Introduction

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Outline

- Mathematical Optimization
- Least-squares
- Linear Programming
- Convex Optimization
- Nonlinear Optimization
- Summary

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Mathematical Optimization (1)

Optimization Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \ldots, m \end{aligned}$$

- Optimization Variable: $x = (x_1, \ldots, x_n)$
- Objective Function: $f_0 : \mathbf{R}^n \to \mathbf{R}$
- Constraint Functions: $f_i : \mathbf{R}^n \to \mathbf{R}$

$x^\star$ is called optimal or a solution

- $f_i(x^\star) \le b_i$, $i = 1, \ldots, m$
- For any z with $f_i(z) \le b_i$, we have $f_0(z) \ge f_0(x^*)$

Mathematical Optimization (2)

- Linear Problem

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)$$

  - for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$

- Nonlinear Program
  - If the optimization problem is not linear

- Convex Optimization Problem

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  - for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1,\ \alpha \geq 0,\ \beta \geq 0$

Applications

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \dots, m \end{aligned}$$

- Abstraction
  - x represents the choice made
  - $f_i(x) \le b_i$ represent firm requirements that limit the possible choices
  - $f_0(x)$ represents the cost of choosing x
  - A solution corresponds to a choice that has minimum cost, among all choices that meet the requirements

Portfolio Optimization (1)

- Variables
  - x_i represents the investment in the i-th asset
  - $x \in \mathbf{R}^n$ describes the overall portfolio allocation across the set of asset
- Constraints
  - A limit on the budget the requirement
  - Investments are nonnegative
  - A minimum acceptable value of expected return for the whole portfolio
- Objective
  - Minimize the variance of the portfolio return

Portfolio Optimization (2)

- We want to spread our money over N different assets; the fraction of our money we invest in asset n is denoted x_n.

$$\sum_{n=1}^{N} x_n = 1, \text{and } 0 \leq x_n \leq 1, \text{for } n = 1, \ldots, N$$

- Denote the return of these investments as $a_1, \ldots, a_N$. The expected return which are usually calculated using some kind of historical average, is $\mu_1, \ldots, \mu_N$. We specify some target expected return ρ, which means:

$$\mathrm{E}\left[\sum_{n=1}^{N} a_n x_n\right] = \sum_{n=1}^{N} \mathrm{E}[a_n] x_n = \sum_{n=1}^{N} \mu_n x_n = \mu^\top x \geq \rho$$

Portfolio Optimization (3)

- We want to solve for the x that achieves this level of return while minimizing the variance of our return

$$\mathrm{Var}\left[\sum_{n=1}^{N} a_n x_n\right] = x^\top \mathrm{Cov}(a) x = x^\top R x = \sum_{m=1}^{N}\sum_{n=1}^{N} R_{m,n} x_m x_n$$

- Our Optimization Program

$$\begin{aligned} \min \quad & x^\top R x \\ \text{s.t.} \quad & \mu^\top x > \rho, \sum_{n=1}^{N} x_n = 1 \\ & 0 \le x_n \le 1, n = 1, \dots, N \end{aligned}$$

  - Quadratic program with linear constraints, convex

Device Sizing

- Variables
  - $x \in \mathbf{R}^n$ describes the widths and lengths of the devices
- Constraints
  - Limits on the device sizes
  - Timing requirements
  - A limit on the total area of the circuit
- Objective
  - Minimize the total power consumed by the circuit

Data Fitting

- Variables
  - $x \in \mathbf{R}^n$ describes parameters in the model
- Constraints
  - Prior information
  - Required limits on the parameters (such as nonnegativity)
- Objective
  - Minimize the prediction error between the observed data and the values predicted by the model

Solving Optimization Problems

General Optimization Problem

- Very difficult to solve
- Constraints can be very complicated, the number of variables can be very lage
- Methods involve some compromise, e.g., computation time, or suboptimal solution

Exceptions

- Least-squares problems
- Linear programming problems
- Convex optimization problems

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Least-squares Problems (1)

The Problem

$$\min \quad \|Ax - b\|_2^2 = \sum_{i=1}^{k} \left(a_i^\top x - b_i\right)^2$$

- $A \in \mathbf{R}^{k\times n}$, $a_i^\top$ is the i-th row of A, $b \in \mathbf{R}^k$
- $x \in \mathbf{R}^n$ is the optimization variable

How to solve it?

Least-squares Problems (1)

The Problem

$$\min \quad \|Ax-b\|_2^2=\sum_{i=1}^{k}\left(a_i^\top x-b_i\right)^2$$

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Setting the gradient to be 0

$$\begin{aligned} & 2A^\top(Ax-b)=0 \\ \Longrightarrow \quad & A^\top Ax=A^\top b \\ \Longrightarrow \quad & x=(A^\top A)^{-1}A^\top b \end{aligned}$$

Least-squares Problems (2)

- A Set of Linear Equations

$$A^\top A x = A^\top b$$

- Solving least-squares problems
  - Reliable and efficient algorithms and software
  - Computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k\times n}$); less if structured
  - A mature technology
  - Challenging for extremely large problems

Using Least-squares

- Easy to Recognize
- Weighted least-squares

$$\sum_{i=1}^{k} w_i \left(a_i^\top x - b_i\right)^2$$

  - Different importance

Using Least-squares

- Easy to Recognize
- Weighted least-squares

$$\sum_{i=1}^{k} w_i (a_i^\top x - b_i)^2 = \sum_{i=1}^{k} (\sqrt{w_i} a_i^\top x - \sqrt{w_i} b_i)^2$$

  - Different importance
- Regularization

$$\sum_{i=1}^{k} (a_i^\top x - b_i)^2 + \rho \sum_{i=1}^{n} x_i^2$$

  - More stable

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Linear Programming

The Problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \qquad i = 1, \ldots, m \end{aligned}$$

- $c, a_1, \ldots, a_m \in \mathbf{R}^n,\ b_1, \ldots, b_m \in \mathbf{R}$

Solving Linear Programs

- No analytical formula for solution
- Reliable and efficient algorithms and software
- Computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- A mature technology
- Challenging for extremely large problems

Using Linear Programming

- Not as easy to recognize
- Chebyshev Approximation Problem

$$\min \quad \max_{i=1,\ldots,k} \left|a_i^\top x - b_i\right|$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & t = \max_{i=1,\ldots,k} \left|a_i^\top x - b_i\right| \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & t \geq \left|a_i^\top x - b_i\right|, i = 1, \ldots, k \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & -t \leq a_i^\top x - b_i \leq t, i = 1, \ldots, k \end{array}$$

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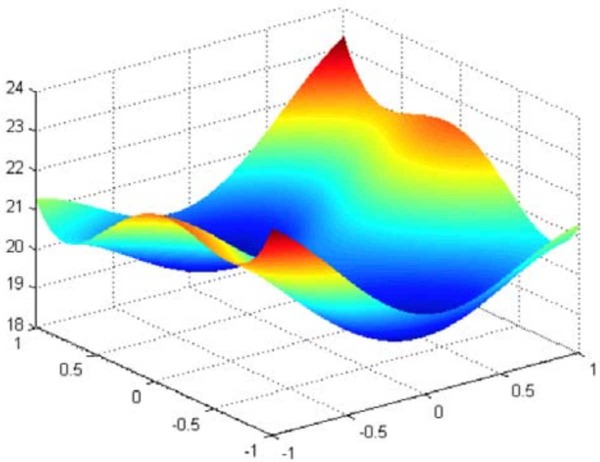
Convex Optimization

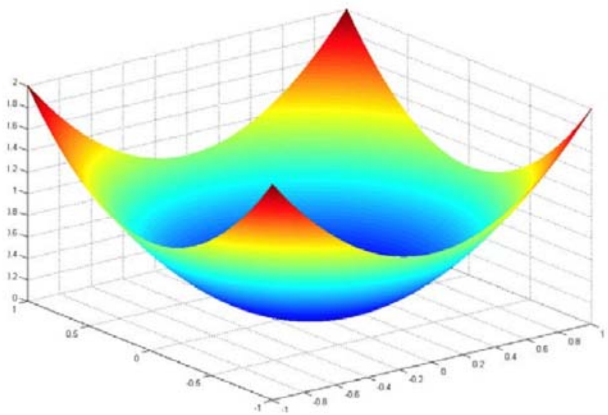
□ Why Convexity?

" The great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

— R. Rockafellar, SIAM Review 1993



Non-Convex Optimization



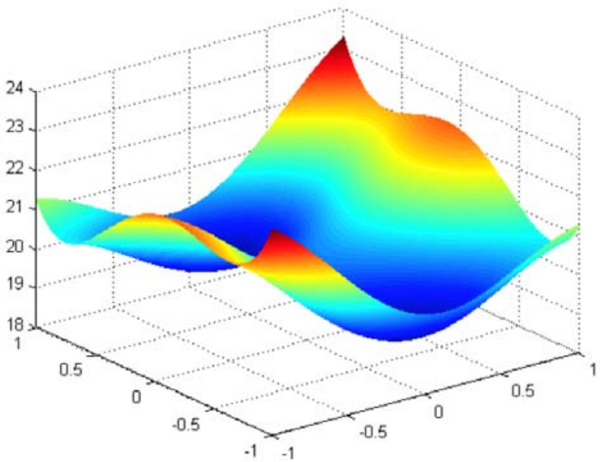
Convex Optimization

Convex Optimization

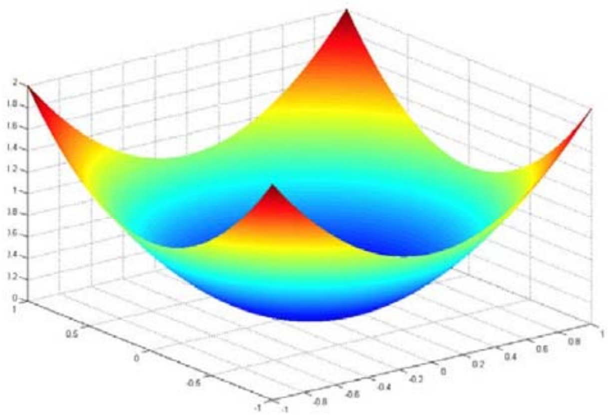
☐ Why Convexity?

“ The great watershed in optimizati nd
nonlinearity, but convexity and nc
— R. 993

Local minimizers are also global minimizers.



Non-Convex Optimization



Convex Optimization

Convex Optimization Problems (1)

The Problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le b_i, \qquad i = 1, \dots, m \end{aligned}$$

- Functions $f_0, \dots, f_m: \mathbf{R}^n \to \mathbf{R}$ are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1,\ \alpha \ge 0,\ \beta \ge 0$

- Least-squares and linear programs as special cases

Convex Optimization Problems (2)

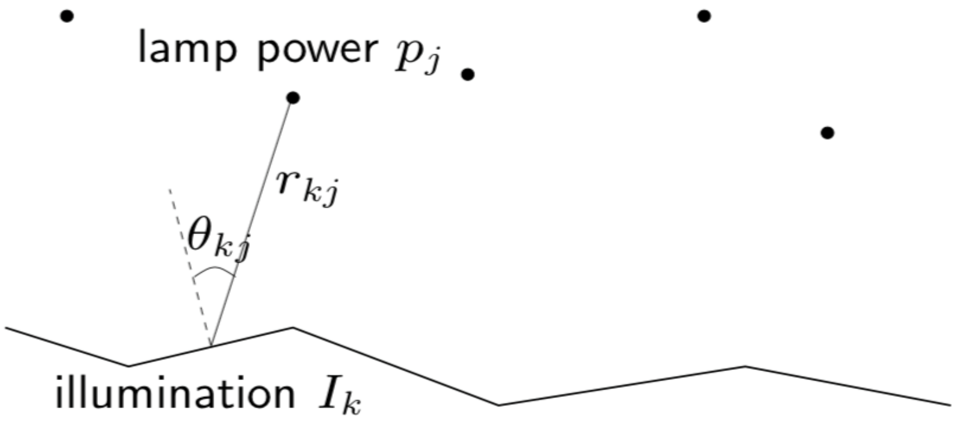
- Solving Convex Optimization Problems
  - No analytical solution
  - Reliable and efficient algorithms (e.g., interior-point methods)
  - Computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$
    - ✓ F is cost of evaluating $f_i's$ and their first and second derivatives
  - Almost a technology

Using Convex Optimization

- Often difficult to recognize
- Many tricks for transforming problems into convex form
- Surprisingly many problems can be solved via convex optimization

An Example (1)

- m lamps illuminating n patches



  - Intensity I_k at patch k depends linearly on lamp powers p_j

$$I_k = \sum_{j=1}^{m} a_{kj} p_j, \qquad a_{kj} = r_{kj}^{-2} \max\{\cos\theta_{kj}, 0\}$$

An Example (2)

- Achieve desired illumination I_{des} with bounded lamp powers

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |\log I_k - \log I_{\mathrm{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

How to solve it?

An Example (3)

1. Use uniform power: $p_j = p$, vary p
2. Use least-squares

$$\min \sum_{i=1}^{k} (I_k - I_{\text{des}})^2 = \sum_{i=1}^{k} \left(\sum_{j=1}^{m} a_{kj} p_j - I_{\text{des}} \right)^2$$

- Round p_j if $p_j > p_{\max}$ or $p_j < 0$

3. Use weighted least-squares

$$\min \sum_{i=1}^{k} (I_k - I_{\text{des}})^2 + \sum_{j=1}^{m} w_j \left(p_j - \frac{p_{\max}}{2} \right)^2$$

- Adjust weights w_j until $0 \le p_j \le p_{\max}$

An Example (4)

4. Use linear programming

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |I_k - I_{\text{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

5. Use convex optimization

$$\begin{array}{ll} \min & \max_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \min & \max_{k=1,\ldots,n} \max\left(\log \frac{I_k}{I_{\text{des}}}, \log \frac{I_{\text{des}}}{I_k}\right) \\ \text{s.t.} & 0 \le p_j \le p_{\max}, j = 1, \ldots, m \end{array}$$

An Example (5)

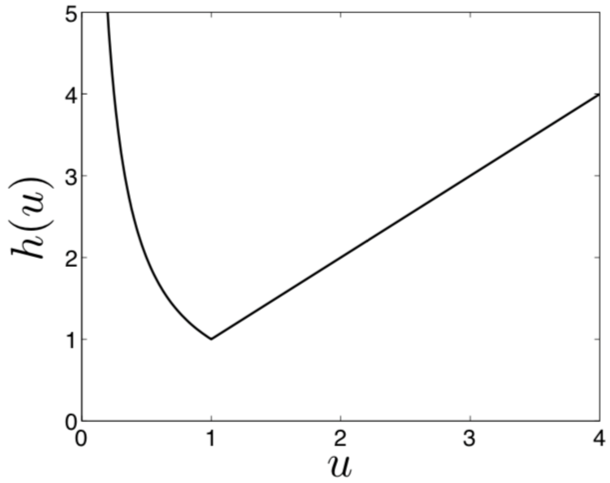
$$\Longleftrightarrow \quad \min \quad \max_{k=1,\ldots,n} \max\left(\frac{I_k}{I_{\text{des}}}, \frac{I_{\text{des}}}{I_k}\right)$$

$$\text{s.t.} \quad 0 \leq p_j \leq p_{\max}, j = 1, \ldots, m$$

$$\Longleftrightarrow \quad \min \quad \max_{k=1,\ldots,n} h\left(\frac{I_k}{I_{\text{des}}}\right)$$

$$\text{s.t.} \quad 0 \leq p_j \leq p_{\max}, j = 1, \ldots, m$$

- $h(u) = \max\left(u, \frac{1}{u}\right)$



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Nonlinear Optimization

- An optimization problem when the objective or constraint functions are not linear, but not known to be convex

- Sadly, there are no effective methods for solving the general nonlinear programming problem
  - Could be NP-hard

- We need compromise

Local Optimization Methods

- Find a point that minimizes f_0 among feasible points near it
  - The compromise is to give up seeking the optimal x
- Fast, can handle large problems
  - Differentiability
- Require initial guess
- Provide no information about distance to (global) optimum
- Local optimization methods are more art than technology

Comparisons

	Problem Formulation	Solving the Problem
Local Optimization Methods for Nonlinear Programming	Straightforward	**Art**
Convex Optimization	**Art**	Standard

Global Optimization

- Find the global solution
  - The compromise is efficiency
- Worst-case complexity grows exponentially with problem size
- Worst-case Analysis
  - Whether the worst-case value is acceptable
  - A local optimization method can be tried

Role of Convex Optimization in Nonconvex Problems

- Initialization for local optimization
  - An approximate, but convex, formulation
- Convex heuristics for nonconvex optimization
  - Sparse solutions (compressive sensing)
- Bounds for global optimization
  - Relaxation
  - Lagrangian relaxation

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Summary

- Mathematical Optimization
- Least-squares
  - Closed-form Solution
- Linear Programming
  - Efficient algorithms
- Convex Optimization
  - Efficient algorithms, Modeling is an art
- Nonlinear Optimization
  - Compromises, Optimization is an Art