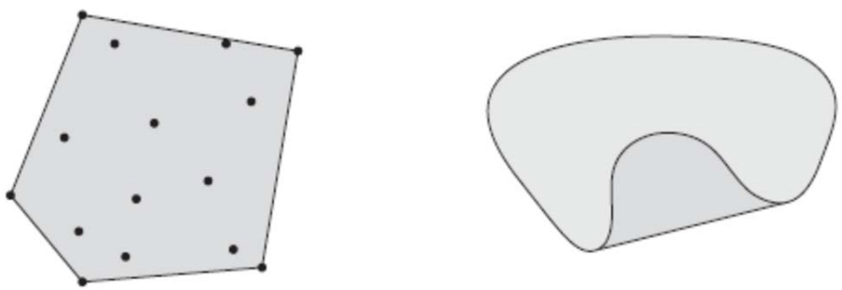
$$\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \in C$$

$$\theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \cdots, k$$

Convex Sets (2)

- Convex hull

$$\mathrm{conv}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_1 + \theta_2 + \cdots + \theta_k = 1, \theta_i \geq 0, i = 1, \cdots, k\}$$

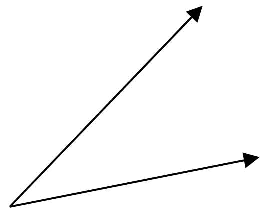


- Infinite sums, integrals

Cone (1)

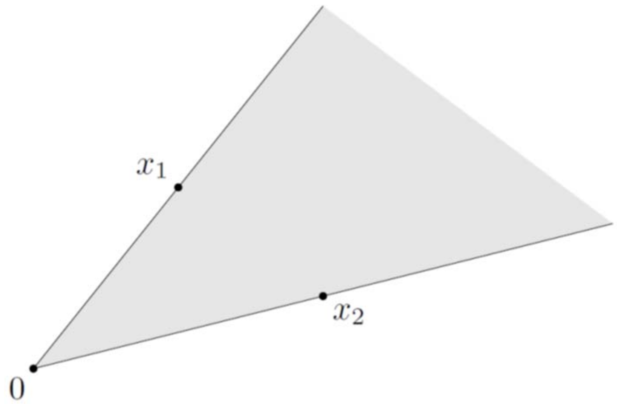
- Cone
  - Cone is a set that

$$x \in C, \theta \geq 0 \Longrightarrow \theta x \in C$$



- Convex cone
  - For any $x_1, x_2 \in C$, $\theta_1, \theta_2 \geq 0$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

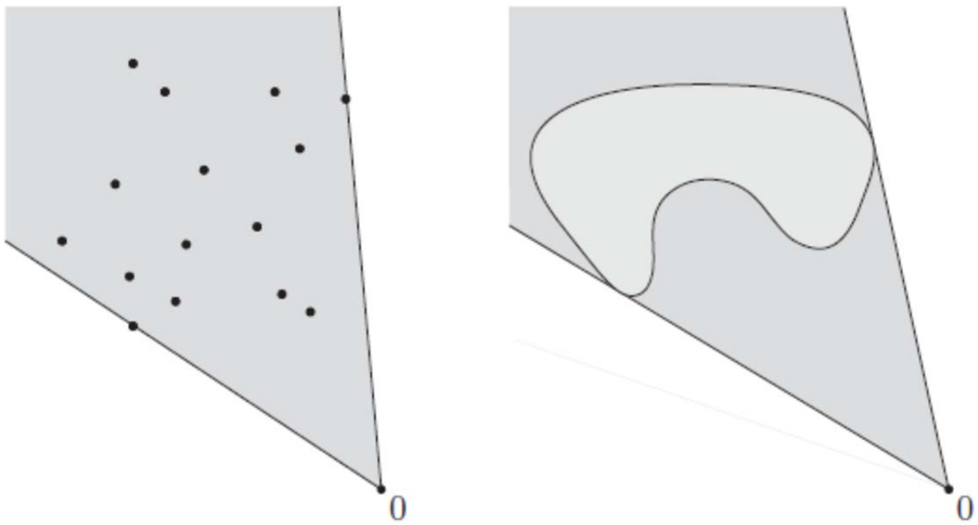


- Conic combination
  - $\theta_1 x_1 + \cdots + \theta_k x_k,\ \theta_i \geq 0, i = 1, \cdots, k$

Cone (2)

- Conic hull

$$\{\theta_1 x_1 + \cdots + \theta_k x_k | x_i \in C,\ \theta_i \geq 0, i = 1, \cdots, k\}$$



Some Examples

- The empty set $\emptyset$, any single point $\{x_0\}$, and the whole space $\mathbf{R}^n$ are affine (hence, convex) subsets of $\mathbf{R}^n$
- Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.
- A line segment is convex, but not affine (unless it reduces to a point).
- A ray, which has the form $\{x_0 + \theta v \mid \theta \geq 0\}$, where $v \neq 0$, is convex, but not affine. It is a convex cone if its base x_0 is 0.
- Any subspace is affine, and a convex cone (hence convex).

Hyperplanes

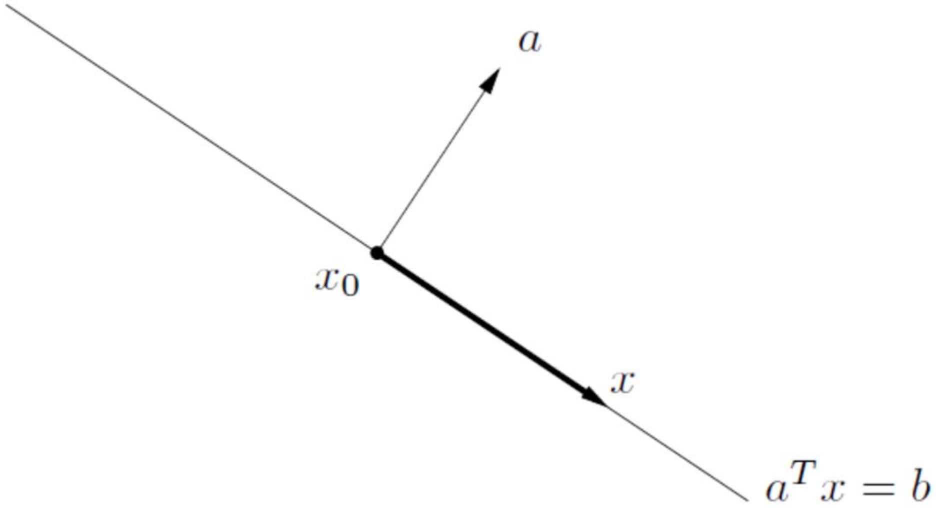
$$\{x \mid a^{\top} x = b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in R$

Other Forms

$$\{x \mid a^{\top}(x - x_0) = 0\}$$

- x_0 is any point such that $a^{\top} x_0 = b$



Hyperplanes

$$\{x | a^\top x = b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in R$

Other Forms

$$\{x | a^\top (x - x_0) = 0\}$$

- x_0 is any point such that $a^\top x_0 = b$

$$\{x | a^\top (x - x_0) = 0\} = x_0 + a^\perp$$

- $a^\perp = \{v | a^\top v = 0\}$

Halfspaces

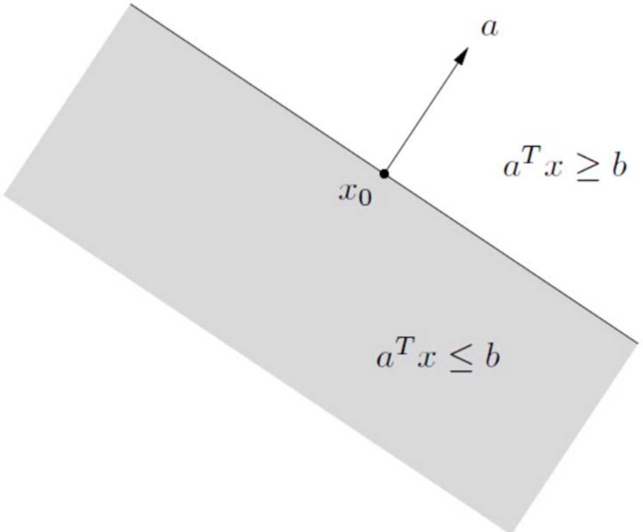
$$\{x | a^\top x \leq b\}$$

- $a \in \mathbf{R}^n$, $a \neq 0$ and $b \in R$

Other Forms

$$\{x | a^\top (x - x_0) \leq 0\}$$

- x_0 is any point such that $a^\top x_0 = b$



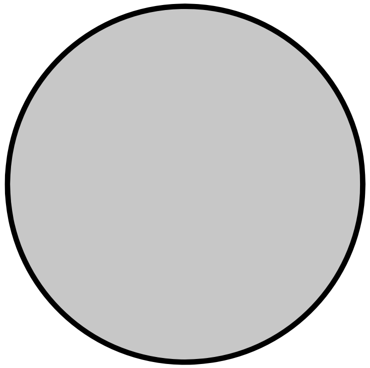
- Convex
- Not affine

Balls

Definition

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}$$
$$= \{x \mid (x - x_c)^\top (x - x_c) \leq r^2\}$$
$$= \{x_c + ru \mid \|u\|_2 \leq 1\}$$

- $r > 0$, and $\|\cdot\|_2$ denotes the Euclidean norm
- Convex

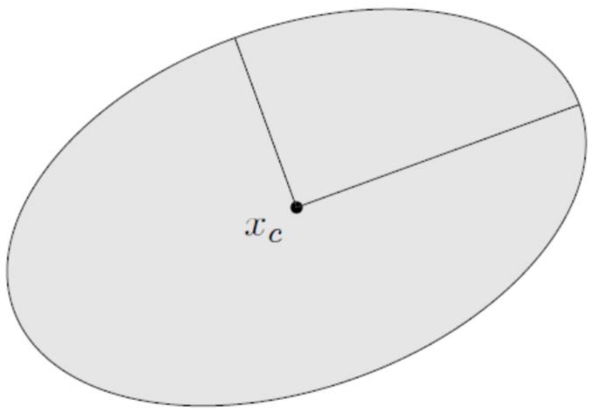


Ellipsoids

Definition

$$\mathcal{E} = \{x \mid (x - x_c)^\top P^{-1}(x - x_c) \le 1\}$$
$$= \{x_c + Au \mid \|u\|_2 \le 1\}$$

- $P \in \mathbf{S}^n_{++}$ determines how far the ellipsoid extends in every direction from x_c;
- Lengths of semi-axes are $\sqrt{\lambda_i}$
- Convex



Norm Balls and Norm Cones

- Norm balls

$$C = \{x \mid \|x - x_c\| \le r\}$$

  - $\|\cdot\|$ is any norm on $\mathbf{R}^n$, x_c is the center

- Norm cones

$$C = \{(x, t) \mid \|x\| \le t\} \subseteq \mathbf{R}^{n+1}$$

  - Second-order Cone

$$\begin{aligned} C &= \{(x, t) \in \mathbf{R}^{n+1} \mid \|x\|_2 \le t\} \\ &= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0, t \ge 0 \right\} \end{aligned}$$

Norm Balls and Norm Cones

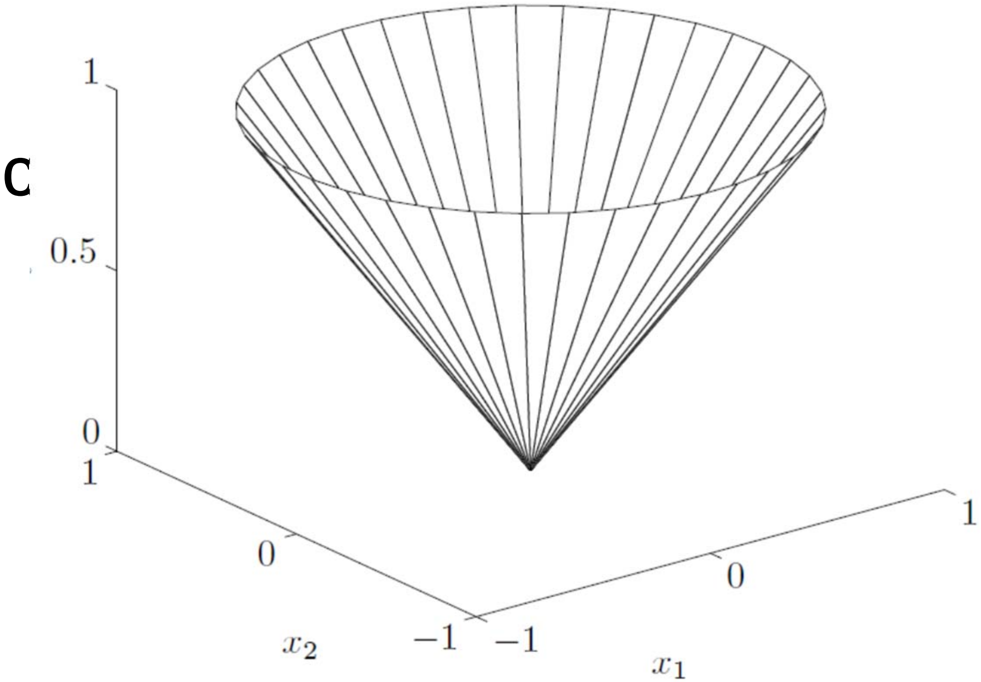
- Norm balls

$$C = \{x \mid \|x - x_c\| \leq r\}$$

  - $\|\cdot\|$ is any norm on $\mathbf{R}^n$, x_c is the center

- Norm cones

  - Sec



Polyhedra (1)

- Polyhedron

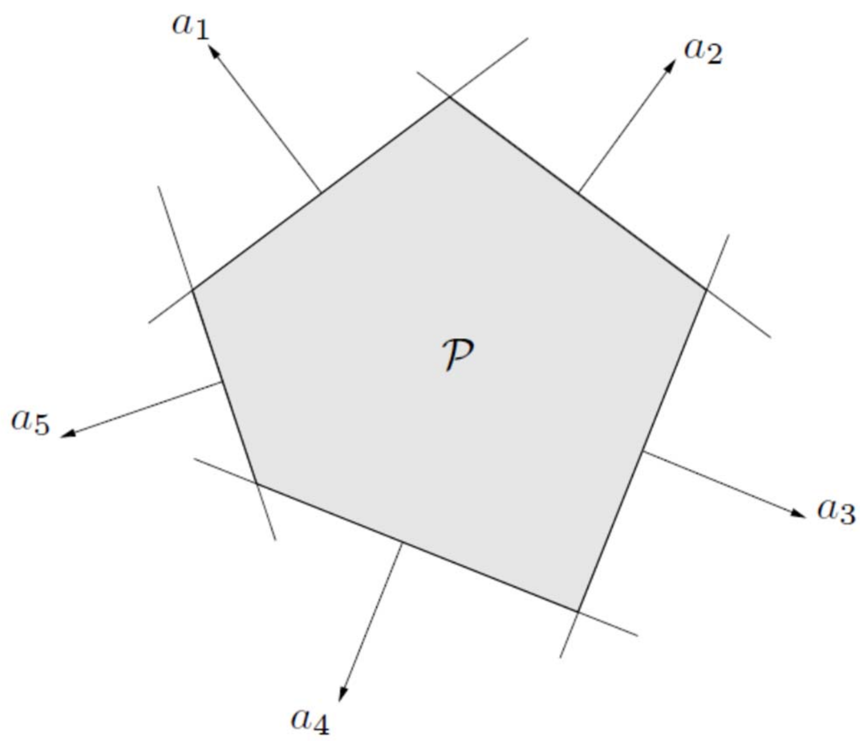
$$\mathcal{P} = \{x \mid a_j^\top x \le b_j, j = 1, \cdots, m, c_j^\top x = d_j, j = 1, \cdots, p\}$$

  - Solution set of a finite number of linear equalities and inequalities
  - Intersection of a finite number of halfspaces and hyperplanes
  - Affine sets (e.g., subspaces, hyperplanes, lines), rays, line segments, and halfspaces are all polyhedra

Polyhedra (2)

- Polyhedron

$$\mathcal{P} = \{x | a_j^\top x \le b_j, j = 1, \cdots, m, c_j^\top x = d_j, j = 1, \cdots, p\}$$



Polyhedra (2)

Polyhedron

$$\mathcal{P} = \{x \mid a_j^\top x \leq b_j, j = 1, \cdots, m, c_j^\top x = d_j, j = 1, \cdots, p\}$$

- Matrix Form

$$\mathcal{P} = \{x \mid Ax \preccurlyeq b, Cx = d\}$$

$$A = \begin{bmatrix} a_1^\top \\ \cdots \\ a_m^\top \end{bmatrix}, \; C = \begin{bmatrix} c_1^\top \\ \cdots \\ c_p^\top \end{bmatrix}$$

$u \preccurlyeq v$ means $u_i \leq v_i$ for all i

Simplexes

Polyhedron?

- An important family of polyhedra

$$C = \mathrm{conv}\{v_0, \cdots, v_k\} = \{\theta_0 v_0 + \cdots + \theta_k v_k | \theta \succcurlyeq 0, 1^\top \theta = 1\}$$

  - $k+1$ points $v_0, \cdots, v_k$ are affinely independent
  - The affine dimension of this simplex is k
- 1-dimensional simplex: line segment
- 2-dimensional simplex: triangle
- Unit simplex: $x \succcurlyeq 0, 1^\top x \leq 1$
  - n-dimensional
- Probability simplex: $x \succcurlyeq 0, 1^\top x = 1$
  - $(n-1)$-dimensional

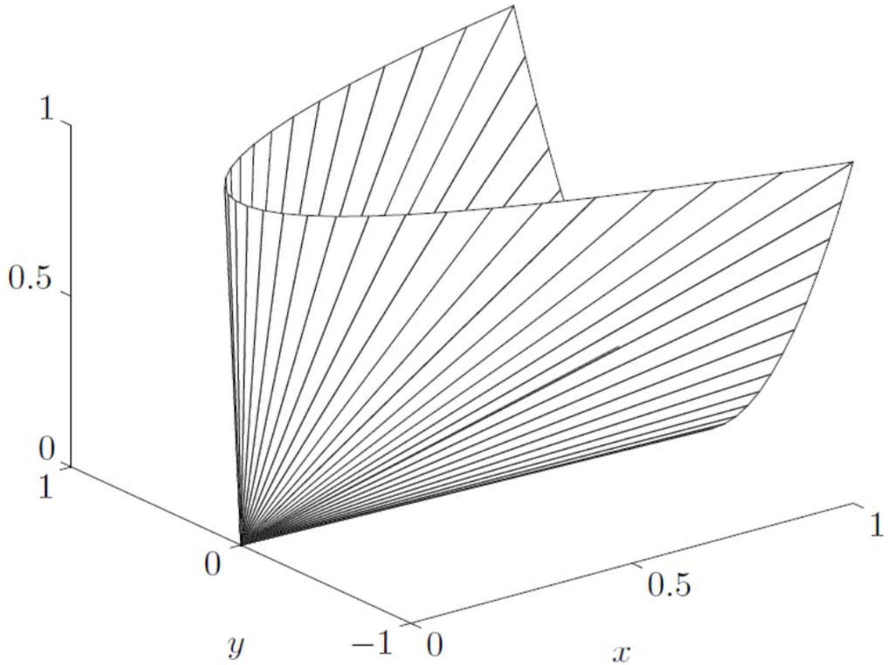
The positive semidefinite cone

- $\mathbf{S}^n = \{X \in \mathbf{R}^{n\times n} \mid X = X^\top\}$ is the set of symmetric $n \times n$ matrices
  - Vector space with dimension $n(n+1)/2$
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succcurlyeq 0\}$ is the set of symmetric positive semidefinite matrices
  - Convex cone
- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$ is the set of symmetric positive definite

The positive semidefinite cone

□ PSD Cone in $\boldsymbol{S}^2$

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2 \iff x \geq 0, z \geq 0, xz \geq y^2$$



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Intersection

- If S_1 and S_2 are convex, then $S_1 \cap S_2$ is convex
  - A polyhedron is the intersection of halfspaces and hyperplanes

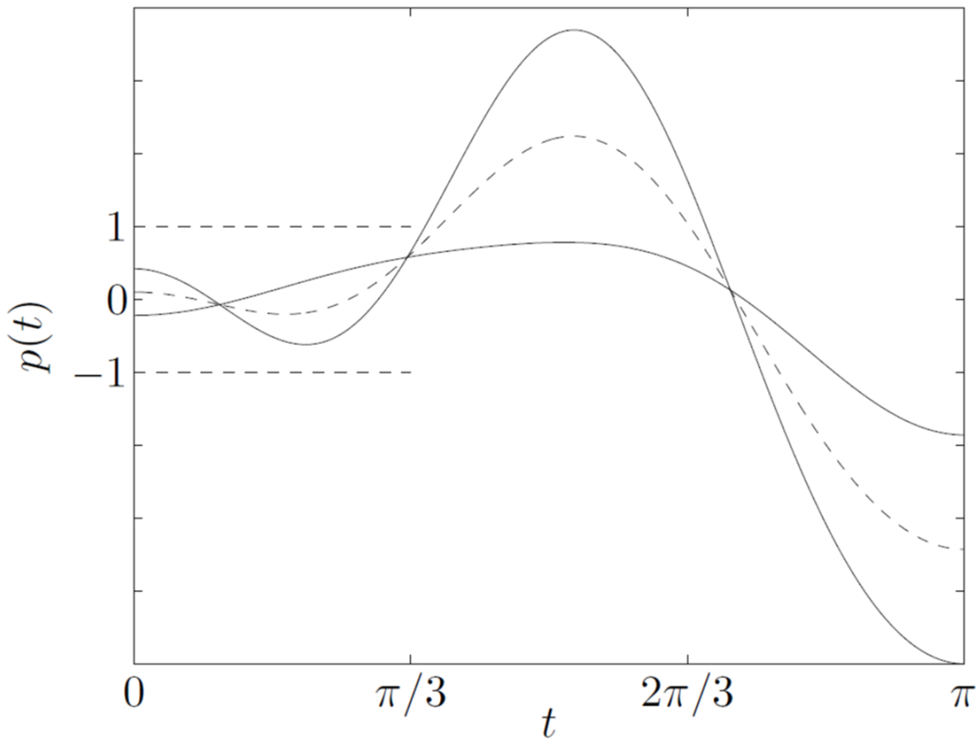
- if S_α is convex for every $\alpha \in \mathcal{A}$, then $\cap_{\alpha \in \mathcal{A}} S_\alpha$ is convex
  - Positive semidefinite cone

$$\mathbf{S}_+^n = \bigcap_{z \neq 0} \{X \in \mathbf{S}^n | z^\top X z \geq 0\}$$

A Complicated Example (1)

$$S = \left\{ x \in \mathbf{R}^m \,\middle|\, |p(t)| \leq 1 \text{ for } |t| \leq \frac{\pi}{3} \right\}$$

- $p(t) = \sum_{k=1}^{m} x_k \cos kt$



A Complicated Example (2)

$$S = \left\{ x \in \mathbf{R}^m \,\middle|\, |p(t)| \le 1 \text{ for } |t| \le \frac{\pi}{3} \right\}$$

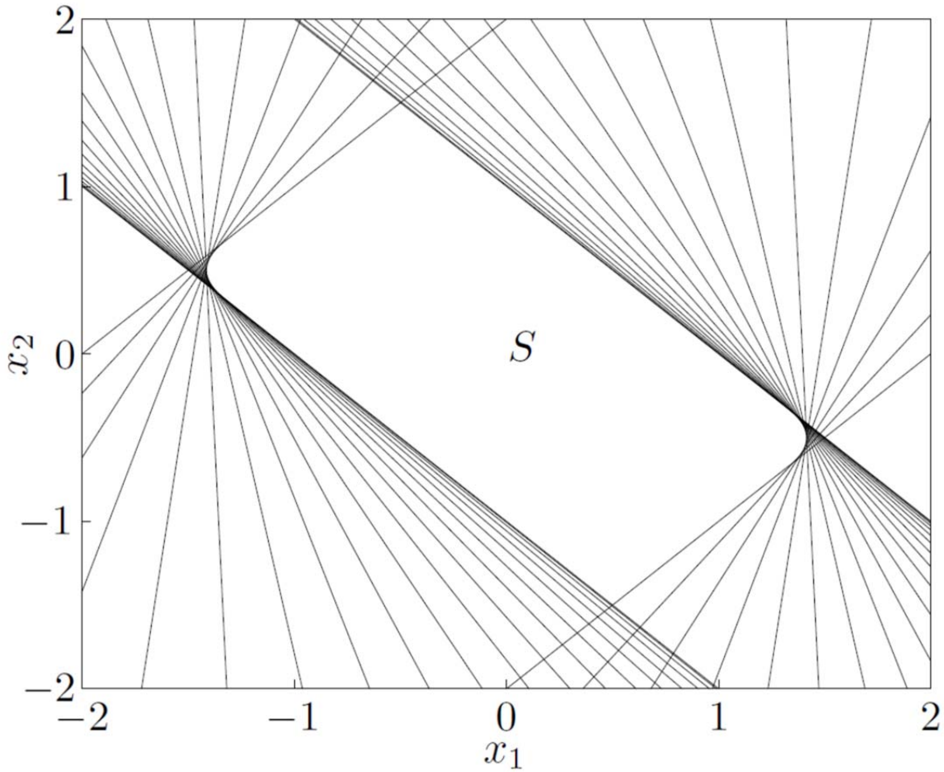
- $p(t) = \sum_{k=1}^{m} x_k \cos kt$

$$S = \bigcap_{|t| \le \pi/3} S_t$$

- $S_t = \{x \mid -1 \le (\cos t, \dots, \cos mt)^\top x \le 1\}$

A Complicated Example (3)

$$S = \bigcap_{|t| \le \pi/3} S_t = \bigcap_{|t| \le \pi/3} \{x \mid -1 \le (\cos t, \dots, \cos mt)^\top x \le 1\}$$



Affine Functions

- Affine function $f: \mathbf{R}^n \to \mathbf{R}^m$

$$f(x) = Ax + b, A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$$

- $S \subseteq \mathbf{R}^n$ is convex

- Then, the image of S under f

$$f(S) = \{f(x) \mid x \in S\}$$

and the inverse image of S under f

$$f^{-1}(S) = \{x \mid f(x) \in S\}$$

are convex

Examples (1)

- Scaling

$$\alpha S = \{\alpha x \mid x \in S\}$$

- Translation

$$S + a = \{x + a \mid x \in S\}$$

- Projection of a convex set onto some of its coordinates

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

  - $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex

Examples (2)

- Sum of two sets

$$S_1 + S_2 = \{x + y | x \in S_1, y \in S_2\}$$

  - Cartesian product: $S_1 \times S_2 = \{(x_1, x_2) | x_1 \in S_1, x_2 \in S_2\}$
  - Linear function: $f(x_1, x_2) = x_1 + x_2$

- Partial sum of $S_1, S_2 \in \mathbf{R}^n \times \mathbf{R}^m$

$$S = \{(x, y_1 + y_2) | (x, y_1) \in S_1, (x, y_2) \in S_2\}$$

  - $m = 0$, intersection of S_1 and S_2
  - $n = 0$, set addition

Examples (3)

- Polyhedron

$$\{x|Ax \preccurlyeq b, Cx = d\} = \{x|f(x) \in \mathbf{R}_+^m \times \{0\}\}$$

  - $f(x) = (b - Ax, d - Cx)$

- Linear Matrix Inequality

$$A(x) = x_1A_1 + \cdots + x_nA_n \preccurlyeq B$$

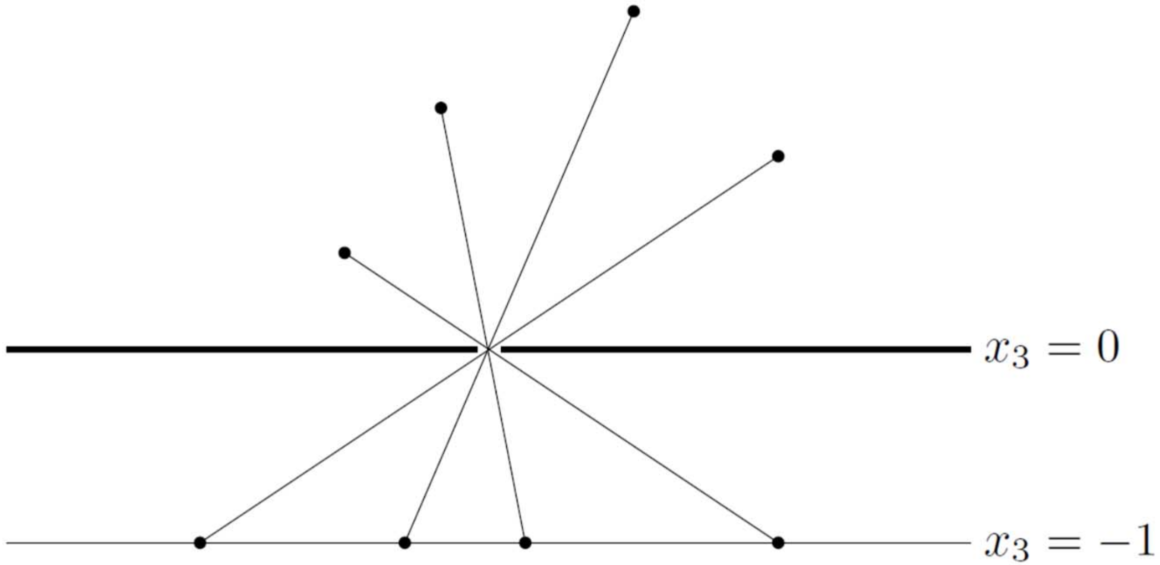
  - The solution set $\{x|A(x) \preccurlyeq B\}$

$$\{x|A(x) \preccurlyeq B\} = \{x|B - A(x) \in \mathbf{S}_+^m\}$$

Perspective Functions (1)

- Perspective function $P: \mathbf{R}^{n+1} \to \mathbf{R}^n$

$$P(z,t) = \frac{z}{t}, \mathrm{dom}\, P = \mathbf{R}^n \times \mathbf{R}_{++}$$



$$(x_1, x_2, x_3) \mapsto -(x_1/x_3, x_2/x_3, 1)$$

Perspective Functions (2)

- Perspective function $P: \mathbf{R}^{n+1} \to \mathbf{R}^n$

$$P(z,t) = \frac{z}{t}, \operatorname{dom} P = \mathbf{R}^n \times \mathbf{R}_{++}$$

- If $C \subseteq \operatorname{dom} P$ is convex, then its image

$$P(C) = \{P(x) | x \in C\}$$

is convex

- If $C \subseteq \mathbf{R}^n$ is convex, the inverse image

$$P^{-1}(C) = \left\{(x,t) \in \mathbf{R}^{n+1} \middle| \frac{x}{t} \in C, t > 0\right\}$$

is convex

Linear-fractional Functions (1)

- Suppose $g: \mathbf{R}^n \rightarrow \mathbf{R}^{m+1}$ is affine

$$g(x) = \begin{bmatrix} A \\ c^\top \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} Ax + b \\ c^\top x + d \end{bmatrix}$$

- The function $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ given by $P \circ g$

$$f(x) = \frac{Ax + b}{c^\top x + d}, \mathrm{dom}\, f = \{c^\top x + d > 0\}$$

Linear-fractional Functions (2)

- If C is convex and $\{c^\top x + d > 0 \text{ for } x \in C\}$, then

$$f(C) = \left\{ \frac{Ax+b}{c^\top x + d} \middle| x \in C \right\}$$

is convex

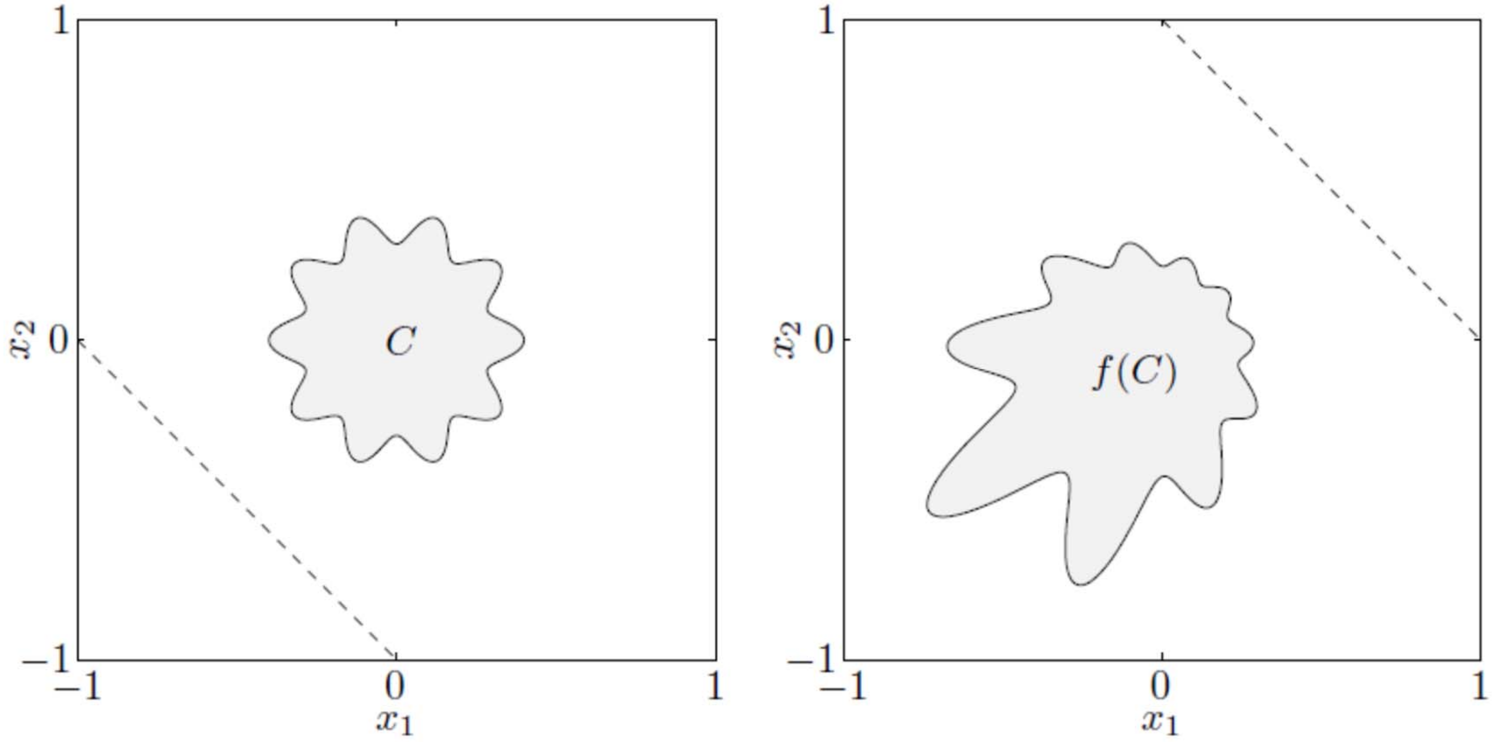
- If $C \subseteq \mathbf{R}^m$ is convex, then the inverse image

$$f^{-1}(C) = \left\{ x \middle| \frac{Ax+b}{c^\top x + d} \in C \right\}$$

is convex

Example

$$f(x) = \frac{1}{x_1 + x_2 + 1} x, \mathrm{dom}\, f = \{(x_1, x_2) | x_1 + x_2 + 1 > 0\}$$



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Proper Cones

- A cone $K \subseteq \mathbf{R}^n$ is called a proper cone if it satisfies the following
  - K is convex.
  - K is closed.
  - K is solid, which means it has nonempty interior.
  - K is pointed, which means that it contains no line ($x \in K, -x \in K \Longrightarrow x = 0$).
- A proper cone K can be used to define a generalized inequality

Generalized Inequalities

- We associate with the proper cone K the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_K y \iff y - x \in K$$

- We define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \operatorname{int} K$$

Examples

- Nonnegative Orthant and Componentwise Inequality
  - $K = \mathbf{R}_+^n$
  - $x \preccurlyeq_K y$ means that $x_i \leq y_i, i = 1, \ldots, n$.
  - $x \prec_K y$ means that $x_i < y_i, i = 1, \ldots, n$.
- Positive Semidefinite Cone and Matrix Inequality
  - $K = \mathbf{S}_+^n$
  - $X \preccurlyeq_K Y$ means that $Y - X$ is PSD
  - $X \prec_K Y$ means that $Y - X$ is positive definite

Properties of Generalized Inequalities

- $\preccurlyeq_K$ is preserved under addition: If $x \preccurlyeq_K y$ and $u \preccurlyeq_K v$, then $x + u \preccurlyeq_K y + v$.
- $\preccurlyeq_K$ is transitive: if $x \preccurlyeq_K y$ and $y \preccurlyeq_K z$, then $x \preccurlyeq_K z$.
- $\preccurlyeq_K$ is preserved under nonnegative scaling: if $x \preccurlyeq_K y$ and $\alpha \geq 0$ then $\alpha x \preccurlyeq_K \alpha y$.
- $\preccurlyeq_K$ is reflexive: $x \preccurlyeq_K x$.
- $\preccurlyeq_K$ is antisymmetric: if $x \preccurlyeq_K y$ and $y \preccurlyeq_K x$, then $x = y$.
- $\preccurlyeq_K$ is preserved under limits: if $x_i \preccurlyeq_K y_i$ for $i = 1, 2, \ldots, x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preccurlyeq_K y$.

Properties of Strict Generalized Inequalities

- If $x \prec_K y$ then $x \preccurlyeq_K y$.
- If $x \prec_K y$ and $u \preccurlyeq_K v$ then $x + u \prec_K y + v$.
- If $x \prec_K y$ and $\alpha > 0$ then $\alpha x \prec_K \alpha y$.
- $x \nprec_K x$.
- If $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

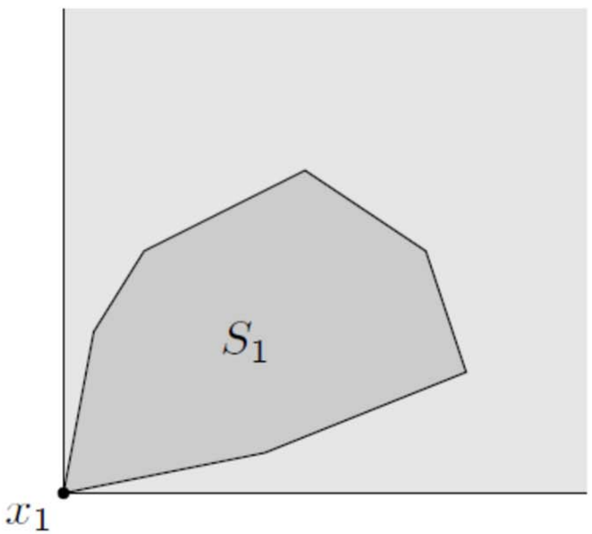
Minimum and Minimal Elements

- $x \in S$ is the minimum element
  - If for every $y \in S$, we have $x \preccurlyeq_K y$
  - $S \subseteq x + K$
  - Minimum element is unique, if exists
- $x \in S$ is a minimal element
  - if $y \in S$, $y \preccurlyeq_K x$ only if $y = x$
  - $(x - K) \cap S = \{x\}$
  - May have different minimal elements
- Maximum, Maximal

Example

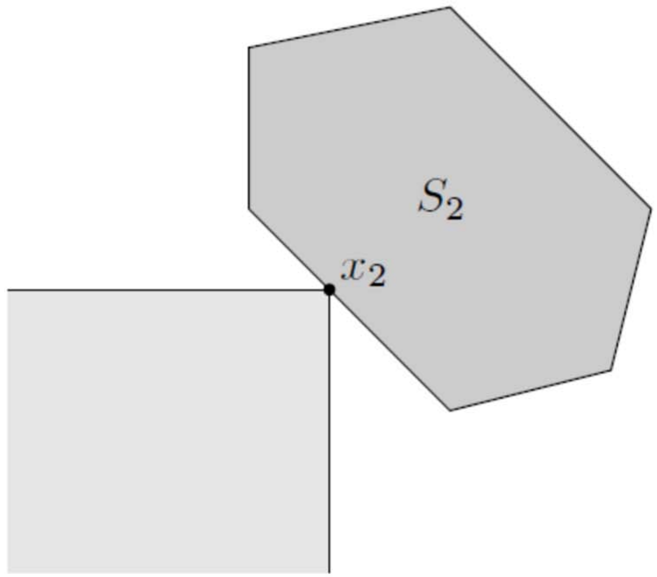
The Cone $\mathbf{R}_+^2$

- $x \preccurlyeq y$ means y is above and to the right of x



$S \subseteq x + K$

$(x - K) \cap S = \{x\}$

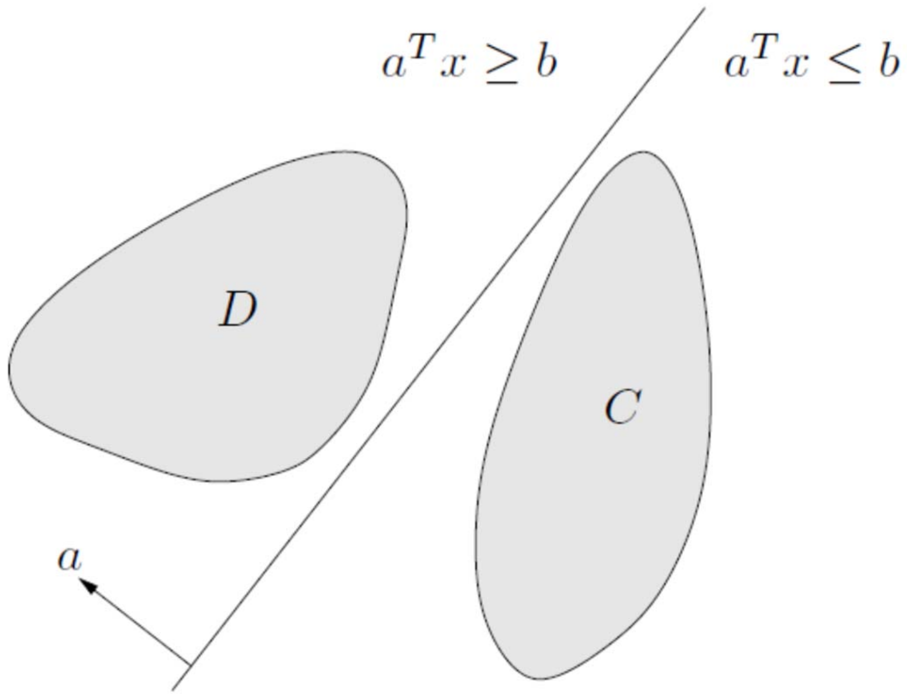


Outline

- Affine and Convex Sets
- Operations That Preserve Convexity
- Generalized Inequalities
- Separating and Supporting Hyperplanes
- Dual Cones and Generalized Inequalities
- Summary

Separating Hyperplane Theorem

- Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then, there exist $a \neq 0$ and b such that



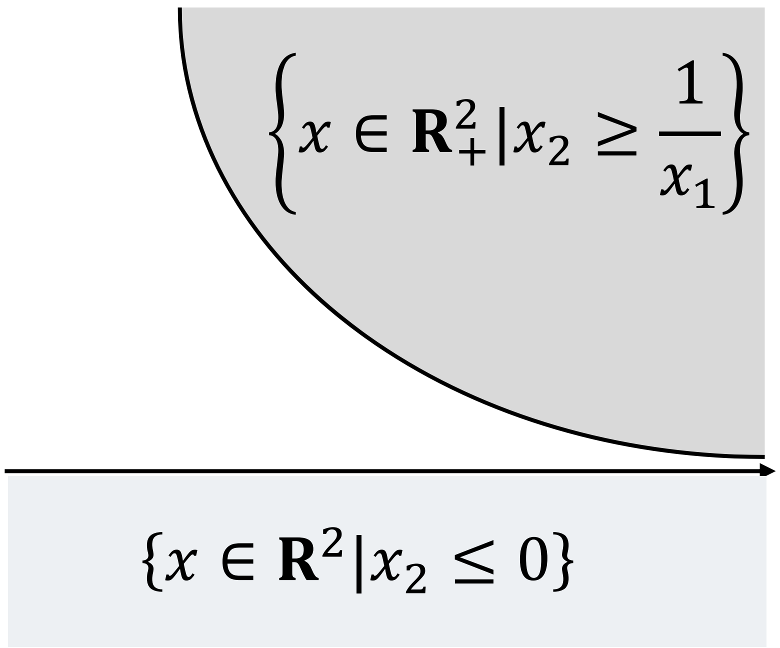
Separating Hyperplane Theorem

- Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then, there exist $a \neq 0$ and b such that $a^\top x \leq b$ for all $x \in C$ and $a^\top x \geq b$ for all $x \in D$.

- $\{x | a^\top x = b\}$ is called a separating hyperplane for the sets C and D.

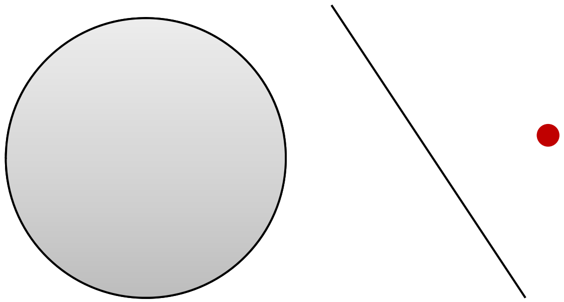
Strict Separation

- $a^\top x < b$ for all $x \in C$ and $a^\top x > b$ for all $x \in D$
- May not be possible in general



Strict Separation

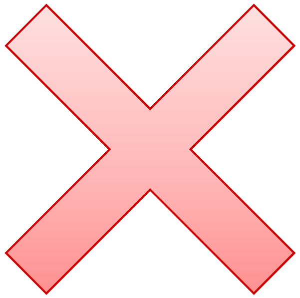
- $a^{\top}x < b$ for all $x \in C$ and $a^{\top}x > b$ for all $x \in D$
- May not be possible in general
- A Point and a Closed Convex Set



- A closed convex set is the intersection of all halfspaces that contain it

Converse separating hyperplane theorems

- Suppose C and D are convex sets, and there exists an affine function f that is nonpositive on C and nonnegative on D. Then C and D are disjoint.



Converse separating hyperplane theorems

- Suppose C and D are convex sets, with C open, and there exists an affine function f that is nonpositive on C and nonnegative on D. Then C and D are disjoint.

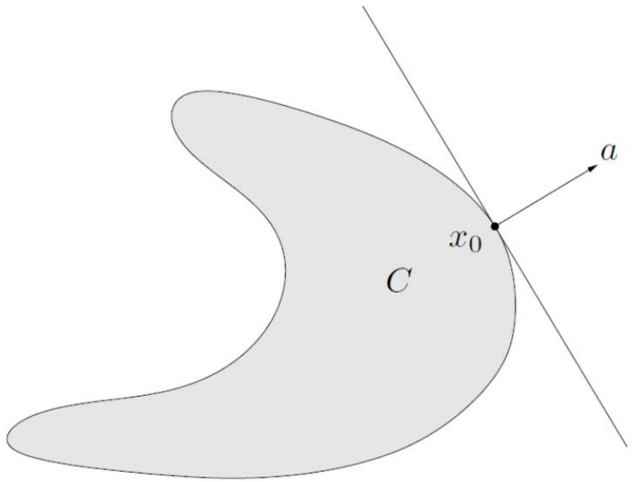
- Any two convex sets C and D, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Supporting Hyperplanes

- Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\mathrm{bd}\, C$, i.e.,

$$x_0 \in \mathrm{bd}\, C = \mathrm{cl}\, C \setminus \mathrm{int}\, C$$

- If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$. The hyperplane $\{x | a^\top x = a^\top x_0\}$ is called a supporting hyperplane to C at the point x_0



Two Theorems

- Supporting Hyperplane Theorem
  - For any nonempty convex set C, and any $x_0 \in \operatorname{bd} C$, there exists a supporting hyperplane to C at x_0.

- Converse Theorem
  - If a set is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex.

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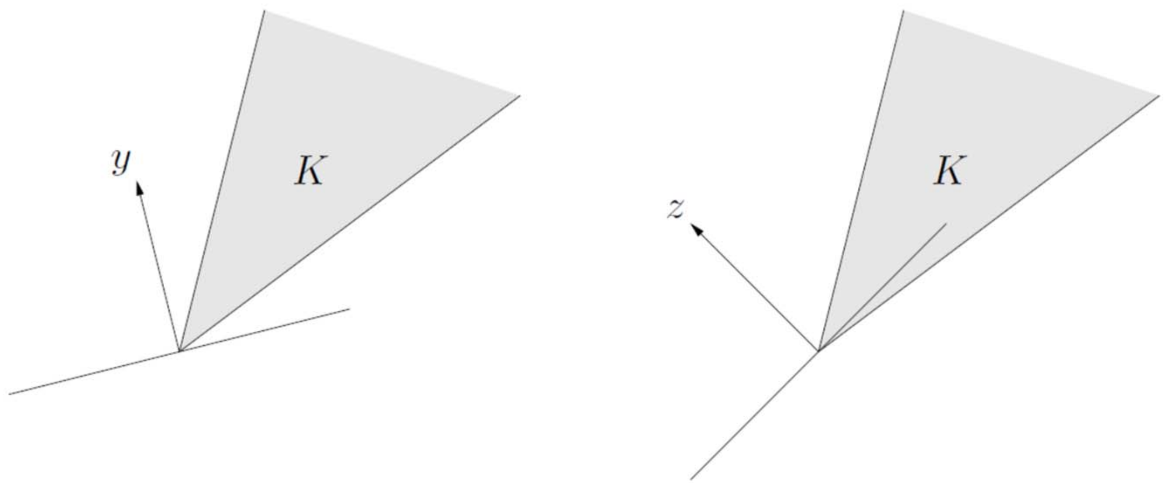
Dual Cone

□ Dual Cone of a Given Cone K

$$K^* = \{y | x^\top y \geq 0 \text{ for all } x \in K\}$$

- K^* is convex, even when K is not
- $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin



Examples

- Subspace
  - The dual cone of a subspace $V \in \mathbf{R}^n$

$$V^{\perp} = \{y | v^{\top} y = 0 \text{ for all } v \in V\}$$

- Nonnegative Orthant
  - The cone $\mathbf{R}^n_+$ is its own dual

$$x^{\top} y \geq 0 \text{ for all } x \succcurlyeq 0 \Leftrightarrow y \succcurlyeq 0$$

- Positive Semidefinite Cone
  - $\mathbf{S}^n_+$ is self-dual

$$\operatorname{tr}(XY) \geq 0 \text{ for all } X \succcurlyeq 0 \Leftrightarrow Y \succcurlyeq 0$$

Properties of Dual Cone

- K^* is closed and convex.
- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$
- If K has nonempty interior, then K^* is pointed.
- If the closure of K is pointed then K^* has nonempty interior.
- K^{**} is the closure of the convex hull of K. (Hence if K is convex and closed, $K^{**} = K$.)

Dual Generalized Inequalities

- Suppose that the convex cone K is proper, so it induces a generalized inequality $\preccurlyeq_K$.
- Its dual cone K^* is also proper. We refer to the generalized inequality $\preccurlyeq_{K^*}$ as the dual of the generalized inequality $\preccurlyeq_K$.

Dual Generalized Inequalities

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- Its dual cone K^* is also proper. We refer to the generalized inequality $\preccurlyeq_{K^*}$ as the dual of the generalized inequality $\preccurlyeq_K$.
  - $x \preccurlyeq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$

$$\left.\begin{array}{l} x \preccurlyeq_K y \Rightarrow y - x \in K \\ \lambda \succcurlyeq_{K^*} 0 \Rightarrow \lambda \in K^* \\ K^* = \{y | x^\top y \geq 0 \text{ for all } x \in K\} \end{array}\right\} \quad \lambda^\top (y - x) \geq 0$$

Dual Generalized Inequalities

- Suppose that the convex cone K is proper, so it induces a generalized inequality $\preccurlyeq_K$.
- Its dual cone K^* is also proper. We refer to the generalized inequality $\preccurlyeq_{K^*}$ as the dual of the generalized inequality $\preccurlyeq_K$.
  - $x \preccurlyeq_K y$ if and only if $\lambda^\top x \leq \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$
  - $x \prec_K y$ if and only if $\lambda^\top x < \lambda^\top y$ for all $\lambda \succcurlyeq_{K^*} 0$, $\lambda \neq 0$
  - These properties hold if the generalized inequality and its dual are swapped

Dual Characterization of Minimum Element

- x is the minimum element of S, with respect to the generalized inequality $\preccurlyeq_K$, if and only if for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^\top z$ over $z \in S$.

- That means, for any $\lambda \succ_{K^*} 0$, the hyperplane $\{z \mid \lambda^\top (z - x) = 0\}$ is a strict supporting hyperplane to S at x.

$$\lambda^\top z \geq \lambda^\top x \Leftrightarrow -\lambda^\top z \leq -\lambda^\top x, \qquad \forall z \in S$$

Dual Characterization of Minimum Element

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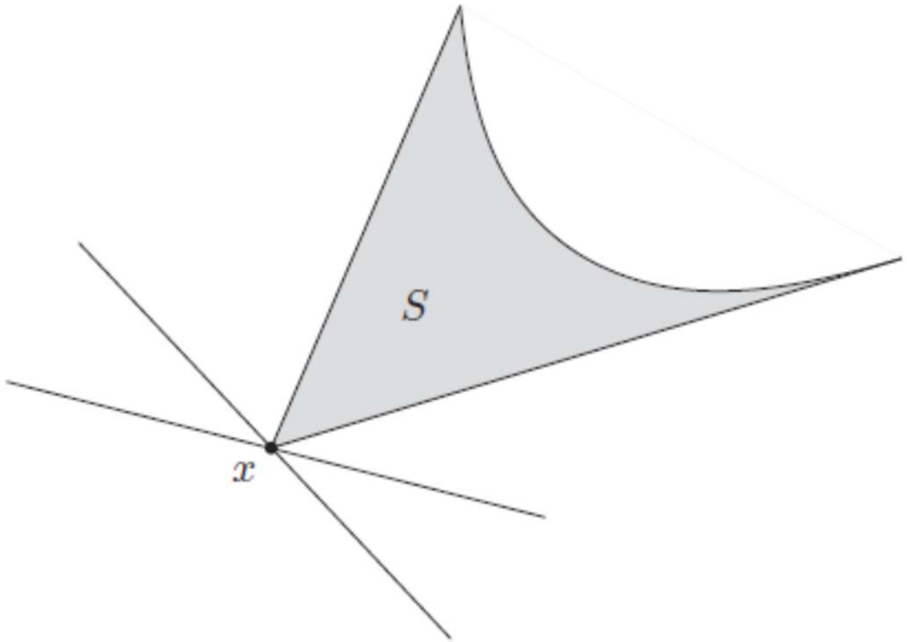
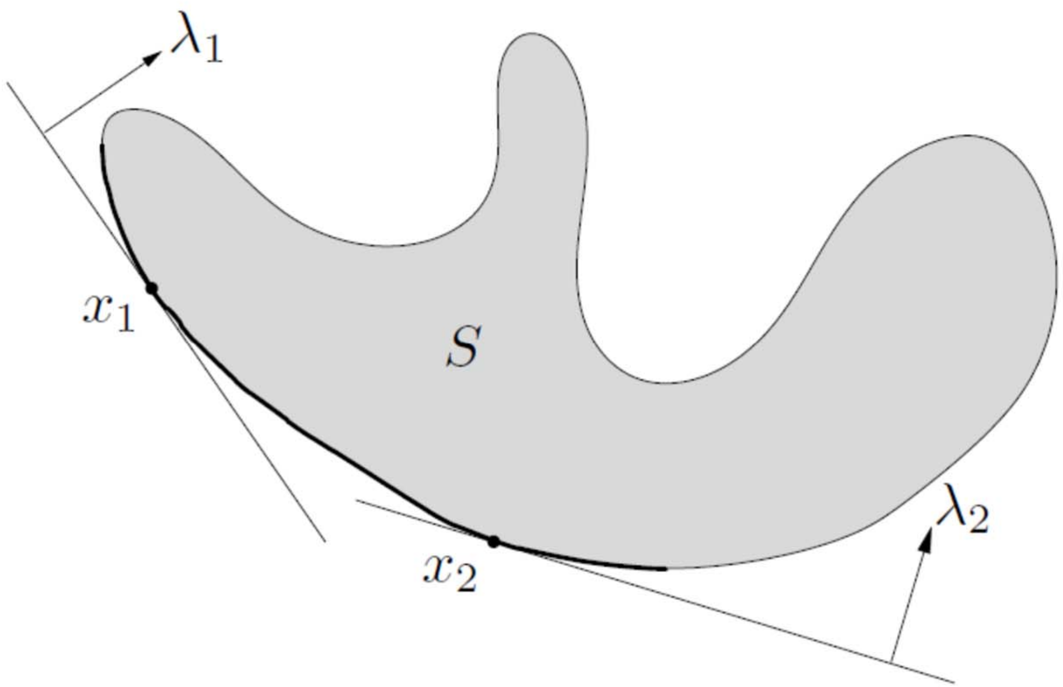


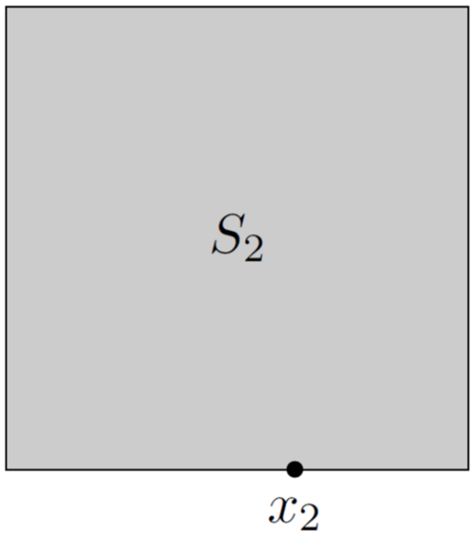
Dual Characterization of Minimal Elements (1)

- If $\lambda \succ_{K^*} 0$, and x minimizes $\lambda^\top z$ over $z \in S$, then x is minimal.



Dual Characterization of Minimal Elements (1)

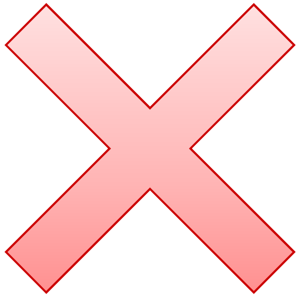
- Any minimizer of $\lambda^{\mathsf{T}} z$ over $z \in S$, with $\lambda \succcurlyeq_{K^*} 0$, is minimal.

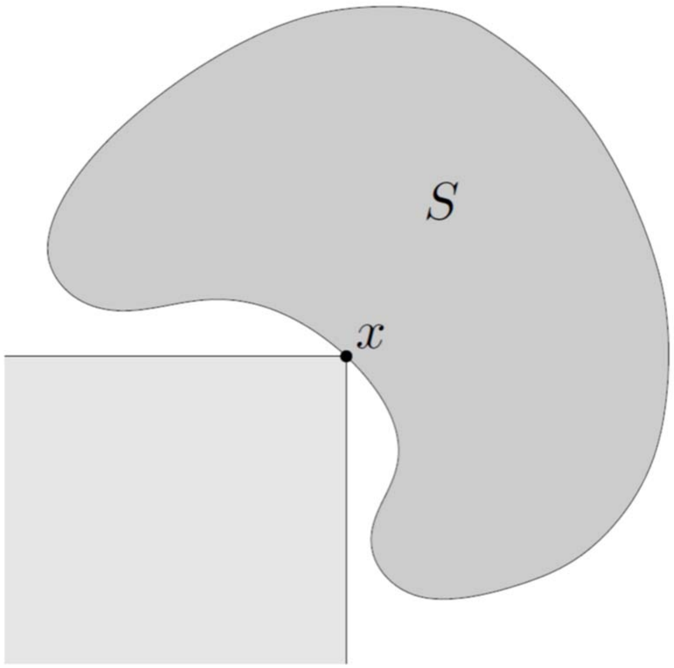


x_2 minimizes $\lambda^{\mathsf{T}} z$ over $z \in S_2$ for $\lambda = (0,1) \succcurlyeq 0$

Dual Characterization of Minimal Elements (2)

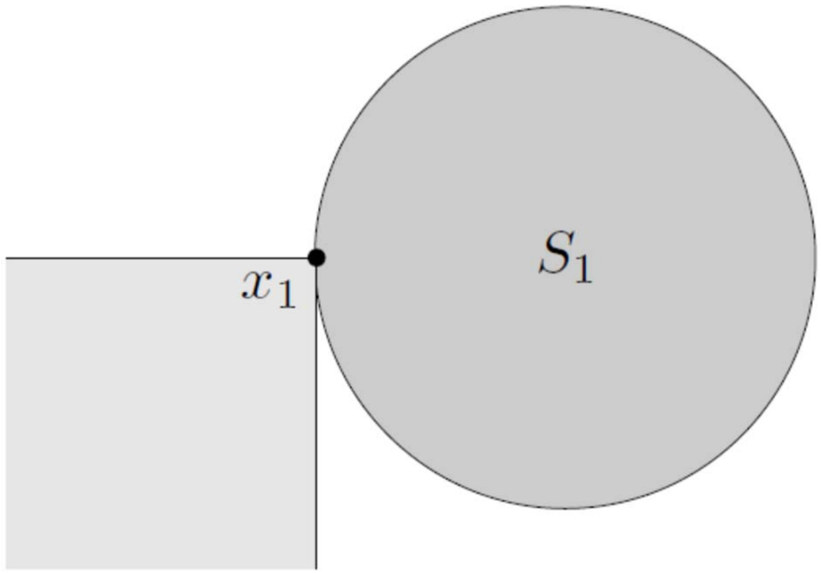
- If x is minimal, then x minimizes $\lambda^\top z$ over $z \in S$ with $\lambda \succ_{K^*} 0$.





Dual Characterization of Minimal Elements (2)

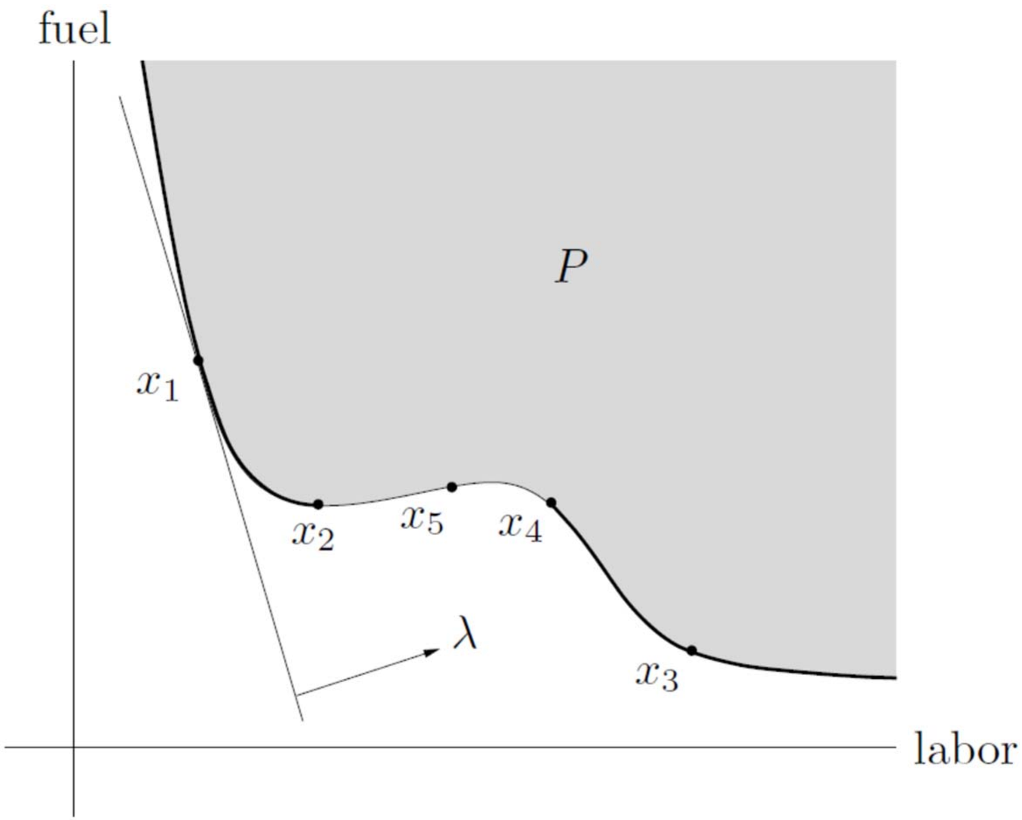
- If S is convex, for any minimal element x there exists a nonzero $\lambda \succcurlyeq_{K^*} 0$ such that x minimizes $\lambda^\top z$ over $z \in S$.



x_1 minimizes $\lambda^\top z$ over $z \in S_1$ for $\lambda = (1,0) \succcurlyeq 0$

Pareto Optimal Production Frontier

- A product which requires n sources
- A resource vector $x \in \mathbf{R}^n$



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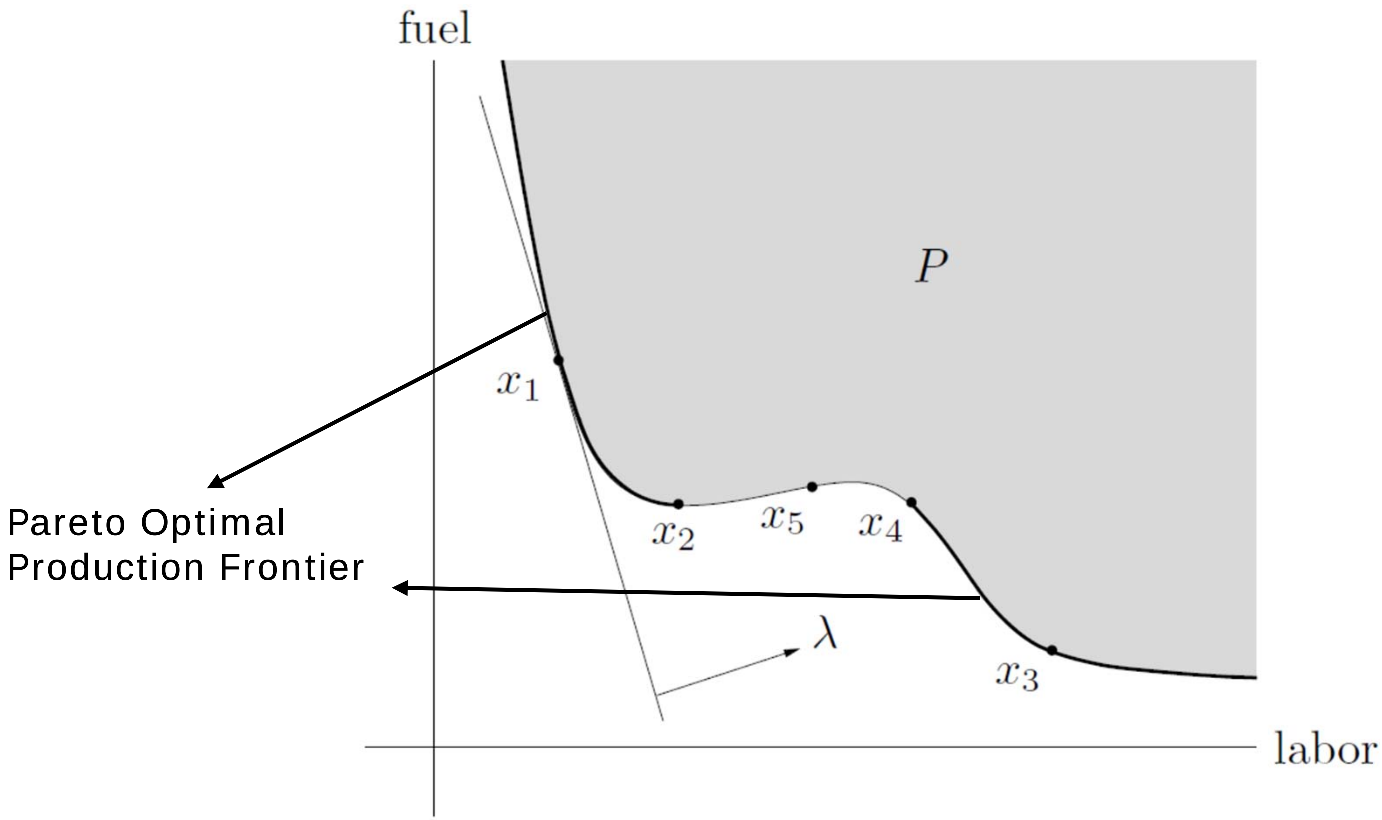

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Summary

- Affine and convex
- Operations that preserve convexity
- Generalized Inequalities
- Separating and supporting hyperplanes
  - Theorems
- Dual cones and generalized inequalities