Optimization Methods **Fall 2025**

Homework 2

Instructor: Lijun Zhang *Name:* Your name, *StudentId:* Your student id

Notice

- The submission email is: **optfall25@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Convex condition (20 points)

a) Let f be twice differentiable, with $\mathbf{dom}(f)$ convex. Prove that f is convex if and only if

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geqslant 0,$$

for all x, y.

b) If $m, n > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, prove that $mn \leq \frac{m^p}{p} + \frac{n^q}{q}$.

Problem 2: Convex function (30 points)

a) Define the Huber function

$$\phi_\delta(t) = \begin{cases} \frac{1}{2}t^2, & |t| \leq \delta, \\ \delta|t| - \frac{1}{2}\delta^2, & |t| > \delta, \end{cases}$$

with $\delta > 0$. Prove that ϕ_δ is convex.

b) Suppose $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$. Show that for $x \in \mathbb{R}^n$,

$$f(x) = \sum_{i=1}^{m} \phi_\delta(a_i^\top x - b_i)$$

is a convex function.

c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable. Show that its running average F, i.e.,

$$F(x) = \frac{1}{x} \int_0^x f(t)dt$$

is convex over $\mathbf{dom}(F) = \mathbb{R}_{++}$.

d) Suppose f and g are both convex, nondecreasing (or nonincreasing), and positive real-valued functions defined on $\mathbb{R}$. Show that fg is convex on $\mathbf{dom}(f) \cap \mathbf{dom}(g)$.

Problem 3: Concave function (10 points)

Show that the function

$$f(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

with $\mathbf{dom}(f) = \mathbb{R}^n_{++}$ is concave.

Problem 4: Conjugate Function (20 points)

a) Suppose f is a closed and convex function. Show that

$$y \in \partial f(x) \quad \Longleftrightarrow \quad x \in \partial f^*(y) \quad \Longleftrightarrow \quad x^\top y = f(x) + f^*(y),$$

where $\partial f(x)$ denotes the set of all subgradients of f at x.

b) Derive the conjugate of $f(x) = \max\{0, 1 - x\}$ on $\mathbb{R}$.

c) Derive the conjugate of $f(x) = \ln\big(1 + e^{-x}\big)$ on $\mathbb{R}$.

d) Derive the conjugate of $f(x) = x^p$ on $\mathbb{R}_{++}$ where $p > 1$.

Problem 5: Projection (20 points)

For any point y, the projection onto a nonempty and closed convex set $\mathcal{X}$ is defined as

$$\Pi_{\mathcal{X}}(y) = \arg\min_{x \in \mathcal{X}} \frac{1}{2}\|x - y\|_2^2.$$

a) Prove that $\|\Pi_{\mathcal{X}}(x) - \Pi_{\mathcal{X}}(y)\|_2^2 \leqslant \langle \Pi_{\mathcal{X}}(x) - \Pi_{\mathcal{X}}(y), x - y\rangle$.

b) Prove that $\|\Pi_{\mathcal{X}}(x) - \Pi_{\mathcal{X}}(y)\|_2 \leqslant \|x - y\|_2$.