Optimization Methods **Fall 2025**

Homework 1

Instructor: Lijun Zhang *Name:* Your name, *StudentId:* Your student id

Notice

- The submission email is: **optfall25@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.
- In this homework, we use boldface letters (e.g., $\mathbf{x} \in \mathbb{R}^n$) to denote vectors, while non-bold letters (e.g., $x \in \mathbb{R}$) denote scalars.

Problem 1: Norms (30 points)

A function $f : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{dom} f = \mathbb{R}^n$ is called a norm if

- f is nonnegative: $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- f is definite: $f(\mathbf{x}) = 0$ only if $\mathbf{x} = \mathbf{0}$
- f is homogeneous: $f(t\mathbf{x}) = |t| f(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

We use the notation $f(\mathbf{x}) = \|\mathbf{x}\|$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|\mathbf{z}\|_* = \sup\left\{\mathbf{z}^{\mathrm{T}}\mathbf{x} \mid \|\mathbf{x}\| \leq 1\right\}.$$

a) Prove that $\|\cdot\|_*$ is a valid norm.

b) Prove that the dual of the Euclidean norm (ℓ_2-norm) is the Euclidean norm, i.e., prove that

$$\|\mathbf{z}\|_{2*} = \sup\left\{\mathbf{z}^T\mathbf{x} \mid \|\mathbf{x}\|_2 \leq 1\right\} = \|\mathbf{z}\|_2.$$

(*Hint*: Use Cauchy-Schwarz inequality.)

c) Prove that the dual of the ℓ_1-norm is ℓ_∞-norm.

d) Next, let's move to the matrix norm. The ℓ_2-norm for a matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$\|A\|_2 = \sup_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}.$$

We define $\rho(A) = \max_i\{|\lambda_i(A)|\}$, where $\lambda_i(A)$ is the eigenvalue of A. Prove that

$$\rho(A) \leq \|A\|_2.$$

Problem 2: Inequalities (10 points)

Let $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^n$, where n is a positive integer. Let $\|\cdot\|$ denote the Euclidean norm.

a) Prove $\|\mathbf{x}+\mathbf{y}\|^2 \leq (1+\epsilon)\|\mathbf{x}\|^2 + (1+\frac{1}{\epsilon})\|\mathbf{y}\|^2$ for any $\epsilon > 0$.

b) Prove $\sum_{i=1}^m \left\|\mathbf{x}_i - \frac{1}{m}\sum_{j=1}^m \mathbf{x}_j\right\|^2 = \frac{1}{2m}\sum_{i=1}^m\sum_{j=1}^m \|\mathbf{x}_i - \mathbf{x}_j\|^2$, where $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$.

(*Hint*: You may need the Young's inequality for products, i.e. if a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.)

Problem 3: Definition of Convexity (20 points)

Which of the following sets are convex? Please provide explanations for your choices.

a) a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$.

b) a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_1^T\mathbf{x}_1 \leq b_1, \mathbf{a}_2^T\mathbf{x} \leq b_2\}$.

c) The set of points closer to a given point than a given set, *i.e.*,

$$\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\| \leq \|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{y} \in S\}$$

where $S \subseteq \mathbb{R}^n$.

d) The set of points closer to one set than another, *i.e.*,

$$\{\mathbf{x} \mid \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\| \mid \mathbf{z} \in S\}.$$

e) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{a}\| \leq \theta\|\mathbf{x} - \mathbf{b}\|\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Problem 4: Convex Functions (25 points)

Which of the following functions are convex? Please provide explanations for your choices.

a) $f(x) = e^{ax}$, $a \in \mathbb{R}$

b) $f(x) = x^a$, $a \geq 1$ or $a \leq 0$

c) $f(x) = x \log x$

d) $f(x_1, \ldots, x_k) = \ln(\sum_{i=1}^k e^{x_i})$

Then prove that, for a convex function $f(x)$ and $\sum_{i=1}^k \theta_k = 1$ with $\theta_i \in [0,1], \forall i, k \geq 2$, the following inequality holds

$$f\left(\sum_{i=1}^k \theta_i x_i\right) \leq \sum_{i=1}^k \theta_i f(x_i).$$

Problem 5: Generalized Inequalities (15 points)

Let K^* be the dual cone of a convex cone K. Prove the following,

a) K^* is indeed a convex cone.

b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.