Orientation rules for incorporating causal knowledge in causal insufficiency: Applications to IDA with latent confounders and MAG listing

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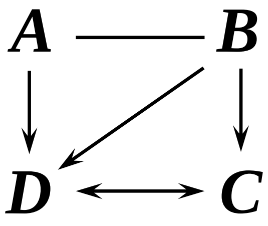
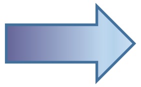
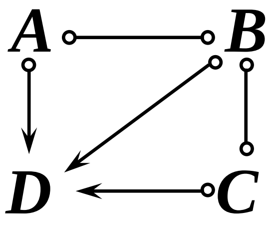
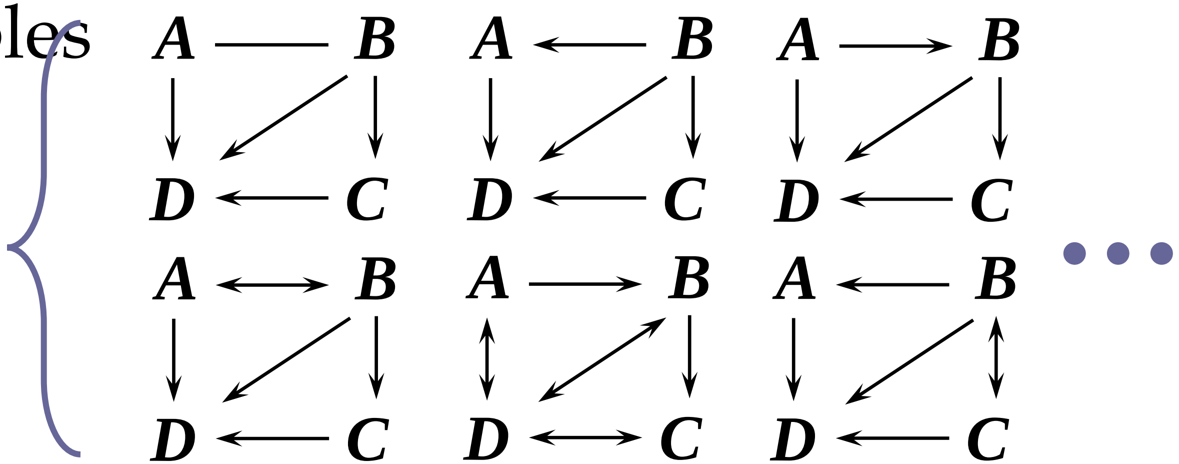
Maximal ancestral graph

- **Maximal ancestral graph (MAGs)** is widely used to characterize causal relations among observable variables in the presence of latent confounders and selection variables









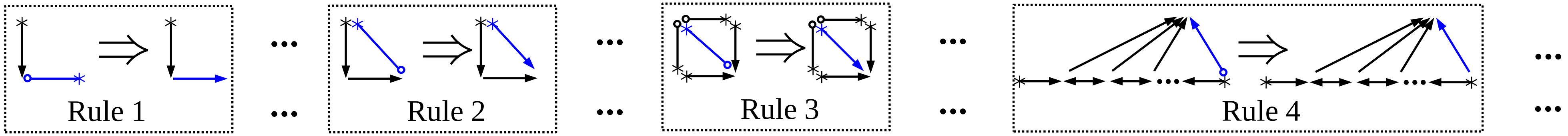
MAG PAG An MEC of MAGs

- With observational data and under mild assumptions, we generally cannot identify a MAG, instead, only a **partial ancestral graph (PAG)**
- A PAG represents a **Markov equivalence class (MEC)** of MAGs

(AOS'02) Thomas Richardson, Peter Spirtes. Ancestral Graph Markov models

Orientation rules

- There are **ten rules** for learning a PAG with observational data by exploiting conditional independence

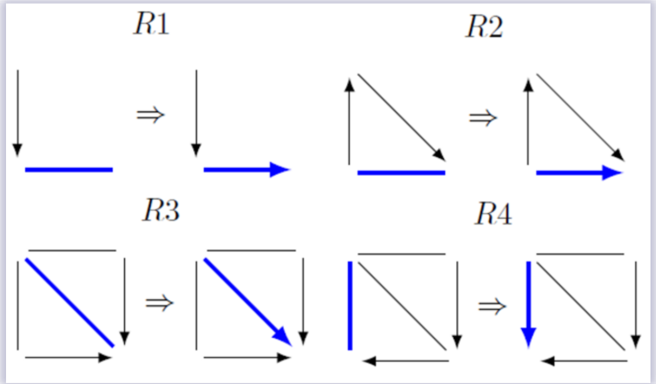


(AIJ 2008) Jiji Zhang. On the completeness of orientation rules for causal discovery in the presence of latent confounders and selection bias

Incorporating background knowledge

- There are **ten rules** for learning a PAG with observational data by exploiting conditional independence
- If there is background knowledge (BK) beyond observational data, it remains an open problem what causal relations are identifiable

Causal sufficiency → Meek rules



(From Perkovic et al. 2017)

- Our research revolves around the orientation rules for incorporating the background knowledge, which is applied in some tasks

Task 1: IDA under latent confounders

IDA under latent confounders

- Fact: The causal effect of X on Y in a PAG is possibly unidentifiable, since the large number of consistent MAGs are possibly associated with different causal effects
- Task: How to determine **the set of possible causal effects by finding adjustment set for (X, Y)** in all the MAGs consistent with the PAG? – **Set determination**
  - Adjustment set : a set $\boldsymbol{Z}$ such that $P(Y|do(X)) = \int P(Z)P(Y|X,Z)dZ$

Enumerating all MAGs is inefficient, with super-exponential complexity

A similar task for CPDAG: IDA

ESTIMATING HIGH-DIMENSIONAL INTERVENTION EFFECTS FROM OBSERVATIONAL DATA

BY MARLOES H. MAATHUIS, MARKUS KALISCH AND PETER BÜHLMANN

ETH Zürich

We assume that we have observational data generated from an unknown underlying directed acyclic graph (DAG) model. A DAG is typically not identifiable from observational data, but it is possible to consistently estimate the equivalence class of a DAG. Moreover, for any given DAG, causal effects can be estimated using intervention calculus. In this paper, we combine these two

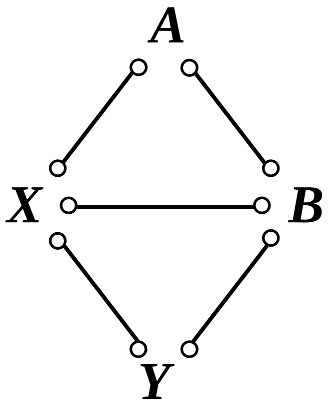
IDA under latent confounders

- Fact: The causal effect of X on Y in a PAG is possibly unidentifiable, since the large number of consistent MAGs are possibly associated with different causal effects
- Task: How to determine **the set of possible causal effects by finding adjustment set for (X, Y)** in all the MAGs consistent with the PAG? – **Set determination**
  - Adjustment set : a set $\boldsymbol{Z}$ such that $P(Y|do(X)) = \int P(Z)P(Y|X,Z)dZ$

Main Idea

We enumerate each **variable set** instead of **MAGs**, and determine **whether each set could be an adjustment set** in some MAGs in the MEC?

An example



PAG

Target: output the set of causal effects of X on Y

Enumerate: $\emptyset, \{A\}, \{B\}, \{A, B\} \subseteq 2^{\{A,B\}}$

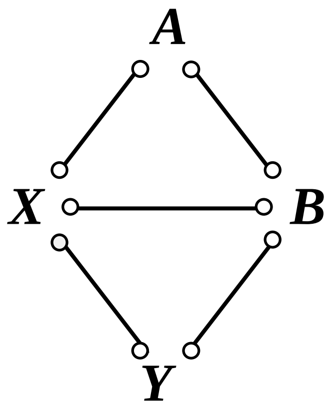
For $\boldsymbol{W} = \emptyset, \{A\}, \{B\}, \{A, B\}$, determine the existence of MAGs represented by the PAG with $\boldsymbol{W}$ as an adjustment set for (X, Y)

The main challenge

Given any a set $\boldsymbol{W} \subseteq \{\boldsymbol{A}, \boldsymbol{B}\}$, how to determine **the existence of** MAGs represented by the PAG with adjustment set for (X, Y) being $\boldsymbol{W}$**?**

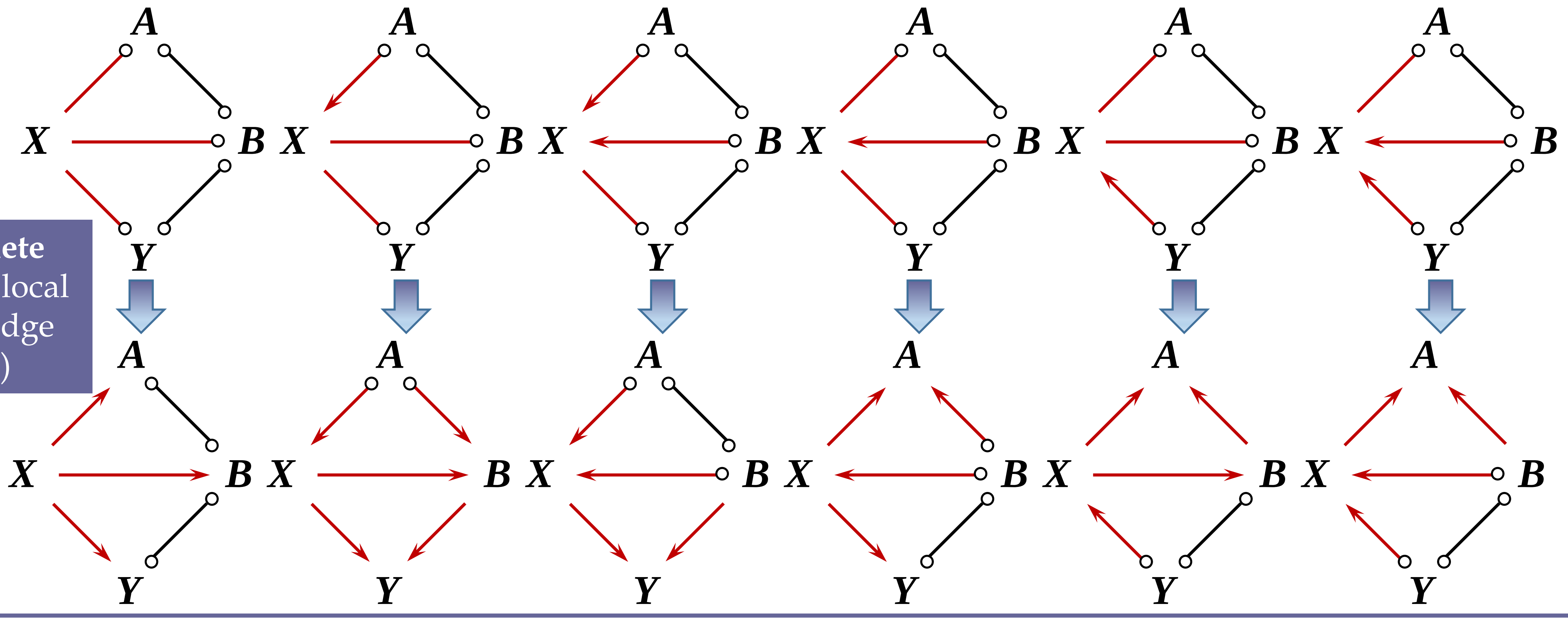
It cannot be achieved based on only a PAG!

Two step procedure



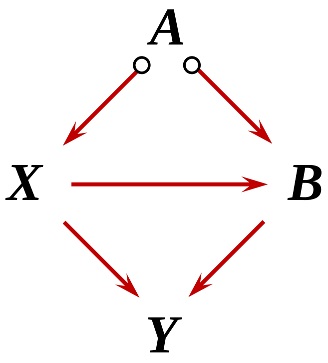
PAG

- Step 1: enumerate the local structures of X
  - We propose the **sufficient** and **necessary** condition for determining the **validity of each local structure**
  - For example, 3 circles -> enumerate 2^3 local structures



Utilizing the **complete orientation rules** for local background knowledge (Wang et al. 2023)

Two step procedure

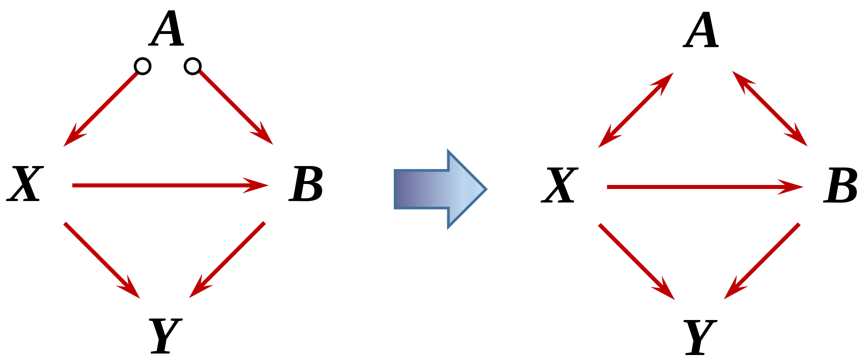


$\mathbb{M}$

- Step 2: for any a set $\boldsymbol{W} \subseteq \{A, B\}$, determining whether $\boldsymbol{W}$ could be an adjustment set in MAG consistent with $\mathbb{M}$
- A key observation: There exists such a MAG $\mathcal{M}$ if and only if we can make some specific vertices not ancestors of Y in $\mathcal{M}$

An example

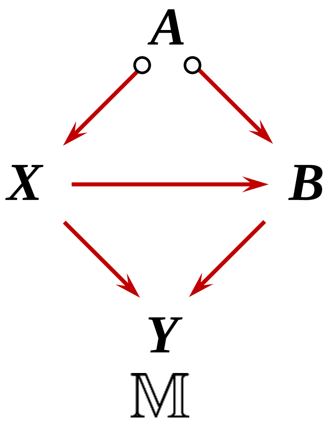
For $\boldsymbol{W} = \emptyset$, if we can construct a MAG with adjustment set $\emptyset$



- We need to prevent $A \in \mathrm{Anc}(Y)$ in the constructed MAG, for otherwise A belongs to the adjustment set for (X, Y) such that $\boldsymbol{W} \neq \emptyset$

⇨ Determining whether we can construct a MAG by adding arrowheads based on $\mathbb{M}$ such that the specific vertices not ancestors of Y

Two step procedure



- Step 2: for any a set $\boldsymbol{W} \subseteq \{A, B\}$, determining whether $\boldsymbol{W}$ could be an adjustment set in MAG consistent with $\mathbb{M}$
- A key observation: There exists such a MAG $\mathcal{M}$ if and only if we can make some specific vertices not ancestors of Y in $\mathcal{M}$

$\boldsymbol{S}$: characterize the variables V where arrowheads need to be introduced

Intuitively, we need to search whether there is a transformation characterize by S such that W is an adjustment set in the obtained MAG

$$\mathrm{Anc}(Y \cup \mathbf{W}, \mathbb{M}) \cap [\mathrm{PossDe}(\bar{\mathbf{W}}, \mathbb{M}) \backslash \bar{\mathbf{W}}] \subseteq \mathbf{S}$$
$$\subseteq \mathrm{PossAn}(Y \cup \mathbf{W}, \mathbb{M}) \cap [\mathrm{PossDe}(\bar{\mathbf{W}}, \mathbb{M}) \backslash \bar{\mathbf{W}}]$$

Exponential number of S

One main theorem

Theorem 2. Given a maximal local MAG $\mathbb{M}$, for any potential adjustment set $\mathbf{W}$, there exists a MAG $\mathcal{M}$ valid to $\mathbb{M}$ such that $\mathbf{W}$ is an adjustment set in $\mathcal{M}$ if there exists a block set $\mathbf{S}$ such that

(1) $\mathrm{PossDe}(\bar{\mathbf{W}}, \mathbb{M}[-\mathbf{S}]) \cap \mathrm{Pa}(\mathbf{S}, \mathbb{M}) = \emptyset$;

(2) $\mathbb{M}[\mathbf{S}_V]$ is a complete graph for any $V \in \bar{\mathbf{W}}$, where $\mathbf{S}_V = \{V' \in \mathbf{S} | V \circ\!\!\rightarrow V' \text{ in } \mathbb{M}\}$;

(3) $\mathbb{M}[\mathrm{PossDe}(\bar{\mathbf{W}}, \mathbb{M}[-\mathbf{S}])]$ is bridged relative to $\mathbf{S}$ in $\mathbb{M}$.

For each S, it can be verified in polynomial time

Algorithm

Algorithm 1: PAGcauses.

Input: PAG $\mathcal{P}$, X, Y

$\widehat{\text{AS}}(\mathcal{P}) = \emptyset$ // Record all the valid adjustment sets;

if $X \notin \text{PossAn}(Y, \mathcal{P})$ **then return** No causal effects;

if *the conditions in Proposition 1 are satisfied for* $\mathcal{P}$ **then**

  return $\widehat{\text{AS}}(\mathcal{P}) \leftarrow \{\text{D-SEP}(X, Y, \mathcal{P}_{\underline{X}})\}$

for *each set* $\mathbf{C} \subseteq \{V \mid V \ast\!\!-\!\!\circ X \text{ in } \mathcal{P}\}$ **do**

  if *the three conditions in Proposition 2 are satisfied* **then**

    Obtain a maximal local MAG $\mathbb{M}$ based on $\mathcal{P}$ and $\mathbf{C}$;

    Find all potential adjustment sets $\mathbf{W}_1, \mathbf{W}_2, \cdots$ given $\mathbb{M}$ according to Definition 6;

    for *each potential adjustment set* $\mathbf{W}_i$ **do**

      for *each block set* $\mathbf{S}$ **do**

        if *the three conditions in Theorem 2 are satisfied given* $\mathbf{S}$ **then**

          $\widehat{\text{AS}}(\mathcal{P}) \leftarrow \widehat{\text{AS}}(\mathcal{P}) \cup \{\mathbf{W}_i\}$;

          break // Break the loop of $\mathbf{S}$;

Output: Set of causal effects via adjustment in the given PAG $\mathcal{P}$ identified with $\widehat{\text{AS}}(\mathcal{P})$ by (1)

If the causal effect is identifiable given a PAG, stop

Enumerate each valid local structure

Enumerate each subset ***W*** possibly being adjustment set

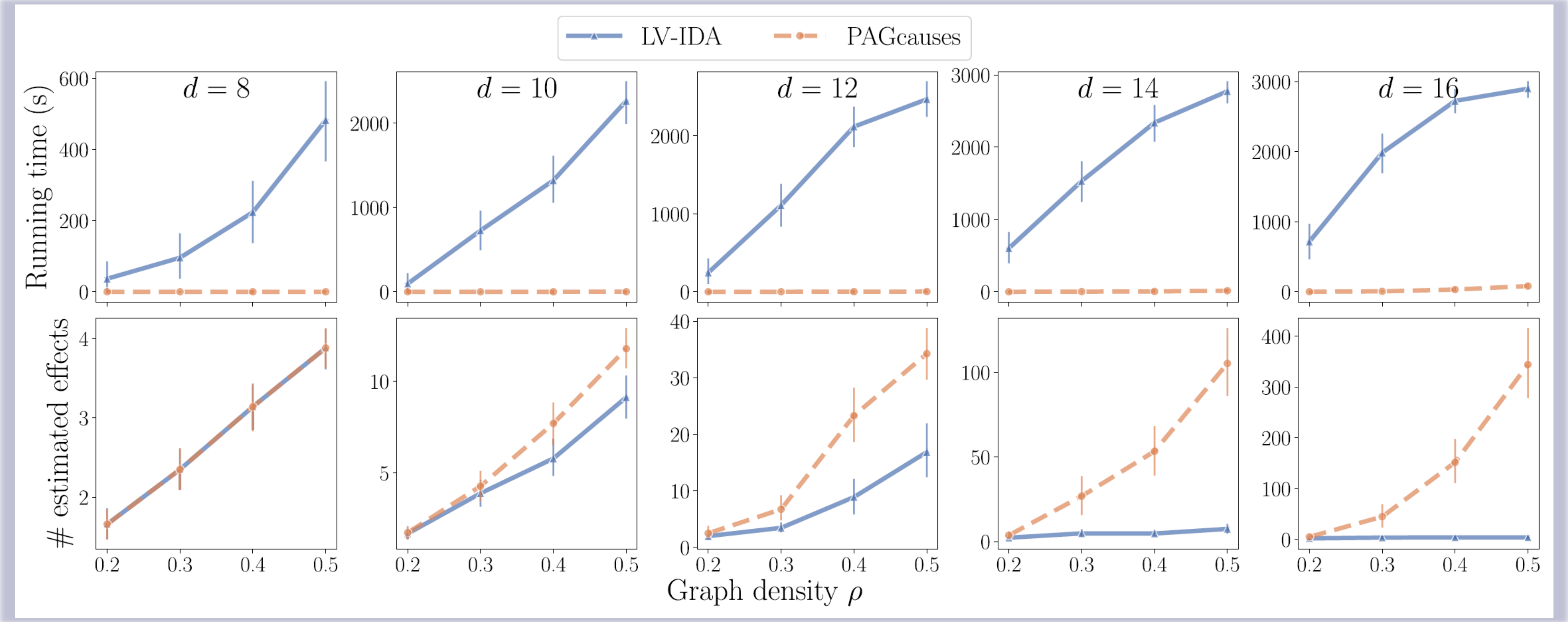
Enumerate each **block set *S***

Determine the existence of MAGs with adjustment set ***W*** given circles transformation dictated by ***S***

Theoretical results

1. Our method can return the **identical** set as the SOTA MAG enumeration method;
2. Our method takes **super-exponentially** less time than SOTA MAG enumeration method

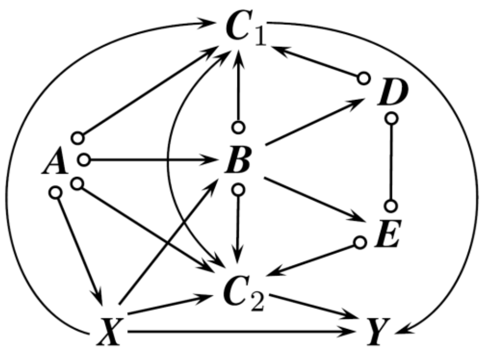
Simulations



Improved version

- We need to enumerate exponential number of $\boldsymbol{S}$ to determine whether we could construct a MAG based on $\mathbb{M}$ by adding arrowheads dictated by $\boldsymbol{S}$ such that the specific vertices not ancestors of Y

- Could we directly find the proper $\boldsymbol{S}$ without the exponential number of enumeration?



$\mathbb{M}$

If $W = \emptyset$ is an adjustment set in some MAG consistent with $\mathbb{M}$, then A cannot be an ancestor of Y, to achieve this, we

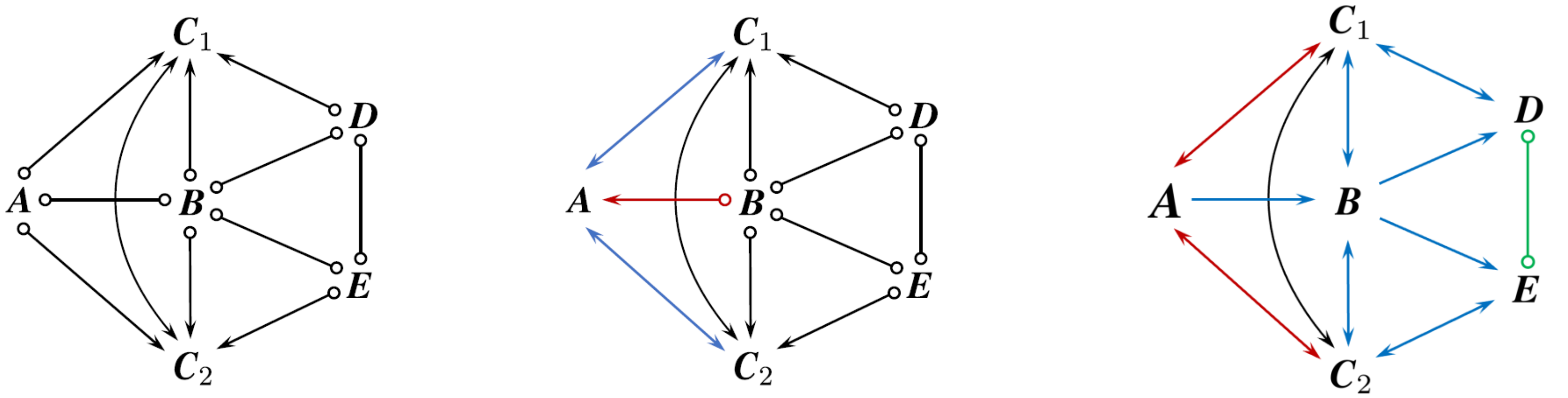
1. Transforming $A \circ\!\!\rightarrow C_1/C_2$ to $A \boxed{\leftrightarrow} C_1/C_2$
2. Some circles can be further oriented given the BK

Background knowledge

Orienting graphs with BK

- What arrowheads are identifiable given the necessary transformation?
  - Existing rules are not complete to incorporate the transformation

Counterexample 1

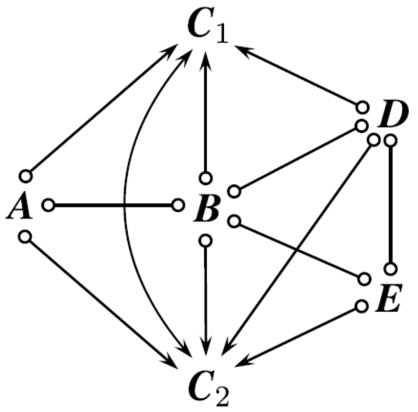


PAG　　Incor. **BK**　　For contradiction

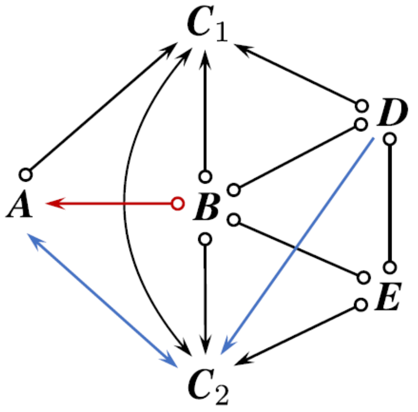
Orienting graphs with BK

- What arrowheads are identifiable given the necessary transformation?
  - Existing rules are not complete to incorporate the transformation

Counterexample 2



PAG

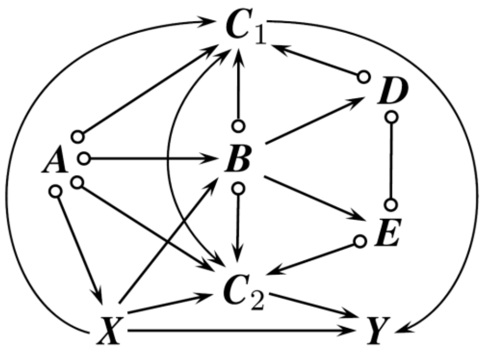


Incor. **BK**

(Proposed by Venkateswaran & Perkovic (2024) and us independently)

Orienting graphs with BK

- What arrowheads are identifiable given the necessary transformation?
  - Existing rules are not complete to incorporate the transformation
- Motivated by the two counterexamples, two orientation rules are proposed to incorporate the transformation, and the updated rules can identify all the arrowheads at the specific variables



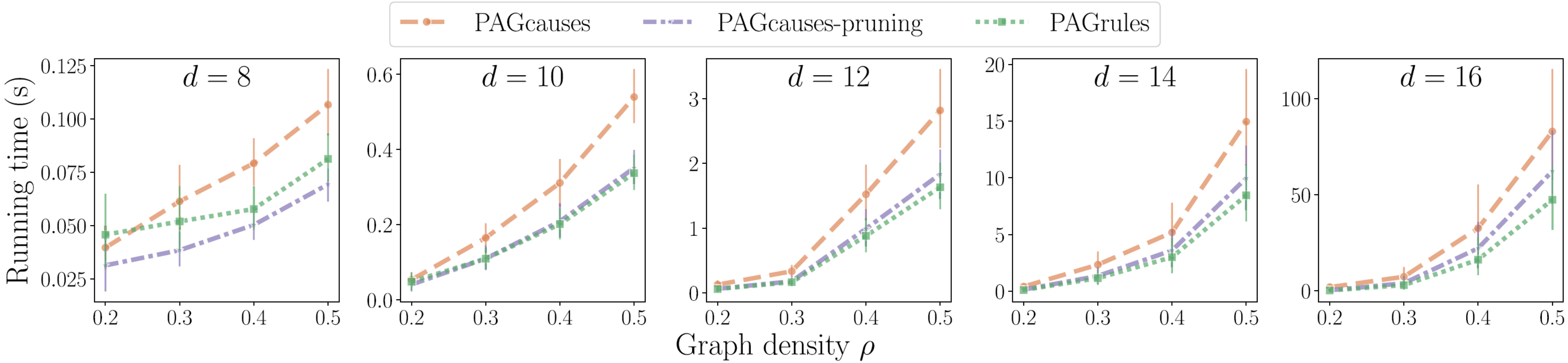
$\mathbb{M}$

Whether $W = \emptyset$ is an adjustment set in some MAG consistent with $\mathbb{M}$

1. Transforming $A \circ\!\!\rightarrow C_1/C_2$ to $A \boxed{\leftrightarrow} C_1/C_2$
2. Some circles can be further oriented given the novel orientation rules: transforming $A \circ\!\!\rightarrow B$ to $A \boxed{\leftrightarrow} B$
3. Obtaining $\boldsymbol{S} = \{C_1, B, C_2\}$, then detecting given $\boldsymbol{W}$ and $\boldsymbol{S}$, whether the conditions in Thm. 2 are satisfied, i.e., we can construct a MAG by adding

Avoiding the enumeration of exponential number of S

Simulations



- All the methods can find the same set of causal effects
- The application of the novel rules can help reduce the complexity

Task 2: MAG Listing

MAG Listing

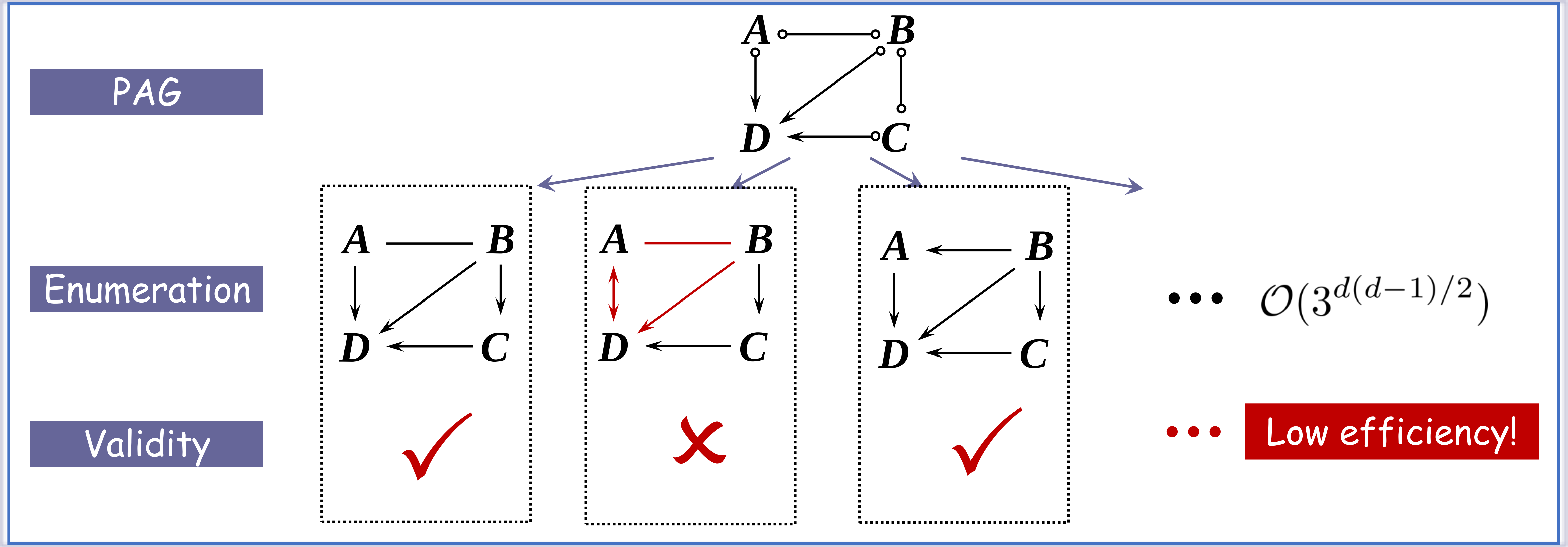
- What does MAG listing do?
  - Outputting all the MAGs consistent with a given PAG
- Why MAG listing?
  1. Causal discovery with active interventions: Select interventional variable

$$H_X = -\sum_{j=1}^{M} \frac{l_j}{L} \log \frac{l_j}{L}$$

  2. Causal effect estimation: LV-IDA, obtaining the distribution of causal effect
  3. Structure learning with additional structural knowledge

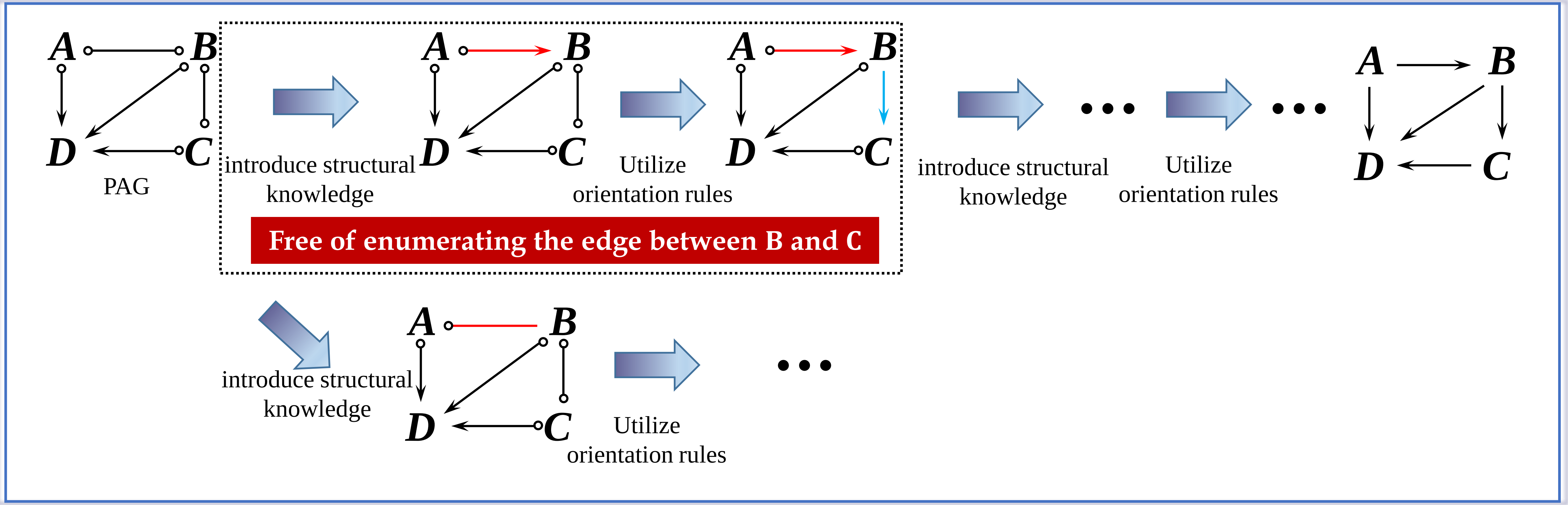
MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)



MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)
  2. Recursively introduce background knowledge **(BK)** and utilize orientation rules



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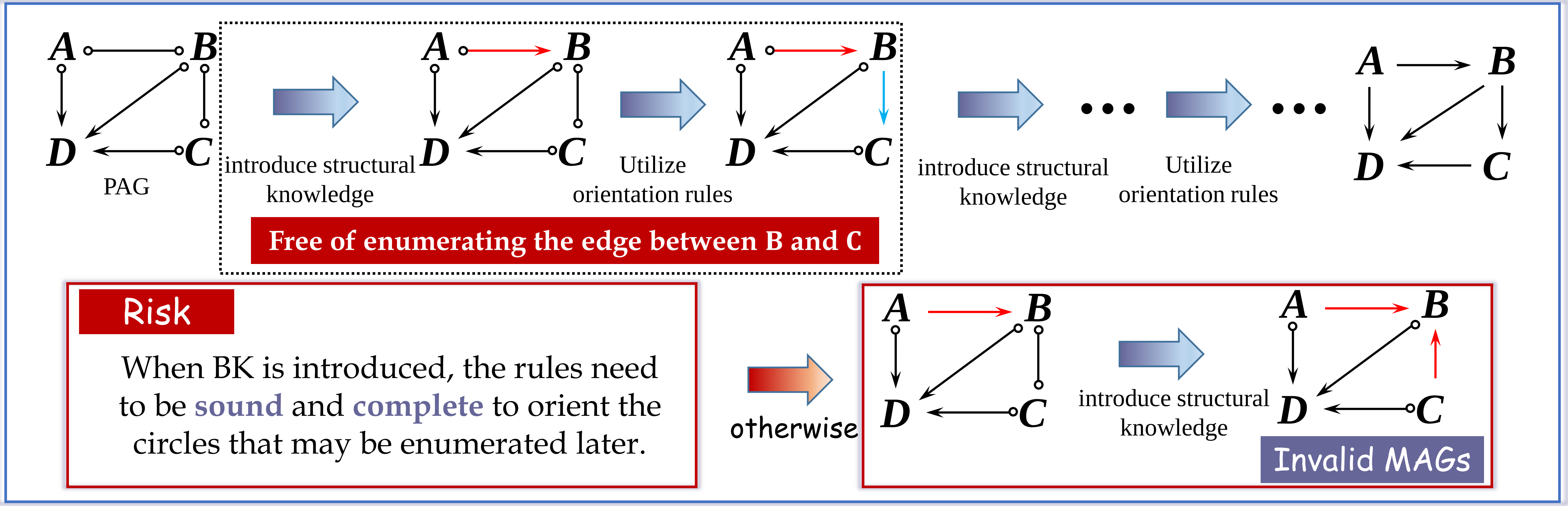
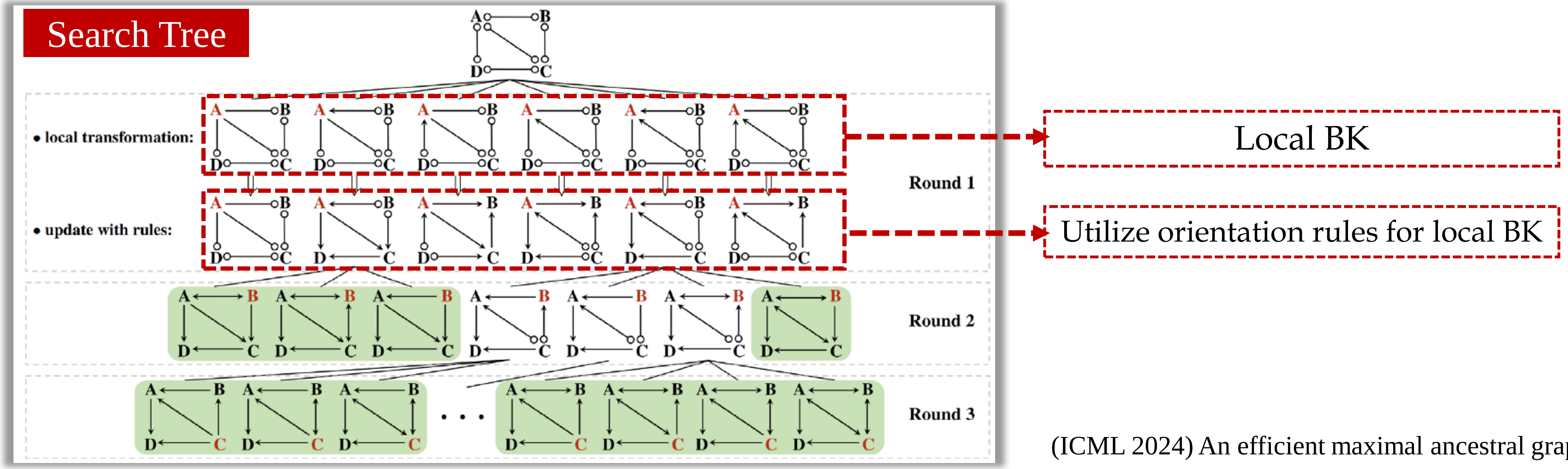


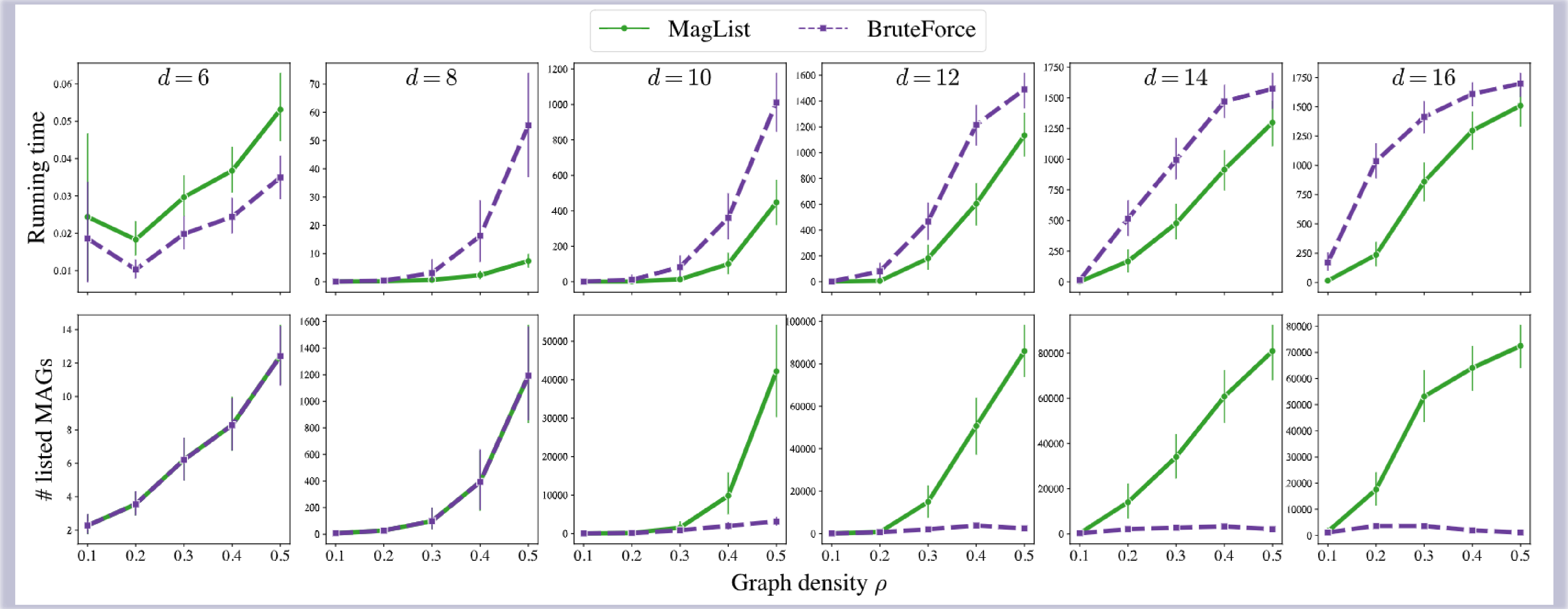
MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)
  2. Recursively introduce background knowledge (**BK**) and utilize orientation rules
     ① MAGList-Wang et al. 2024 (introduce local BK and utilize existing rules)



(ICML 2024) An efficient maximal ancestral graph listing algorithm

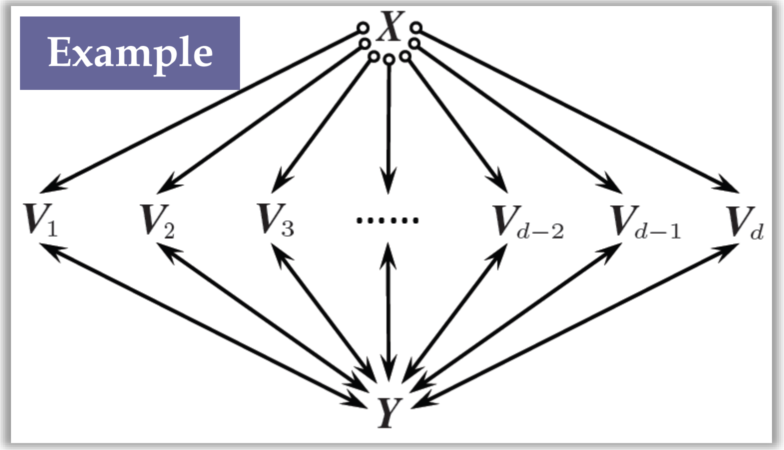
Simulations



MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)
  2. Recursively introduce background knowledge (**BK**) and utilize orientation rules
     ① MAGList-Wang et al. 2024 (introduce **local BK** and utilize existing rules)
     - Suffers an exponential **delay** (time for outputting each MAG)

Example



The search of local BK leads to the **exponential delay!**

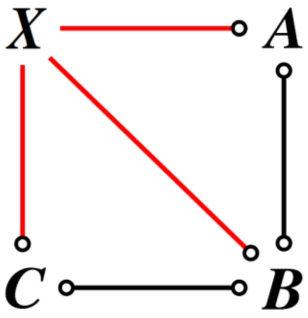
MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)
  2. Recursively introduce background knowledge (**BK**) and utilize orientation rules
     a) MAGList-Wang et al. 2024 (introduce **local BK** and utilize existing rules)
        - Suffers an exponential **delay** (time for outputting each MAG)

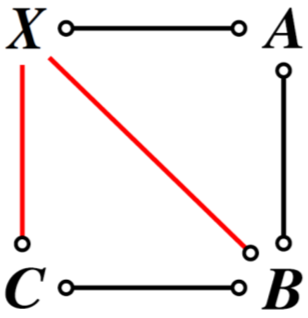
     b) (**Our method**) Transform each circle at the selected variable one by one

MAG Listing

- How?
  1. Directly enumerating the orientation combinations of all circles (brute-force)
  2. Recursively introduce background knowledge (**BK**) and utilize orientation rules
     a) MAGList-Wang et al. 2024 (introduce **local BK** and utilize existing rules)
        - Suffers an exponential **delay** (time for outputting each MAG)
     b) (**Our method**) Transform each circle at the selected variable one by one
        - Establish the sound and **locally complete** orientation rules for **singleton BK**

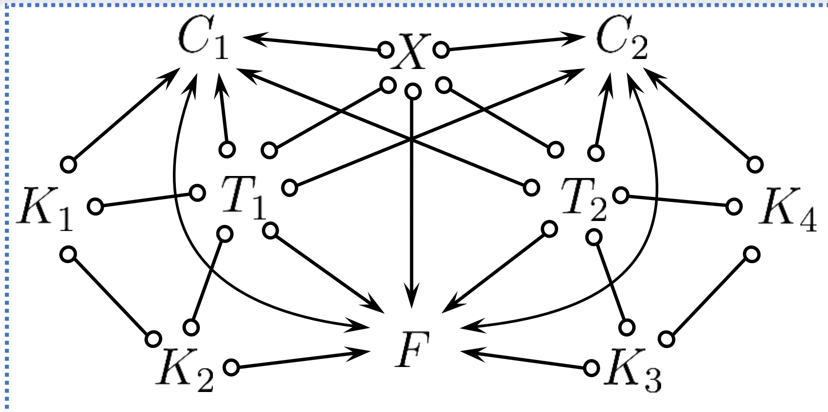


Local BK

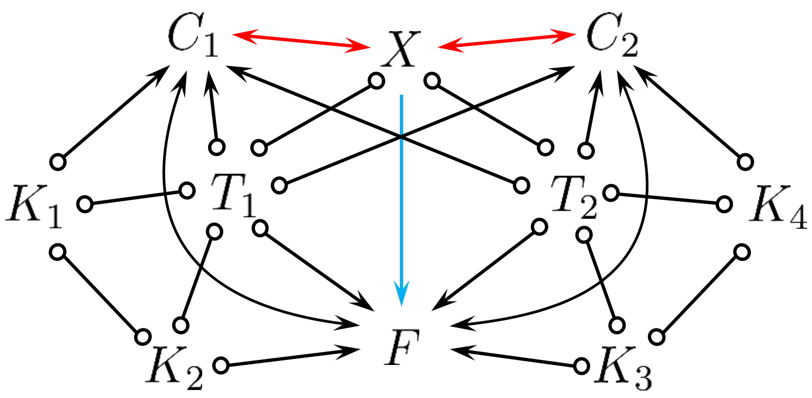


Singleton BK

The novel rules



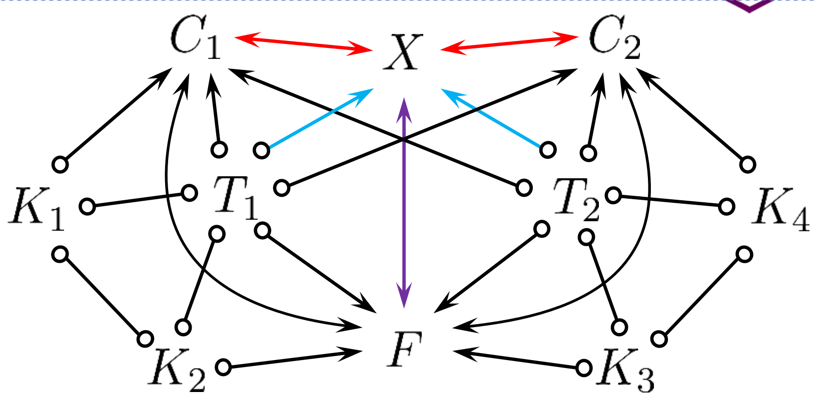
PAG



Incorporate BK

BK: $C_1 \leftrightarrow X \leftrightarrow C_2$

Identification: $X \to F$



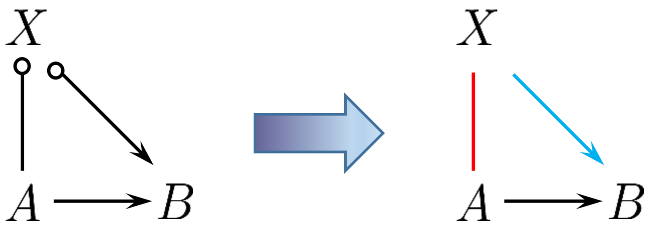
For contradiction

Suppose: $X \leftrightarrow F$

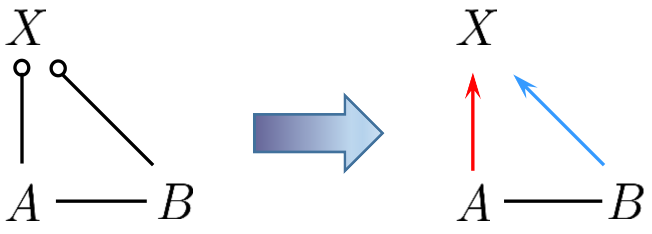
Identification: $T_1 \circ\!\!\rightarrow X \leftarrow\!\!\circ T_2$

(New unshielded colliders!)

$\mathcal{R}_{14}$



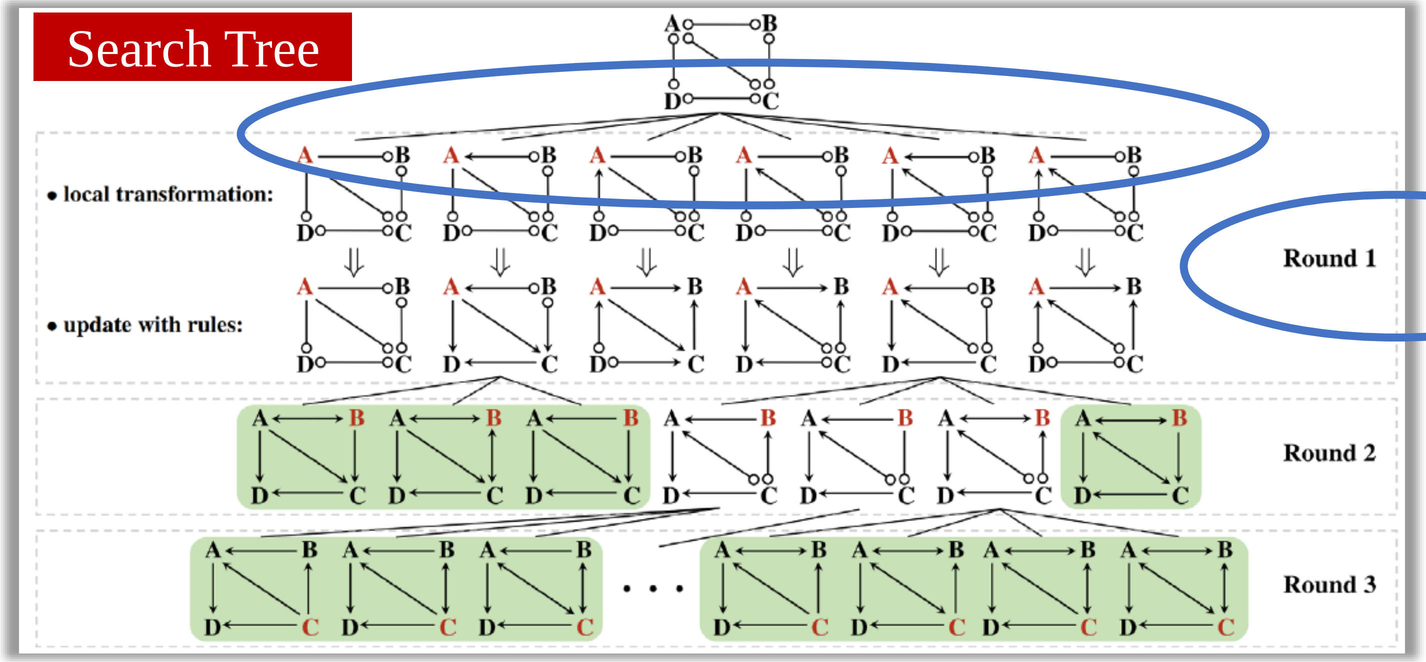
$\mathcal{R}_{15}$



$\mathcal{R}_{16}$

MAG Listing procedure

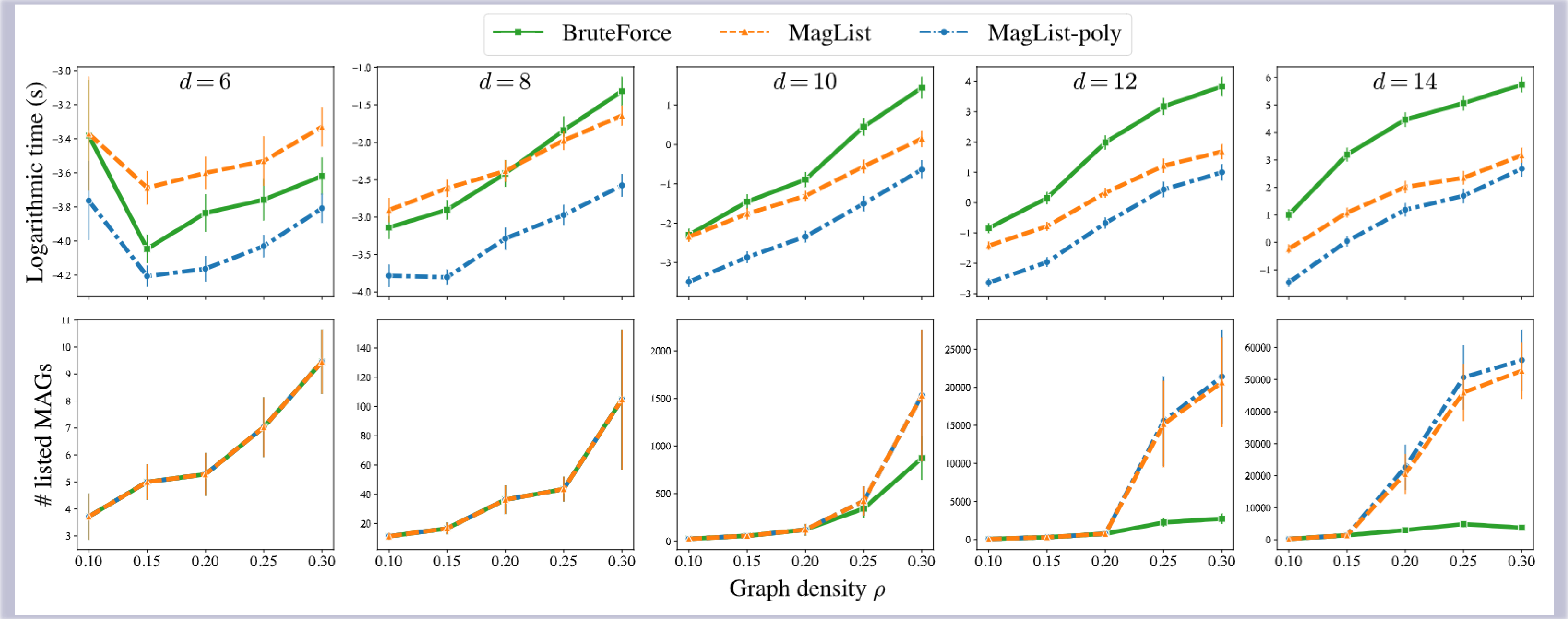
- We have established the sound and locally complete rules for singleton BK



Select A to orient circles **one by one and use the rules**

- We can prove that the method has a polynomial delay

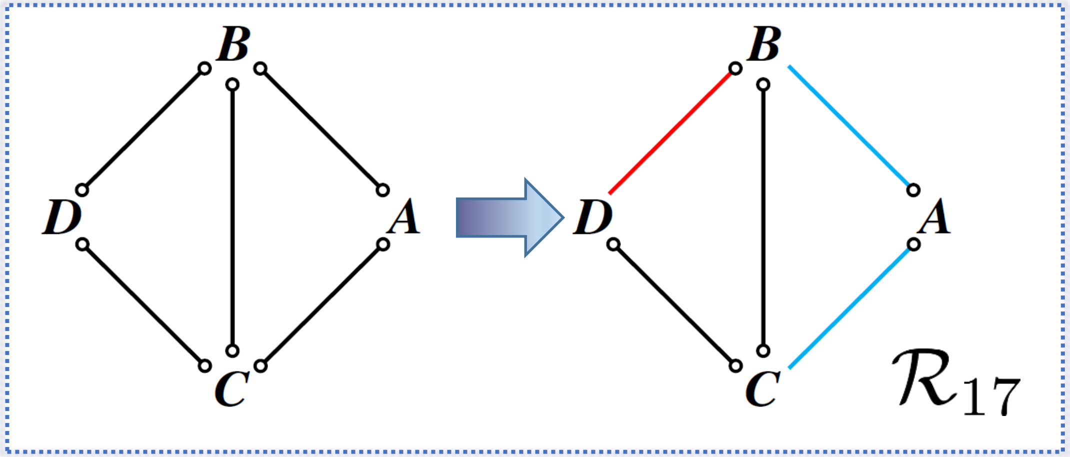
Simulations

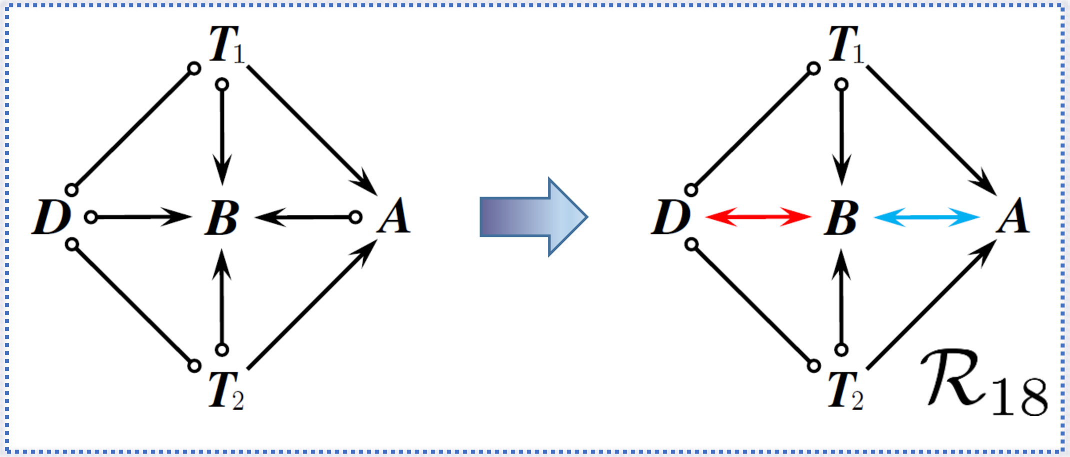


Additional rules

- Are the rules in the current complete?

No!





Joint works with and References

Wen-Bo Du
(NJU)

Lue Tao
(NJU)

Tian Qin
(NJU → Optiver)

Zhi-Hua Zhou
(NJU)

- ✓ Tian-Zuo Wang, Lue Tao, Tian Qin, Zhi-Hua Zhou. Estimating possible causal effects with latent variables via adjustment and novel rule orientation. **Artificial Intelligence**, 2025.
- ✓ Tian-Zuo Wang, Wen-Bo Du, Zhi-Hua Zhou. Polynomial-delay MAG listing with novel locally complete orientation rules. **ICML 2025**. Oral (0.99%).
- ✓ Tian-Zuo Wang, Wen-Bo Du, Zhi-Hua Zhou. An efficient maximal ancestral graph listing algorithm. **ICML 2024**. Spotlight (3.5%).
- ✓ Tian-Zuo Wang, Tian Qin, Zhi-Hua Zhou. Sound and complete causal identification with latent variables given local background knowledge. **Artificial Intelligence**, 2023.

Thanks for listening!