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On Discrete Presheaf Monads

Gao Zhang [1] and Shaoqun Zhang [2,*]

[1] School of Mathematical Science, Nanjing Normal University, Nanjing 210046, China; gaozhang@nnu.edu.cn
[2] School of Intelligence Science and Technology, Nanjing University, Nanjing 210023, China
* Correspondence: zhangsq@nju.edu.cn

Abstract: For a quantale I, which is a unit interval endowed with a continuous triangular norm and the Barr extension $\overline{\beta}_I$ of the ultrafilter monad to I-Rel, a characterization of the discrete presheaf monad associated to $\overline{\beta}_I$ is given. It is also proved that, when & is the Łucasiewicz triangular norm, the discrete presheaf monad is isomorphic to the saturated prefilter monad, and when & is the product triangular norm, the prime functional ideal monad is isomorphic to a submonad of the discrete presheaf monad.

Keywords: discrete presheaf monad; continuous triangular norm; Barr extension of the ultrafilter monad

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1. Introduction and Preliminaries

The lax-algebraic method is efficient in axiomatizing various types of spaces in terms of convergency, such as Barr's relational presentation of topological spaces [1] and Lawvere's characterization of generalized metric spaces as categories enriched over the Lawvere quantale $\mathsf{P}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ [2]. In addition, Clementino and Hofmann [3] extended the ultrafilter monad to the category P_+-Rel and obtained a lax-algebraic characterization of Lowen's approach spaces [4]. More examples can be found in [5–9].

Given an associated lax extension $\hat{T}$ of monad (T, m, e) to the category V-Rel of relations valued in a quantale V, a new monad (Π, n, d) named a discrete presheaf monad associated to $\hat{T}$ arises. We have the following isomorphism:

$$(\Omega, 2)\text{-}\mathsf{Cat} \cong \Pi\text{-}\mathsf{Mon} \cong (T, \mathsf{V})\text{-}\mathsf{Cat},$$

where (T, V)-Cat is the category of lax algebras for $\hat{T}$, Π-Mon is the category of Kleisli monoids of the discrete presheaf monad and $(\Pi, 2)$-Cat is the category of lax algebras for the Kleisli extension of (Ω, n, d).

The above isomorphism provides different characterizations of the same object. When (T, m, e) is the ultrafilter monad and $\mathsf{V} = 2$, the discrete presheaf monad is isomorphic to the filter monad. Thus, the above isomorphism implies that topological spaces are the lax algebras for the filter monad [10], the Kleisli monoids of the filter monad [11] and the lax algebras for the ultrafilter monad; that is, we can characterize a topological space using the filter convergence, the neighborhood system and the ultrafilter convergence.

Furthermore, the discrete presheaf monad (Ω, n, d) is V-power-enriched [12]; hence, it satisfies that (i) the RegMono-injective separated Kleisli monoids are exactly the Eilenberg–Moore algebras for (Ω, n, d); (ii) there is a derived monad on the category V-Cat that has the same Eilenberg–Moore algebras with the original monad (Ω, n, d). These two facts will be helpful in studying injective objects. For example, when the discrete presheaf monad is the filter monad, the derived monad on 2-Cat $=$ Ord is the composite monad of the ordered-filter monad and down-set monad on Ord; thus, its Elenberg–Moore algebras are complete and continuous posets (continuous lattices). Therefore, by (i), we come to Scott's result [13]: the injective T_0 spaces are continuous lattices endowed with their Scott topology.

The paper aims to study the discrete presheaf monad associated to the Barr extensions of the ultrafilter monad to I-Rel, where I is the unit interval endowed with a continuous triangular norm. When the continuous t-norm is a product t-norm, there is a characterization of the discrete presheaf monad, which is presented in [14] (Subsection IV.3.3). We present a similar characterization in this paper for a general continuous triangular norm. As for the Eilenberg–Moore algebras for these monads, it is clear only when the continuous t-norm is the Łukasiewicz triangular norm.

Some preparations about continuous triangular norms, monads and lax extensions are given in the remainder of this section. In Section 2, we extend the ultrafilter monad to the category I-Rel of I-relations, and we prove that these extensions are associated. For a general continuous triangular norm, a characterization of the discrete presheaf monad is presented in Section 2 too. Section 3 focuses on the case where the continuous triangular norm is Archimedean. It is proved that, when the continuous triangular norm is the Łucasiewicz triangular norm, the discrete presheaf monad is isomorphic to the saturated prefilter monad [15]; when the continuous triangular norm is the product triangular norm, the prime functional ideal monad [16,17] is isomorphic to a submonad of the discrete presheaf monad.

1.1. Continuous Triangular Norms and I-Rel

A triangular norm (t-norm for short) is a binary operation & on the unit interval I such that $(I, \&, 1)$ is a commutative monoid and $a\&(-)$ is monotone for all $a \in I$.

A t-norm & is said to be continuous if the function $(-)\&(-)\colon I \times I \to I$ is continuous with respect to the standard topology. Given a continuous t-norm &, for every $a \in I$, since $a\&(-)\colon I \to I$ preserves arbitrary joints, it admits a right adjoint $a \to (-)\colon I \to I$ that is determined by

$$a\&b \le c \iff b \le a \to c$$

for all $b, c \in I$. The map $\to$ is called the implication of &.

The following proposition is easy to check.

Proposition 1. *Let & be a continuous t-norm. Then, for any $a, b, c \in I$ and $\{a_i\}_i \subset I$, we have that*

(1) $a \to b = 1 \iff a \le b$;
(2) $(a\&b) \to c = a \to (b \to c)$;
(3) $a \to (\bigwedge_i a_i) = \bigwedge_i (a \to a_i)$;
(4) $(\bigvee_i a_i) \to a = \bigwedge_i (a_i \to a)$.

Example 1. *There are three basic continuous t-norms.*

1. *The Łukasiewicz t-norm $a\&_L b = \max\{a + b - 1, 0\}$. Its implication is given by $a \to b = \min\{1 - a + b, 1\}$.*
2. *The product t-norm $a\&_P b = ab$. Its implication is given by $a \to b = \min\{1, \frac{b}{a}\}$.*
3. *The Gödel t-norm $a\&b = a \wedge b$. Its implication is given by*

$$a \to b = \begin{cases} 1, & a \le b, \\ b, & a > b. \end{cases}$$

Since & is associated, we can denote by a^n the

$$\overbrace{a\&\cdots\&a}^{n}.$$

An element $a \in I$ is called idempotent if $a^2 = a$.

The Łukasiewicz t-norm and the product t-norm are Archimedean, which means that, for any $a, b \in (0, 1)$, there is an $n \in \mathbb{N}$ such that $a^n < b$. Moreover, the Łukasiewicz t-norm is nilpotent, which means that, for any $a \in (0, 1)$, there is an $n \in \mathbb{N}$ such that $a^n = 0$.

Similarly, we can define a t-norm & on any closed interval $[a,b]$. Two t-norms $([a,b],\&,b)$ and $([a',b'],\&',b')$ are said to be isomorphic if there exists an order isomorphism $f\colon [a,b] \to [a',b']$ such that $c\&d = f(c)\&' f(d)$ holds for all $c,d \in [a,b]$.

Example 2. *The Lawvere quantale* $\mathsf{P}_+ = ([0,\infty]^{\mathrm{op}},+,0)$ *is isomorphic to the product t-norm* $(I,\&,1)$ *by*

$$f\colon [0,\infty] \to [0,1], \quad x \mapsto e^{-x}.$$

Assumption 1. *From now on, we always give a t-norm by a triple* $([a,b],\&,b)$ *and assume that it is continuous. In particular,* I *denotes the unit interval endowed with a continuous t-norm* $(I,\&,1)$.

A continuous t-norm is Archimedean if and only if it is isomorphic either to the Łucasiewicz t-norm or to the product t-norm [18]. The following celebrated ordinal sum decomposition theorem gives a characterization of continuous t-norms.

Theorem 1 ([18,19])**.** *Let* $(I,\&,1)$ *be a continuous t-norm. Then, there exists a family of disjoint* $(a_n,b_n) \subset I$ *such that:*

(1) *For any* n, a_n, b_n *are idempotent and* $([a_n,b_n],\&,b_n)$ *is isomorphic to the unit interval endowed with a continuous Archimedean t-norm;*

(2) $a\&b = a \wedge b$ *for any* $(x,y) \notin \bigcup_n [a_n,b_n]^2$.

If $a \in (0,1)$ lies in some above-mentioned (a_n,b_n), we let a^- denote a_n and a^+ denote b_n.

An I-relation $r\colon X \nrightarrow Y$ is a map $r\colon X \times Y \to I$. Given two I-relations $r\colon X \nrightarrow Y$, $s\colon Y \nrightarrow Z$, the composition $s \cdot r\colon X \nrightarrow Z$ is given by

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y)\&s(y,z).$$

Sets and I-relations form a category

$$\mathsf{I}\text{-}\mathsf{Rel}.$$

Let $f\colon X \to Y$ be a map. Then, there exist two I-relations $f_\circ\colon X \nrightarrow Y$, $f^\circ\colon Y \nrightarrow X$ given by

$$f_\circ(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y \end{cases} \quad \text{and} \quad f^\circ(x,y) = f_\circ(y,x).$$

Usually, we simply write $f_\circ$ as f.

1.2. Monads and Their Lax Extensions

1.2.1. Monad

A monad on a category A is a triple (T,m,e) consisting of an endfunctor $T\colon \mathsf{A} \to \mathsf{A}$ and two natural transformations: the unit $e\colon \mathrm{id}_{\mathsf{A}} \to T$ and the multiplication $m\colon T^2 \to T$, making the diagrams

$$\begin{array}{ccccc} T & \xrightarrow{eT} & T^2 & \xleftarrow{Te} & T \\ & \underset{\mathrm{id}}{\searrow} & \downarrow{\scriptstyle m} & \underset{id}{\swarrow} & \\ & & T & & \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{mT} & T^2 \\ {\scriptstyle Tm}\downarrow & & \downarrow{\scriptstyle m} \\ T^2 & \xrightarrow{m} & T \end{array}$$

commutative.

An adjunction $F \dashv G\colon \mathsf{B} \to \mathsf{A}$ gives rise to a monad $(GF, G\epsilon F, \eta)$ on A, where η and ϵ are the unit and the counit, respectively. Many of the monads discussed in this paper are determined by adjunctions constructed as follows:

Proposition 2. *Let* A *be a locally small category and* A *be the object of* A. *The functor* $\mathsf{A}(-,A)\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Set}$ *is right adjoint if and only if the product* $\prod_{x \in X} A_x$ *in* A *with* $A_x = A$ *for all* $x \in X$ *exists for every set* X.

Proof. Let B be an object in A. A map $f\colon X \to \mathsf{A}(B, A)$ is equivalent to a source $(B, f(x)\colon B \to A_x)$ with $A_x = A$ for all $x \in X$. In addition, f is universal from X to $\mathsf{A}(-, A)$; that is, for any object C in A and map $g\colon X \to \mathsf{A}(C, A)$, there is a morphism $g'\colon C \to B$ such that $g = \mathsf{A}(-, A)(g') \cdot f$ if and only if the source $(B, f(x))$ is universal. That means that B is the product of $\{A_x\}_{x \in X}$ in which $A_x = A$ for all $x \in X$.

Since $\mathsf{A}(-, A)$ is right adjoint if and only if, for each set X, there exist an object F_0X in A and a universal map $\eta_X\colon X \to \mathsf{A}(F_0X, A)$ from X to $\mathsf{A}(-, A)$, the conclusion follows. In this case, the assignment $X \mapsto \prod_{x \in X} A_x$ is functorial and left adjoint to $\mathsf{A}(-, A)$. □

Example 3.

(1) *Let* 2 *be the set* $\{0, 1\}$. *Then, the functor* $\mathsf{Set}(-, 2)\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$, *known as the contravariant powerset functor, is right adjoint, and the induced monad is called the double-powerset monad.*

(2) *Let* 2 *be the lattice* $(\{0, 1\}, \leq)$, *in which* $0 \leq 1$. *The functor* $\mathsf{Lat}(-, 2)\colon \mathsf{Lat}^{\mathrm{op}} \to \mathsf{Set}$ *is right adjoint, where* Lat *is the category lattices and lattice morphisms. For each set* X, *the product* $\prod_{x \in X} 2_x$ *is exactly the power set* $\mathsf{P}X$ *endowed with the inclusion order. This adjunction induces the ultrafilter monad* $(\beta, \Sigma, (\dot{-}))$. *We spell it out here: the functor* β *defined on objects by* $\beta X = \{U \mid U \text{ is an ultrafilter on } X\}$ *on morphisms by* $\beta(f)\colon U \mapsto \{A \subset Y \mid f^{-1}(A) \in U\}$. *The two natural transformations are given by:*

$$(\dot{-})_X\colon X \to \beta X, \quad x \mapsto \{A \subset X \mid x \in A\};$$
$$\Sigma_X\colon \beta^2 X \to \beta X, \quad \mathfrak{U} \mapsto \{A \subset X \mid A^{\beta} \in \mathfrak{U}\},$$

where $A^{\beta} = \{U \in \beta X \mid A \in U\}$.

A morphism (isomorphism) $\kappa\colon (T, m, e) \to (T', m', e')$ of monads is a natural transformation (isomorphism) $\kappa\colon T \to T'$ such that

$$m = m' \cdot (\kappa * \kappa) \quad \text{and} \quad e' = \kappa \cdot e,$$

where $*$ is the horizontal composition of natural transformations.

Let (T, m, e) be a monad on Set. If T' is a subfunctor of T and the inclusion transformation $i\colon T' \to T$ satisfies that, for any $x \in X, \Phi \in T'T'X$,

$$e_X(x) \in T'X \quad \text{and} \quad (m \cdot (i * i))_X(\Phi) \in T'X,$$

then $(T', m \cdot (i * i), e)$ is called a submonad of (T, m, e). Usually, we simply write m for $m \cdot (i * i)$.

1.2.2. Lax Extension

A lax extension of a functor $T\colon \mathsf{Set} \to \mathsf{Set}$ to I-Rel is an assignment $\hat{T}\colon \mathsf{Set} \to \mathsf{Set}$ and a family of maps $\hat{T}_{X,Y}\colon \mathsf{I}\text{-}\mathsf{Rel}(X, Y) \to \mathsf{I}\text{-}\mathsf{Rel}(X, Y)$ (X, Y run through all the sets, and we usually simply write $\hat{T}_{X,Y}$ as $\hat{T}$) such that

(1) $\hat{T}X = X$;
(2) $\hat{T}(r) \leq \hat{T}(r')$ and $\hat{T}(s \cdot r) \leq \hat{T}(s) \cdot \hat{T}(r)$;
(3) $Tf \leq \hat{T}(f)$ and $(Tf)^{\circ} \leq \hat{T}(f^{\circ})$.

for any I-relations $r \leq r'\colon X \nrightarrow Y, s\colon Y \nrightarrow Z$ and map $f\colon X \to Y$.

A lax extension of a monad (T, m, e) on Set to I-Rel is a triple $(\hat{T}, m, e)$ such that $\hat{T}$ is a lax extension of T and e, m are op-lax; that is,

(4) $m_Y \cdot \hat{T}\hat{T}r \leq \hat{T}r \cdot m_X$;
(5) $e_Y \cdot r \leq \hat{T}r \cdot e_X$.

for any I-relations $r\colon X \nrightarrow Y$.

A (T,I)-relation $r\colon X \rightharpoondown Y$ is an I-relation $r\colon TX \nrightarrow Y$. The Kleisli convolution of (T,I)-relations $r\colon X \rightharpoondown Y$, $s\colon Y \rightharpoondown Z$ is a (T,I)-relation $s \circ r\colon X \rightharpoondown Z$ given by

$$s \circ r = s \cdot (\hat{T}r) \cdot m_X^{\circ}.$$

In general, sets with (T,I)-relations (composed with the Kleisli convolution) do not form a category. There are two problems: (i) in general, the Kleisli convolution is not associated; (ii) in general, the Kleisli convolution does not allow for identity morphisms.

For (ii), we take a subclass of (T,I)-relations: a (T,I)-relation $r\colon X \rightharpoondown Y$ is unitary if it satisfies that

$$e_Y^{\circ} \circ r = r \quad \text{and} \quad r \circ e_X^{\circ} = r.$$

For every unitary (T,I)-relation $r\colon X \rightharpoondown Y$, the unitary (T,I)-relations $1_X^{\sharp} = e_X^{\circ} \circ e_X^{\circ}$ are identities of the Kleisli convolution; that is,

$$r \circ 1_X^{\sharp} = r \quad \text{and} \quad 1_Y^{\sharp} \circ r = r;$$

see [14] (Subsection III.1.8) for more detail.

A lax extension is said to be associated if the Kleisli convolution of unitary (T,I)-relations is associated. In this case, we obtain a category

$$(T,\mathsf{I})\text{-}\mathsf{URel}.$$

2. The Barr Extension to I-Rel and Induced Monad

We consider the strata extensions of the Barr extension $\overline{\beta}$ along $\mathsf{Rel} \to \mathsf{I}$-$\mathsf{Rel}$:

$$\overline{\beta}_I r\colon \beta X \nrightarrow \beta Y, \quad \overline{\beta}_I r(F,G) = \bigvee \{a \mid F\,(\overline{\beta} r_a)\,G\}$$

for any $F \in \beta X, G \in \beta Y$ and $r\colon X \nrightarrow Y$, where $r_a = \{(x,y) \mid r(x,y) \geq a\}$ and $\overline{\beta}$ is the Barr extension of the ultrafilter monad to Rel; that is,

$$F\,(\overline{\beta} r_a)\,G \iff \forall A \in F,\ G \ni r_a(A) = \{y \in Y \mid \exists x \in A, r(x,y) \geq a\}.$$

When r is dummy in one variable—that is $X = \{*\}$ or $Y = \{*\}$—we simply write $r(x)$ for $r(x,*)$ or $r(*,x)$. In this case, it is easy to check that ([20] (1.1.5 Lemma))

$$\overline{\beta}_I r(U) = \bigvee_{A \in U} \bigwedge_{x \in A} r(x) = \bigwedge_{A \in U} \bigvee_{x \in A} r(x).$$

Proposition 3. *For each relation $\mu, \nu\colon \{*\} \rightharpoondown X$ and $a \in [0,1]$, we have that*

(1) $\overline{\beta}_I(\mu \vee \nu) = \overline{\beta}_I(\mu) \vee \overline{\beta}(\nu)$;
(2) $\overline{\beta}_I(a \& \mu) = a \& \overline{\beta}_I(\mu)$;
(3) $\overline{\beta}_I(\mu \wedge \nu) = \overline{\beta}_I(\mu) \wedge \overline{\beta}_I(\nu)$.

Proof. For (1), we have that

$$\begin{aligned} \bigwedge_{A \in U} \bigvee_{x \in A} (\mu \vee \nu)(x) &= \bigwedge_{A \in U} \Big(\bigvee_{x \in A} \mu(x)\Big) \vee \Big(\bigvee_{x \in A} \nu(x)\Big) \\ &= \overline{\beta}_I(\mu) \vee \overline{\beta}_I(\nu) \qquad (U \text{ is a filter}). \end{aligned}$$

(2) results directly from the continuity of &. The proof of (3) is similar to that of (1). □

Since the unit interval is a completely distributive lattice, the strata extensions $\overline{\beta}_I$ are lax extensions of an ultrafilter monad to I-Rel [14] (IV.2.4.3 Propostion). We call these lax extensions Barr extensions to I-Rel and use the same notation $\overline{\beta}$ to denote them if no confusion would arise. These lax extensions also appeared in [6,8].

Next, we prove that the Barr extensions of the ultrafilter monad to I-Rel are associated. The proof for the case that $(I,\&,1)$ is the product t-norm is given in [14] (III.2.4.3 Proposition) and relies on the fact that the unit interval is completely distributive and the product t-norm is continuous. Thus, the proof also works well here with a slight modification.

Proposition 4. *The Barr extensions of an ultrafilter monad to* I-Rel *are associated.*

Proof. Thanks to [14] (III.1.9.4 Proposition), it suffices to show that $\overline{\beta}$ preserves the composition of I-relations and that $m^\circ\colon \overline{\beta} \to \overline{\beta\beta}$ is natural.

Let $r\colon X \nrightarrow Y, s\colon Y \nrightarrow Z$ be I-relations, F be an ultrafilter on X and H be an ultrafilter on Z. For any $a < a' < (\overline{\beta}s \cdot \overline{\beta}r)(F,H)$, there exists some $G_0 \in \beta Y$ such that $a < a' \leq \overline{\beta}s(G_0,H)\&\overline{\beta}r(F,G_0)$. Since & is continuous, we can pick b,c such that $G_0\ (\overline{\beta}s_b)\ H$, $F\ (\overline{\beta}r_c)\ G_0$ and $b\&c = a$. For any $A \in F$, since $s_b(r_c(A)) \in H$ and $s_b(r_c(A)) \subset (s\cdot r)_a(A)$, we have that $F\ \overline{\beta}(s\cdot r)_a\ H$. Thus, $\overline{\beta}(s\cdot r)(F,H) \geq a$.

Since m is op-lax, we only need to show that $\overline{\beta\beta}r \geq m_Y^\circ \cdot \overline{\beta}r \cdot m_X$. Let $\mathfrak{F} \in \beta\beta X$, $\mathfrak{G} \in \beta\beta Y$. Note that the Barr extension of an ultrafilter monad to Rel is associated. For any $a < \overline{\beta}r(m_X(\mathfrak{F}), m_Y(\mathfrak{G}))$, we have that

$$m_X\,(\mathfrak{F})(\overline{\beta}r_a)\,m_Y(\mathfrak{G}) \iff \mathfrak{F}\,(\overline{\beta\beta}r_a)\,\mathfrak{G} \implies \mathfrak{F}\,(\overline{\beta}(\overline{\beta}r)_a)\,\mathfrak{G} \implies \mathfrak{F}\,(\overline{\beta\beta}r)_a\,\mathfrak{G},$$

so $\overline{\beta\beta}r(\mathfrak{F},\mathfrak{G}) \geq a$. □

Since the Barr extensions of the ultrafilter monad to I-Rel are associated, there is a functor $(\beta,\mathsf{I})\text{-URel}(-,\{*\})\colon (\beta,\mathsf{I})\text{-URel}^{\mathrm{op}} \to \mathsf{Set}$. For each set X, the product $\prod_{x\in X}\{*\}_x$ is given by

$$\prod_{x\in X}\{*\}_x = X, \quad p_x = (c_x)^\sharp = c_x^\circ \cdot e_X^\circ\colon X \nrightarrow \{*\}_x,$$

where c_x is the map from $\{*\}$ to X that maps $*$ to $x \in X$. Thus, the left adjoint of $(\beta,\mathsf{I})\text{-URel}(-,*)$ is identical on objects and maps every map $f\colon X \to Y$ to the unitary $(\overline{\beta},\mathsf{I})$-relation $f^\sharp = f^\circ \cdot e_Y^\circ$. We denote it by $(-)^\sharp$. The monad determined by this adjunction is referred to as the discrete presheaf monad associated to $\overline{\beta}$. It is denoted as (Π, n, d).

In order to characterize the discrete presheaf monad associated to $\overline{\beta}$, we introduce some notions first. An I-category [21,22] (X,r) is a set X with an I-relation $r\colon X \nrightarrow X$ that satisfies

$$r(x,x) = 1 \quad \text{and} \quad r(x,y)\&r(y,z) \leq r(x,z)$$

for any $x,y,z \in X$. For convenience, we denote by $X(-,-)$ the I-relation r if no confusion would arise. The underlying order $\leq$ of X is defined by

$$x \leq y \iff X(x,y) = 1.$$

Let X be an I-category. The tensor [21,22] $x \otimes_X a$ of $x \in X$ and $a \in I$ is an element of X such that, for any $y \in X$, it holds that

$$X(x \otimes_X a, y) = a \to X(x,y).$$

The subscript X of $\otimes_X$ is omitted if there is no danger of ambiguity. An I-category X is called tensored if, for any $x \in X$ and $a \in I$, the tensor $x \otimes a$ exists.

An $\mathsf{I}_\vee$-category is a tensored I-category whose underlying order has all finite joins. An $\mathsf{I}_\vee$-functor is a map between $\mathsf{I}_\vee$-categories that preserves tensors and all finite joins. Since the composition of $\mathsf{I}_\vee$-functors is again an $\mathsf{I}_\vee$-functor, we obtain a category whose objects are $\mathsf{I}_\vee$-categories and morphisms are $\mathsf{I}_\vee$-functors and denote it by

$$\mathsf{I}\text{-}\mathsf{Cat}^\vee_\otimes.$$

Example 4. *Let $d_L(x,y) = x \to y$. Then, (I, d_L) is an $\mathsf{I}_\vee$-category tensored by $x \otimes a = a\&x$ for any $a, x \in I$. Given a set X, the product $\prod_{x\in X}(I, d_L)_x$ in $\mathsf{I}\text{-}\mathsf{Cat}^\vee_\otimes$ is (I^X, sub_X), where*

$$\mathrm{sub}_X \colon I^X \nrightarrow I^X, \quad (\mu, \nu) \mapsto \bigwedge_{x\in X} \mu(x) \to \nu(x).$$

The tensor of (I^X, sub_X) is given by

$$\mu \otimes a = a\&\mu$$

for any $a \in I, \mu \in I^X$, and its underlying order is pointwise order, and hence is complete.

By the above example, the functor $\mathsf{I}\text{-}\mathsf{Cat}^\vee_\otimes(-, (I, d_L)) \colon \mathsf{I}\text{-}\mathsf{Cat}^{\vee\,\mathrm{op}}_\otimes \to \mathsf{Set}$ is left adjoint. The right adjoint $(\mathsf{P}^\bullet_I)^{\mathrm{op}} \colon \mathsf{Set} \to \mathsf{I}\text{-}\mathsf{Cat}^{\vee\,\mathrm{op}}_\otimes$ is given by

$$(\mathsf{P}^\bullet_I)^{\mathrm{op}}(X) = (X, \mathrm{sub}_X), \quad (\mathsf{P}^\bullet_I)^{\mathrm{op}}(f)(\mu) = \mu \cdot f$$

for any map $f \colon X \to Y$ and $\mu \in I^Y$. We denote the induced monad by $(\mathbf{F}^\vee_\otimes, m_\vee, e_\vee)$ and explicitly state its form here:

$$(e_\vee)_X(x) \colon I^X \to I, \quad \mu \mapsto \mu(x);$$
$$(m_\vee)_X(\Phi) \colon I^X \to I, \quad \mu \mapsto \Phi(\widetilde{\mu})$$

for any $x \in X$ and $\Phi \in (\mathsf{F}^\vee_\otimes)^2 X$, where $I^{\mathsf{F}^\vee_\otimes X} \ni \widetilde{\mu} \colon \phi \mapsto \phi(\mu)$.

For the rest of this section, we prove that the discrete presheaf monad is isomorphic to $(\mathbf{F}^\vee_\otimes, m_\vee, e_\vee)$. At first, we show that each element of $\mathsf{F}^\vee_\otimes X$ is determined by its effect on $\{1_A \mid A \subset X\}$, where

$$1_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

This conclusion is proved in [14] (IV.3.3.1 Theorem) for the Lawvere quantale P_+ and hence for the product t-norm. Our strategy is that we prove the result for the Archimedean case first, and then, with the help of the ordinal sum decomposition theorem, we prove the result for general cases.

An element μ of I^X is called bounded if $\bigwedge \mu > 0$.

Lemma 1. *Let $\phi \colon (I^X, \mathrm{sub}_X) \to (I, d_L)$ be an $\mathsf{I}_\vee$-functor. Then, ϕ is determined by its effect on bounded elements.*

Proof. If $\phi(1_X) = 0$, we are finished. If $\phi(1_X) > 0$, then, by the continuity of &,

$$\phi(\mu) = \bigwedge_{a>0} \phi(\mu) \vee a = \bigwedge_{b>0} \phi(\mu) \vee (b\&\phi(1_X)) = \bigwedge_{b>0} \phi(\mu \vee b_X)$$

for any μ. □

Lemma 2. *Let $\phi \colon (I^X, \mathrm{sub}_X) \to (I, d_L)$ be an $\mathsf{I}_\vee$-functor. If $(I, \&, 1)$ is Archimedean, then ϕ is fully determined by its effect on $\{1_A \mid A \subset X\}$.*

Proof. *Case 1.* $(I, \&, 1)$ is isomorphic to the Łukasiewicz t-norm.

Let $a_n = 1 - \frac{1}{n}, n \in \mathbb{N}$ and $\mu \in I^X$. Fix n. Since $(I, \&, 1)$ is the Łukasiewicz t-norm, there exists an $M_n \in \mathbb{N}$ such that $(a_n)^{M_n - 1} > (a_n)^{M_n} = 0$. Let

$$b_n = \bigvee_{i=1}^{M_n} (a_n)^i \& \phi(1_{A_i}) \quad \text{and} \quad c_n = \bigvee_{i=1}^{M_n} (a_n)^{i-1} \& \phi(1_{A_i}),$$

where $A_1 = \{x \mid a_n \leq \mu(x) \leq 1\}$ and $A_i = \{x \mid (a_n)^i \leq \mu(x) < (a_n)^{i-1}\}$ for $i = 2, 3, \cdots, M_n$. Since ϕ preserves the tensors and all finite joins, it holds that

$$b_n = \bigvee_{i=1}^{M_n} (a_n)^i \& \phi(1_{A_i}) \leq \phi(\mu) \leq \bigvee_{i=1}^{M_n} (a_n)^{i-1} \& \phi(1_{A_i}) = c_n.$$

Since $(I, \&, 1)$ is the Łukasiewicz t-norm,

$$c_n - b_n = \Big(\bigvee_{i=1}^{M_n} (a_n)^{i-1} \& \phi(1_{A_i})\Big) - \Big(a_n \& \bigvee_{i=1}^{M_n} (a_n)^{i-1} \& \phi(1_{A_i})\Big) \leq 1 - a_n$$

for any $n \in \mathbb{N}$. Note that, for any $n \in \mathbb{N}$, it holds that (i) b_n, c_n are determined by $\{\phi(1_A) \mid A \subset X\}$; (ii) $\phi(\mu) \in [b_n, c_n]$. This completes the proof.

Case 2. $(I, \&, 1)$ is isomorphic to the product t-norm. The proof for the product t-norm case is in [14] (IV.3.3.1 Theorem). We sketch the proof here as follows: Step 1. use Lemma 1 to assume that μ is bounded; Step 2. since $(I, \&, 1)$ is the product t-norm, there is an M_n such that $a_n^{M_n} < \bigwedge \mu$. Then, the proof is similar to Case 1. □

From the proof, we can see that, if the range of μ falls within some $[a, b]$ for which $([a, b], \&, b)$ is a continuous Archimedean t-norm, then $\phi(\mu)$ can be determined by μ and $\{\phi(1_A) \mid A \subset X\}$.

In order to prove the above conclusion for a general continuous t-norm, we introduce some notations and prove an easy lemma.

Let ϕ be an $\mathsf{I}_\vee$-functor and μ an element of I^X. For convenience, we denote by a the infimum of μ and by b the supremum of μ. For each $t \in [a, b]$, let $A_t = \{x \mid \phi(x) > t\}$ and

$$\mu_t(x) = \begin{cases} \mu(x), & x \in A_t \\ 0, & x \notin A_t. \end{cases}$$

It is easy to check that

$$\mu = (t \wedge \mu) \vee \mu_t \quad \text{and} \quad t \& 1_{A_t} \leq \mu_t \leq b \& 1_{A_t}$$

for all $t \in [a, b]$. The following function α is determined by μ and $\{\phi(1_A) \mid A \subset X\}$:

$$\alpha \colon [a, b] \to [0, 1], \quad x \mapsto \phi(1_{A_x}).$$

Lemma 3. *The function α satisfies the following statements:*

(1) *α is decreasing and $\alpha(b) = 0$;*
(2) *For each idempotent $t \in (a, b)$, if $\alpha(t) \geq t$, then $\phi(\mu) = \phi(t \vee \mu)$ and $\phi(\mu) \geq t$;*
(3) *For each idempotent $t \in (a, b)$, if $\alpha(t) < t$, then $\phi(\mu) = \phi(t \wedge \mu)$ and $\phi(\mu) \leq t$.*

Proof. Note that ϕ preserves the empty joint, (1) is trivial.

To see (2), since $t = t \wedge \alpha(t) \leq \phi(\mu_t)$, we have that $\phi(\mu) = \phi(t \wedge \mu) \vee \phi(\mu_t) = \phi(\mu_t)$. Then, we conclude that

$$\phi(t \vee \mu) = \phi(t \vee \mu_t) = (t \wedge \phi(1_X)) \vee \phi(\mu_t) = t \vee \phi(\mu_t) = \phi(\mu).$$

To see (3), since $t \wedge \alpha(t) = \alpha(t) \leq \phi(\mu_t) \leq \alpha(t)$, it holds that $\phi(\mu_t) = \alpha(t) \leq (t \wedge \phi(\mu)) = \phi(t \wedge \mu)$. Hence, we have that $\phi(\mu) = \phi(t \wedge \mu) \vee \phi(\mu_t) = \phi(t \wedge \mu)$. □

Now, we can prove the following proposition.

Proposition 5. *Let $\phi \colon (X, \mathrm{sub}_X) \to (I, d_L)$ be an $\mathsf{I}_\vee$-functor. Then, it is fully determined by its effect on $\{1_A \mid A \subset X\}$.*

Proof. Given an $\mu \in I^X$, we adopt the notation of Lemma 3 and let $B = \{x \mid \mu(x) = a\}$.

Case 1. a is not idempotent and $\alpha(a) \leq a$.

Since $\alpha(a^+) < a^+$, by Lemma 3, we have that $\phi(\mu) = \phi(a^+ \wedge \mu)$. Then, as the range of $a^+ \wedge \nu$ is contained within $[a^-, a^+]$, we can apply Lemma 2 to reach the desired conclusion.

Case 2. a is idempotent and $\alpha(a) \leq a$.

In this case, we have that $\alpha(a) = a \wedge \alpha(a) \leq \phi(\mu_a) \leq \alpha(a)$ and $\mu = (a \wedge 1_B) \vee \mu_a$. Hence, the conclusion follows from

$$\phi(\mu) = (a \wedge \phi(1_B)) \vee \phi(\mu_a) = (a \wedge \phi(1_B)) \vee \alpha(a).$$

Case 3. $\alpha(a) > a$.

As $\alpha(a) > a, \alpha(b) = 0 < b$ and α is decreasing, one can find some $c \in (a, b)$ such that

$$a \leq t < c \implies \alpha(t) > t \quad \text{and} \quad b \geq t > c \implies \alpha(t) < t.$$

By the ordinal sum decomposition theorem, we distinguish four subcases.

Subcase 1. $\alpha(c) \geq c$ and there exists some $d > c$ such that the elements of $[c, d]$ are idempotent. In this case, $\phi(\mu) = c$.

By Lemma 3, we have that

$$\phi(\mu) \leq \bigwedge_{c<t<d} t = c \quad \text{and} \quad \phi(\mu) \geq c.$$

Subcase 2. $\alpha(c) < c$ and there exists some $d < c$ such that the elements of $[d, c]$ are idempotent. In this case, $\phi(\mu) = c$.

By Lemma 3, we have that

$$\phi(\mu) \geq \bigvee_{d<t<c} t = c \quad \text{and} \quad \phi(\mu) \leq c.$$

Subcase 3. $\alpha(c) \geq c$ and there exists some $d > c$ such that the elements of $(c, d]$ are non-idempotent.

In this case, we have that $\alpha(d^-) \geq d^-$ and $\alpha(d^+) < d^+$. Thus, one can use Lemma 3 twice to obtain that

$$\phi(\mu) = \phi((\mu \wedge d^+) \vee d^-).$$

It follows from Lemma 2 that $\phi((\mu \wedge d^+) \vee d^-)$ is determined by $\{\phi(1_A) \mid A \subset X\}$.

Subcase 4. $\alpha(c) < c$ and there exists some $d < c$ such that the elements of $[d, c)$ are non-idempotent.

The proof is similar to that of Subcase 3. □

The unitariness of a (β, I)-relation $r\colon \{*\} \nrightarrow X$ is trivial; thus, there is a bijection between (β, I)-$\mathsf{URel}(\{*\}, X)$ and I^X.

By endowing (β, I)-$\mathsf{URel}(\{*\}, X)$ with the I-relation sub_X, by Proposition 3, every unitary $(\overline{\beta}, \mathsf{I})$-relation $r\colon X \nrightarrow Y$ gives an $\mathsf{I}_\vee$-functor

$$r \circ (-)\colon (\beta, \mathsf{I})\text{-}\mathsf{URel}(\{*\}, X) \to (\beta, \mathsf{I})\text{-}\mathsf{URel}(\{*\}, Y).$$

Thus, the functor $K = (\beta, \mathsf{I})$-$\mathsf{URel}(\{*\}, -)\colon (\beta, \mathsf{I})$-$\mathsf{URel} \to \mathsf{Set}$ can be lifted to

$$K\colon (\beta, \mathsf{I})\text{-}\mathsf{URel} \to \mathsf{I}\text{-}\mathsf{Cat}^{\vee}_{\otimes}.$$

For every map $f\colon X \to Y$, the $\mathsf{I}_\vee$-functor $K^{\mathrm{op}}(f^\sharp)$ maps the unitary $(\overline{\beta}, \mathsf{I})$-relation $u\colon 1 \nrightarrow Y$ to

$$f^\sharp \circ \mu = f^\circ \cdot e_Y^\circ \cdot \overline{\beta}\mu = (\mathsf{P}_I^\bullet)^{\mathrm{op}}(f)(\mu).$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} (\beta, \mathsf{I})\text{-}\mathsf{URel}^{\mathrm{op}} & \xrightarrow{\quad K^{\mathrm{op}} \quad} & (\mathsf{I}\text{-}\mathsf{Cat}^{\vee}_{\otimes})^{\mathrm{op}}. \\ & {\scriptstyle (-)^{\sharp}} \nwarrow \quad \mathsf{Set} \quad \nearrow {\scriptstyle (\mathsf{P}^{\bullet}_{I})^{\mathrm{op}}} & \end{array}$$

In addition, K^{op} induces a monad morphism $\delta\colon \Pi \to \mathsf{F}^{\vee}_{\otimes}$ given by

$$\begin{aligned} \delta_X\colon & (\beta, \mathsf{I})\text{-}\mathsf{URel}(X, \{*\}) \to \mathsf{I}\text{-}\mathsf{Cat}^{\vee}_{\otimes}((\mathsf{P}^{\bullet}_{I})^{\mathrm{op}}(X), (I, d_L)) \\ & : r \mapsto r \circ (-). \end{aligned}$$

For every $\mathsf{I}_{\vee}$-functor ϕ, let

$$r_\phi\colon X \nrightarrow \{*\}, \quad (U, *) \mapsto \bigwedge_{A \in U} \phi(1_A).$$

Since, for every $\mathfrak{U} \in \beta^2 X$ and $A \in \Sigma(\mathfrak{U})$, it holds that

$$\overline{\beta}(r_\phi)(\mathfrak{U}) = \bigvee_{\mathfrak{a} \in \mathfrak{U}} \bigwedge_{U \in \mathfrak{a}} r_\phi(U) \leq \bigvee_{\mathfrak{a} \in \mathfrak{U}} \bigwedge_{U \in (\mathfrak{a} \cap A^\beta)} r_\phi(U) \leq \phi(1_A),$$

r_ϕ is unitary.

Theorem 2. *The monads* $(\mathsf{F}^{\vee}_{\otimes}, m_{\vee}, e_{\vee})$ *are isomorphic to the discrete presheaf monads associated to* $\overline{\beta}$.

Proof. It suffices to show that $\phi = r_\phi \circ (-)$ and $r = r_{r \circ (-)}$ hold for any $\mathsf{I}_{\vee}$-functor ϕ and unitary (β, I)-relation $r\colon X \nrightarrow \{*\}$.

To see $\phi = r_\phi \circ (-)$, by Proposition 5, we only need to prove that ϕ coincides with $r_\phi \circ (-)$ on $\{1_A \mid A \subset X\}$. Given an $A_0 \subset X$, we have that

$$r_\phi \circ 1_{A_0} = \bigvee_{U \in \beta X} (\bigwedge_{A \in U} \phi(1_A)) \& \overline{\beta}(1_{A_0})(U) \leq \bigvee_{U \in A_0^\beta} \bigwedge_{A \in U} \phi(1_A) \leq \phi(1_{A_0}).$$

If $\phi(1_{A_0}) = 0$, we are done. If $\phi(1_{A_0}) > 0$, the set $\{A \mid \phi(1_A) < \phi(1_{A_0})\}$ is directed because ϕ preserves finite joins; hence, there exists an ultrafilter U_0 that extends $\{A_0\}$ and excludes $\{A \mid \phi(1_A) < \phi(1_{A_0})\}$. Thus,

$$r_\phi \circ 1_{A_0} \geq \bigwedge_{A \in U_0} \phi(1_A) = \phi(1_{A_0}).$$

For $r = r_{r \circ (-)}$, given an ultrafilter U_0 on X, it holds that

$$r_{r \circ (-)}(U_0) = \bigwedge_{A \in U_0} \bigvee_{U \in \beta X} \overline{\beta}(1_A)(U) \& r(U) = \bigwedge_{A \in U_0} \bigvee_{U \in A^\beta} r(U) \geq r(U_0).$$

For the other direction, let $a < \bigwedge_{A \in U_0} \bigvee_{U \in A^\beta} r(U)$; then, the set

$$\{A^\beta \cap \{U \in \beta X \mid r(U) > a\} \mid A \in U_0\}$$

is a filter base. By the axiom of choice, there exists an ultrafilter $\mathfrak{U}$ on βX that extends the aforementioned filter base. It is easy to check that $\Sigma(\mathfrak{U}) = U_0$ and there exists some $\mathfrak{a}_0 \in \mathfrak{U}$ such that $\mathfrak{a}_0 \cap \{U \in \beta X \mid r(U) \leq a\} = \varnothing$. By the unitariness of r, we have that

$$r(U_0) \geq \overline{\beta} r(\mathfrak{U}) = \bigvee_{\mathfrak{a} \in \mathfrak{U}} \bigwedge_{U \in \mathfrak{a}} r(U) \geq \bigwedge_{U \in \mathfrak{a}_0} r(U) > a. \quad \square$$

Example 5. *A (β, I)-algebra is a pair (X, c) consisting of a set X and a (β, I)-relation $c\colon X \nrightarrow X$ satisfying*

$$c(\dot{x}, x) = 1 \quad \text{and} \quad \overline{\beta}(c)(\mathfrak{U}, U)\&c(U, x) \le c(\Sigma(\mathfrak{U}), x)$$

for any $\mathfrak{U} \in \beta^2 X, U \in \beta X$ and $x \in X$. These objects were investigated in [8] under the name of I-valued topological spaces.

An element ϕ of $\mathsf{F}^{\vee}_{\otimes} X$ is a map $\phi\colon (I^X, \mathrm{sub}_X) \to (I, d_L)$ subject to

(1) $\phi(0_X) = 0$;

(2) $\phi(\mu \vee \nu) = \phi(\mu) \vee \phi(\nu)$ *for any $\mu, \nu \in I^X$;*

(3) $\phi(a\&\mu) = a\&\phi(\mu)$ *for any $a \in I$ and $\mu \in I^X$.*

The elements of $\mathsf{F}^{\vee}_{\otimes} X$ play the role of many-valued filters.

An $\mathsf{F}^{\vee}_{\otimes}$-monoid is a pair $(X, n\colon X \to \mathsf{F}^{\vee}_{\otimes} X)$ such that

$$n(x) \le (e_{\vee})_X(x) = (-)(x) \quad \text{and} \quad n \le (m_{\vee})_X \cdot \mathsf{F}^{\vee}_{\otimes}(n) \cdot n.$$

Based on [14] (IV.3.2.2 Theorem), the maps

$$conv\colon \mathsf{Set}(X, \mathsf{F}^{\vee}_{\otimes} X) \to \mathsf{I}\text{-}\mathsf{Rel}(\beta X, X), \quad conv(n)(U, x) = r_{n(x)}(U)$$
$$nbhd\colon \mathsf{I}\text{-}\mathsf{Rel}(\beta X, X) \to \mathsf{Set}(X, \mathsf{F}^{\vee}_{\otimes} X), \quad nbhd(c)(x) = c(-, x) \circ (-)$$

are bijections between (β, I)-algebras and $\mathsf{F}^{\vee}_{\otimes}$-monoids. Therefore, with the help of Theorem 2, we can describe I-valued toplogical spaces in terms of their neighborhood systems.

3. When & Is Archimedean

In order to give another characterization of the discrete presheaf monad, we introduce a new type of I-category. Let X be an I-category. The cotensor $a \rightarrowtail_X x$ of $a \in I$ and $x \in X$ is an element of X such that, for any $y \in X$, it holds that

$$X(y, a \rightarrowtail_X x) = a \to X(y, x).$$

Usually, we omit the subscript X of $\rightarrowtail_X$. An I-category X is called cotensored if, for any $a \in I$ and $x \in X$, the cotensor $a \rightarrowtail x$ exists.

An $\mathsf{I}_{\wedge}$-category is a cotensored I-category whose underlying order is closed under finite meets. An $\mathsf{I}_{\wedge}$-functor is a map between $\mathsf{I}_{\wedge}$-categories that preserves cotensors and all finite meets. $\mathsf{I}_{\wedge}$-categories and $\mathsf{I}_{\wedge}$ functors assemble into a category

$$\mathsf{I}\text{-}\mathsf{Cat}^{\wedge}_{\rightarrowtail}.$$

The I-category (I, d_L) is contensored by $a \rightarrowtail x = a \to x$ for any $a, x \in I$. Given a set X, the product $\prod_{x \in X}(I, d_L)_x$ in $\mathsf{I}\text{-}\mathsf{Cat}^{\wedge}_{\rightarrowtail}$ is (I^X, sub_X), which is cotensored by $a \rightarrowtail \mu = a \to \mu$ for any $a \in I$ and $\mu \in I^X$. Thus, the functor $\mathsf{I}\text{-}\mathsf{Cat}^{\wedge}_{\rightarrowtail}(-, (I, d_L))\colon (\mathsf{I}\text{-}\mathsf{Cat}^{\wedge}_{\rightarrowtail})^{\mathrm{op}} \to \mathsf{Set}$ is right adjoint. The induced monad is denoted as $(\mathsf{F}^{\wedge}_{\rightarrowtail}, m_{\wedge}, e_{\wedge})$.

The following lemma gives a useful characterization of $\mathsf{I}_{\wedge}$-functors.

Lemma 4 ([23] (Proposition 2.11))**.** *Let $\phi\colon X \to (I, d_L)$ be an $\mathsf{I}_{\wedge}$-functor. Then,*

$$\phi = \bigvee_{\phi(t)=1} X(t, -).$$

Furthermore, the implication $\to$ is continuous at the second variable and

$$X(z, x \wedge y) = X(z, x) \wedge X(z, y)$$

holds for all $x, y, z \in X$. Given a filtered subset $F \subset X$, the map $\psi = \bigvee_{t \in F} X(t, -)$ is an $\mathsf{I}_{\wedge}$ functor.

Proof. That ϕ preserves all finite meets implies that $\{t \mid \phi(t) = 1\}$ is not empty. For every x, y such that $\phi(x) = 1$, note that $X(x, y) \rightarrowtail y \geq x$; then, it holds that

$$X(x,y) \to \phi(y) = \phi(X(x,y) \rightarrowtail y) \geq \phi(x) = 1.$$

For the other direction, since $\phi(\phi(y) \rightarrowtail y) = 1$, it holds that

$$\phi(y) \to X(\phi(y) \rightarrowtail y, y) = X(\phi(y) \rightarrowtail y, \phi(y) \rightarrowtail y) = 1.$$

We can conclude the second statement from direct calculation:

$$\begin{aligned}
\psi(x \wedge y) &= \bigvee_{t \in F} X(t, x \wedge y) \\
&= \bigvee_{t \in F} X(t, x) \wedge X(t, y) \\
&= \psi(x) \wedge \psi(y); && (F \text{ is filtered}) \\
\psi(a \rightarrowtail x) &= \bigvee_{t \in F} X(t, a \rightarrowtail x) \\
&= \bigvee_{t \in F} a \to X(t, x) \\
&= a \to \psi(x), && (a \to (-) \text{ is continuous}) \quad \square
\end{aligned}$$

Therefore, when the continuous t-norm is Archimedean, the monad $(\mathsf{F}^{\wedge}_{\rightarrowtail}, m_{\wedge}, e_{\wedge})$ is exactly the conical I-semifilter monad, and isomorphic to the saturated prefilter monad [15].

3.1. The Łucasiewicz t-Norm

The Łucasiewicz t-norm has nice properties such as the implication $\to$ of it being continuous,

$$(a \to 0) \to 0 = a \quad \text{and} \quad a \& (b \to c) = (a \to b) \to c$$

for any $a, b, c \in I$.

Theorem 3. *We have the following isomorphism:*

$$(\Pi, m', e') \cong (\mathsf{F}^{\vee}_{\otimes}, m_{\vee}, e_{\vee}) \cong (\mathsf{F}^{\wedge}_{\rightarrowtail}, m_{\vee}, e_{\vee}).$$

Proof. Given a $\phi \in \mathsf{F}^{\vee}_{\otimes} X$, $\phi((-) \to 0) \to 0$ is an $\mathsf{I}_{\wedge}$-functor because, for each $\mu, \nu \in I^X$ and $a \in I$,

$$\begin{aligned}
\phi((a \to \mu) \to 0) \to 0 &= \phi(a \& (\mu \to 0)) \to 0 \\
&= (a \& \phi(\mu \to 0)) \to 0 \\
&= a \to (\phi(\mu \to 0) \to 0); \\
\phi((\mu \wedge \nu) \to 0) \to 0 &= \phi((\mu \to 0) \vee (\nu \to 0)) \to 0 \\
&= (\phi(\mu \to 0) \to 0) \wedge (\phi(\nu \to 0) \to 0).
\end{aligned}$$

It is easy to check that $\{\delta_X\}_X$ give rise to an isomorphism of monads in which

$$\delta_X \colon \mathsf{F}^{\vee}_{\otimes} X \to \mathsf{F}^{\wedge}_{\rightarrowtail} X, \quad \phi \mapsto \phi((-) \to 0) \to 0. \quad \square$$

Example 6. *Combining Example 5 and the above Theorem, we describe* I*-valued topological spaces in terms of neighborhood systems via not only* $(\mathsf{F}^{\vee}_{\otimes}, m_{\vee}, e_{\vee})$ *but also* $(\mathsf{F}^{\wedge}_{\rightarrowtail}, m_{\wedge}, e_{\wedge})$*. In addition, it is shown in [24] that the algebras for* $(\mathsf{F}^{\wedge}_{\rightarrowtail}, m_{\wedge}, e_{\wedge})$ *are exactly continuous* I*-lattices. Thus, by [14] (IV.4.6.5 Corollary), one can obtain that the injective* I*-valued topological spaces are exactly the continuous* I*-lattices.*

3.2. The Product t-Norm

For the product t-norm, $a \to (-)$ is continuous for any a and $(-) \to b$ for any $b \neq 0$.

Recall that an element $\mu \in I^X$ is called bounded if $\bigwedge \mu > 0$. The set BX of bounded elements endowed with sub_X is an $\mathsf{I}_\wedge$-category but not an $\mathsf{I}_\vee$-category since $(BX, \leq)$ does not admit a bottom element. (BX, sub_X) is tensored and its underlying order admits all nonempty finite joints.

Denote by $\mathsf{B}^\vee_\otimes X$ the set of maps $\phi\colon (BX, \mathrm{sub}_X) \to (I, d_L)$ for which ϕ preserves tensors and all nonempty finite joints. For any $f\colon X \to Y$ and $\phi \in \mathsf{B}^\vee_\otimes X$, defining $\mathsf{B}^\vee_\otimes(f)(\phi) = \phi(- \cdot f)$ establishes $\mathsf{B}^\vee_\otimes$ as a functor. By Lemma 1, $\delta_X\colon \mathsf{F}^\vee_\otimes X \to \mathsf{B}^\vee_\otimes X$ forms a natural isomorphism, where $\delta_X(\phi) = \phi\mid_{BX}$. Since $\mathsf{F}^\vee_\otimes$ and $\mathsf{B}^\vee_\otimes$ are isomorphic, we can obtain a monad $(\mathsf{B}^\vee_\otimes, m_\vee, e_\vee)$, where

$$(e_\vee)_X(x)\colon BX \to I, \quad \mu \mapsto \mu(x);$$
$$(m_\vee)_X(\Phi)\colon BX \to I, \quad \mu \mapsto \Phi(\widetilde{\mu})$$

for any $x \in X$ and $\Phi \in (\mathsf{B}^\vee_\otimes)^2 X$, where $B\mathsf{B}^\vee_\otimes X \ni \widetilde{\mu}\colon \phi \mapsto \phi(\mu)$.

Turn to the monad $(\mathsf{F}^\wedge_\rightarrowtail, m_\wedge, e_\wedge)$. For each set X, we denote the set of $\mathsf{I}_\wedge$-functors from (BX, sub_X) to (I, d_L) by $\mathsf{B}^\wedge_\rightarrowtail X$ and let

$$\mathsf{B}^\wedge_\rightarrowtail(f)\colon \mathsf{B}^\wedge_\rightarrowtail X \to \mathsf{B}^\wedge_\rightarrowtail Y, \quad \phi \mapsto \phi(- \cdot f);$$
$$(e_\wedge)_X(x)\colon BX \to I, \quad \mu \mapsto \mu(x);$$
$$(m_\wedge)_X(\Phi)\colon BX \to I, \quad \mu \mapsto \Phi(\widetilde{\mu}).$$

for any $f\colon X \to Y, x \in X$ and $\Phi \in (\mathsf{B}^\wedge_\rightarrowtail)^2 X$, where $B\mathsf{B}^\wedge_\rightarrowtail X \ni \widetilde{\mu}\colon \phi \mapsto \phi(\mu)$. It is straightforward to verify that $\mathsf{B}^\wedge_\rightarrowtail$ is functorial and $\{(m_\wedge)_X\}_X, \{(e_\wedge)_X\}_X$ give rise to natural transformations.

Proposition 6. *The triple* $(\mathsf{B}^\wedge_\rightarrowtail, m_\wedge, e_\wedge)$ *is a monad.*

Proof. To keep the notation simple, we omit the subscripts $\wedge$ of $(m_\wedge)_X$ and $(e_\wedge)_X$. They give rise to a monad since

$$(m_X \cdot e_{\mathsf{B}^\wedge_\rightarrowtail X})(\phi) = (\widetilde{-})(\phi) = \phi;$$
$$(m_X \cdot \mathsf{B}^\wedge_\rightarrowtail e_X)(\phi) = \phi((\widetilde{-}) \cdot e_X) = \phi;$$
$$(m_X \cdot \mathsf{B}^\wedge_\rightarrowtail(m_X))(\mathbf{\Phi})(\mu) = \mathbf{\Phi}((\widetilde{\mu}) \cdot m_X) = \mathbf{\Phi}(\Phi \mapsto \Phi(\widetilde{\mu}));$$
$$(m_X \cdot m_{\mathsf{B}^\wedge_\rightarrowtail X})(\mathbf{\Phi})(\mu) = \mathbf{\Phi}(\widetilde{\widetilde{\mu}}) = \mathbf{\Phi}(\Phi \mapsto \Phi(\widetilde{\mu})). \quad \square$$

This monad is isomorphic to the bounded I-semifilter monad [15], and hence is isomorphic to the functional ideal monad [16,17]; refer to [15] (Section 7) for more information.

An element of $\mathsf{B}^\wedge_\rightarrowtail X$ is called prime if it preserves all finite joints. $1_{BX} \in \mathsf{B}^\wedge_\rightarrowtail X$ is prime. A prime $\phi \in \mathsf{B}^\wedge_\rightarrowtail X$ is called proper if $\phi \neq 1_{BX}$. Since $\mathsf{B}^\wedge_\rightarrowtail(f)(\phi)$ is prime for every prime $\phi \in \mathsf{B}^\wedge_\rightarrowtail X$ and $f\colon X \to Y$, we obtain a subfunctor $\mathsf{P}^\wedge_\rightarrowtail$.

Proposition 7. *Prime* $\mathsf{I}_\wedge$*-functors give rise to a submonad of* $(\mathsf{B}^\wedge_\rightarrowtail, m_\wedge, e_\wedge)$.

Proof. $(e_\wedge)_X(x) = (-)(x)$ is prime obviously. Given a $\Phi \in (\mathsf{P}^\wedge_\rightarrowtail)^2 X$, since

$$(m \cdot (i * i))_X(\Phi)(\mu \vee \nu) = \Phi((\widetilde{\mu \vee \nu}) \cdot i_X) = \Phi((\widetilde{\mu} \cdot i_X) \vee (\widetilde{\nu} \cdot i_X)) = \Phi(\widetilde{\mu} \cdot i_X) \vee \Phi(\widetilde{\nu} \cdot i_X),$$

the proof is finished. $\square$

The monad$(\mathsf{P}^\wedge_\rightarrowtail, m_\wedge, e_\wedge)$ is exactly the monad (P_π, m^π, e^π) in [25], and hence is isomorphic to the prime functional ideal monad.

Combining [17] (Theorem 4.1) and [25] (Theorem 3.5), there is a bijection between proper elements of $\mathsf{P}^{\wedge}_{\rightarrowtail}X$ and $\beta X \times (0,1]$. Given a pair $(U, a) \in \beta X \times (0,1]$, the corresponding proper prime $\mathsf{I}_{\wedge}$-functor is given by

$$\phi = \bigvee_{\substack{A \in U_0 \\ a > a_0}} \mathrm{sub}_X(a \& 1_A, -).$$

We further compute:

$$\begin{aligned}
\phi(\mu) &= \bigvee_{\substack{A \in U_0 \\ a > a_0}} \mathrm{sub}_X(a \& 1_A, \mu) \\
&= \bigvee_{\substack{A \in U_0 \\ a > 0}} a \to \Big(\bigwedge_{x \in A} \mu(x)\Big) \\
&= \bigvee_{A \in U_0} \Big(\bigwedge_{a > a_0} a\Big) \to \Big(\bigwedge_{x \in A} \mu(x)\Big) && (\mu \text{ is bounded}) \\
&= a_0 \to \Big(\bigvee_{A \in U} \bigwedge_{x \in A} \mu(x)\Big) \\
&= \bigwedge_{U \in \beta X} r(U) \to \overline{\beta}\mu(U),
\end{aligned}$$

in which the (β, I)-relation $r \colon X \nrightarrow \{*\}$ is given by

$$r(U) = \begin{cases} a_0, & U = U_0, \\ 0, & U \neq U_0. \end{cases}$$

A (β, I)-relation $r \colon X \nrightarrow \{*\}$ is called prime if it is of the above type, and we denote it by (U_0, a_0). Then, there is a bijection between prime (β, I)-relations $r \colon X \nrightarrow \{*\}$ and proper elements of $\mathsf{P}^{\wedge}_{\rightarrowtail}X$.

It is easy to show that prime (β, I)-relations are unitary. An element of $\mathsf{B}^{\vee}_{\otimes}X$ is called prime if it preserves all nonempty finite meets. $0_{BX} \in \mathsf{B}^{\vee}_{\otimes}X$ is prime. A prime $\phi \in \mathsf{B}^{\vee}_{\otimes}X$ is called proper if $\phi \neq 0_{BX}$. The proof of the following proposition is similar to that of Proposition 7.

Proposition 8. *Prime $\mathsf{I}_{\vee}$-functors give rise to a submonad of $(\mathsf{B}^{\vee}_{\otimes}, m_{\vee}, e_{\vee})$.*

Lemma 5. *There is a bijection between prime (β, I)-relations $r \colon X \nrightarrow \{*\}$ and proper elements ϕ of $\mathsf{P}^{\vee}_{\otimes}X$.*

Proof. When transitioning from $\mathsf{F}^{\vee}_{\otimes}X$ to $\mathsf{B}^{\vee}_{\otimes}X$, the bijections in Theorem 2 are modified as follows:

$$r \mapsto r \circ (-) \colon BX \to I \quad \text{and} \quad \mathsf{B}^{\vee}_{\otimes}X \ni \phi \mapsto r_\phi = \bigwedge_{A \in (-)} \bigwedge_{a > 0} \phi(1_A \vee a).$$

It suffices to prove that $r \circ (-)$ is prime and r_ϕ is prime whenever r and ϕ are prime.

That $r \circ (-)$ is prime follows from Proposition 3. To see that r_ϕ is prime, suppose that there are two ultrafilters U_1, U_2 such that $r(U_1) > 0$ and $r(U_2) > 0$. Since $U_1 \neq U_2$, there is a $B \in U_1$ and $X \setminus B \in U_2$. Then, for any $a > 0$, we have that

$$
\begin{aligned}
a\&\phi(1_X) &= \phi(0_X \vee a) = \phi((1_B \vee a) \wedge (1_{X\setminus B} \vee a)) \\
&= \phi(1_B \vee a) \wedge \phi(1_{X\setminus B} \vee a) \\
&\geq r(U_1) \wedge r(U_2) \\
&> 0.
\end{aligned}
$$

By the arbitrariness of a, we reach a contradiction. □

With Lemma 5 at hand, we can prove the following results.

Theorem 4. *There is an isomorphism*

$$(\mathsf{P}^{\wedge}_{\rightarrowtail}, m_{\wedge}, e_{\wedge}) \cong (\mathsf{P}^{\vee}_{\otimes}, m_{\vee}, e_{\vee}).$$

Proof. For each prime (β, I)-relation $r\colon X \nrightarrow \{*\}$, we denote by $\iota(r)$ and $c(r)$ the correspondent ultrafilter on X and number in $(0, 1]$, respectively. The corresponding $\mathsf{I}_{\wedge}$-functor and $\mathsf{I}_{\vee}$-functor are given by

$$c(r) \to \overline{\beta}(-)(\tau(r)) \quad \text{and} \quad c(r)\&\overline{\beta}(-)(\tau(r)),$$

respectively. For convenience, we use the notation r for the corresponding $\mathsf{I}_{\wedge}$-functor and $\mathsf{I}_{\vee}$-functor as well.

Let $R\colon (\beta, \mathsf{I})\text{-}\mathsf{PRel}(X, \{*\}) \nrightarrow \{*\}$ be a prime (β, I)-relation, where $(\beta, \mathsf{I})\text{-}\mathsf{PRel}(X, \{*\})$ denotes the set of prime (β, I)-relations from X to $\{*\}$. By [17] (Propositon 5.3 5.4), we have that

$$c(m_{\wedge}(R)) = c(R)\& \bigvee_{\mathfrak{a}\in\iota(R)} \bigwedge_{r\in\mathfrak{a}} c(r) = c(R)\& \bigwedge_{\mathfrak{a}\in\iota(R)} \bigvee_{r\in\mathfrak{a}} c(r),$$

and

$$\iota(m_{\wedge}(R)) = \bigcup_{\mathfrak{a}\in\iota(R)} \bigcap_{r\in\mathfrak{a}} \iota(r).$$

Let $a_0 = c(m_{\vee}(R))$ and $U_0 = \iota(m_{\vee}(R))$. Then, for any $\mu \in BX$, it holds that

$$a_0\&\overline{\beta}(\mu)(U_0) = m_{\vee}(R)(\mu) = R \circ (\widetilde{\mu}) = c(R)\&\overline{\beta}(\widetilde{\mu})(\iota(R)),$$

where

$$\widetilde{\mu}\colon \{*\} \nrightarrow (\beta, \mathsf{I})\text{-}\mathsf{PRel}(X, \{*\}), \quad (*, r) \mapsto c(r)\&\overline{\beta}(\mu)(\iota(r)).$$

It suffices to demonstrate that $U_0 = \iota(m_{\wedge}(R))$ and $a_0 = c(m_{\wedge}(R))$.

Suppose, on the contrary, that $U_0 \neq \iota(m_{\wedge}(R))$. It is routine to check that $\mathfrak{U} = \{\{\iota(r) \mid r \in \mathfrak{a}\} \mid \mathfrak{a} \in \iota(R)\}$ is an ultrafilter on βX and $\Sigma\mathfrak{U} = \iota(m_{\wedge}(R))$. Thus, there exists an $A \in U_0$ and an $\mathfrak{a} \in \iota(R)$ such that $A^{\beta} \cap \{\iota(r) \mid r \in \mathfrak{a}\} = \varnothing$. Thus, for any $a > 0$, it holds that

$$\overline{\beta}(\widetilde{1_A \vee a})(\iota(R)) \leq \bigvee_{r\in\mathfrak{a}_0} c(r)\&\overline{\beta}(1_A \vee a)(\iota(r)) = a\& \bigvee_{r\in\mathfrak{a}_0} c(r) \leq a.$$

By the arbitrariness of a, we conclude that $a_0\&\overline{\beta}(\mu)(U_0) = 0$, which leads to a contradiction.

For $a_0 = c(m_\wedge(R))$, we can compute directly as follows:

$$\begin{aligned} a_0 &= m_\vee(R)(1_X) \\ &= R \circ (\widetilde{1_X}) \\ &= c(R)\&\overline{\beta}(\widetilde{1_X})(\iota R) \\ &= c(R)\& \bigvee_{\mathfrak{a}\in\iota(R)} \bigwedge_{r\in\mathfrak{a}} c(r)\&\overline{\beta}(1_X)(\iota(r)) = c(R)\&\overline{\beta}(\widetilde{1_X})(\iota R) \\ &= c(R)\& \bigvee_{\mathfrak{a}\in\iota(R)} \bigwedge_{r\in\mathfrak{a}} c(r) \quad \square \end{aligned}$$

Corollary 1. *The prime functional ideal monad is isomorphic to a submonad of the discrete presheaf monad associated to $\overline{\beta}$.*

Proof. Follow from Proposition 8 and Theorem 4 directly. □

4. Conclusions

In this paper, we extended the ultrafilter monad to the category I-Rel, where I is the unit interval endowed with a continuous t-norm. This lax extension is associated, and hence induces a new monad named the discrete presheaf monad (Π, n, d).

For a general continuous t-norm, a characterization of the discrete presheaf monad is presented: $(\Pi, n, d) \cong (\mathsf{F}^\vee_\otimes, m_\vee, e_\vee)$.

When $(I, \&, 1)$ is the Łucasiewicz t-norm, with the help of the natural isomorphism

$$\delta_X \colon \mathsf{F}^\vee_\otimes X \to \mathsf{F}^\wedge_\rightarrowtail X, \quad \phi \mapsto \phi((-) \to 0) \to 0$$

, we obtain another characterization: $(\Pi, n, d) \cong (\mathsf{F}^\wedge_\rightarrowtail, m_\wedge, e_\wedge)$. The Eilenberg–Moore algebras of the latter have a good characterization.

When $(I, \&, 1)$ is the product t-norm, the (β, I)-algebras are precisely Lowen's approach spaces. It is shown that the approach spaces are lax algebras for the functional ideal monad and the prime functional ideal monad. A natural question arises: what is the relationship between them and the discrete presheaf monad? Here, we only proved that the prime functional ideal monad is isomorphic to a submonad of the discrete presheaf monad.

Problem 1. *Is the discrete presheaf monad isomorphic to the functional ideal monad?*

As mentioned earlier, in this paper, we only provide a characterization of the Eilenberg–Moore algebras for the discrete presheaf monad when the t-norm used is the Łukasiewicz t-norm. However, for other continuous t-norms, further research is still needed to explore the Eilenberg–Moore algebras for the discrete presheaf monad.

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