Lecture 4. Gradient Descent Method

Advanced Optimization (Fall 2022)

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Outline

- Gradient Descent
- Convex and Lipschitz
  - Polyak Step Size
  - Convergence without Optimal Value
  - Optimal Time-Varying Step Sizes
- Strongly Convex and Lipschitz

Part 1. Gradient Descent

- Convex Optimization Problem
- Gradient Descent
- Performance Measure
- The First Gradient Descent Lemma

Convex Optimization Problem

- We adopt a minimization language

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \end{aligned}$$

  - optimization variable $\mathbf{x} \in \mathbb{R}^d$
  - objective function $f : \mathbb{R}^d \mapsto \mathbb{R}$: convex and continuously differentiable
  - feasible domain $\mathcal{X} \subseteq \mathbb{R}^d$: convex

Gradient Descent

- GD Template:

$$\mathbf{x}_{t+1} = \Pi_{\mathcal{X}}\left[\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)\right]$$

  - $\mathbf{x}_1$ can be an arbitrary point inside the domain.
  - $\eta_t > 0$ is the potentially time-varying ***step size*** (or called ***learning rate***).
  - Projection $\Pi_{\mathcal{X}}[\mathbf{y}] = \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x} - \mathbf{y}\|$ ensures the feasibility.

Goal

To output a sequence $\{\bar{\mathbf{x}}_t\}_{t=1}^T$ such that $\bar{\mathbf{x}}_t$ approximates $\mathbf{x}^\star$ when t goes larger.

- Function-value level: $f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^\star) \leq \varepsilon(T)$
- Optimizer-value level: $\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\| \leq \varepsilon(T)$

where $\{\bar{\mathbf{x}}_t\}_{t=1}^T$ can be ***statistics*** of the original sequence $\{\mathbf{x}_t\}_{t=1}^T$,

and $\varepsilon(T)$ is the ***approximation error*** and is a function of iterations T.

Goal

- In general, there are two performance measures (essentially same).

Convergence: $f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^\star) \leq \varepsilon(T),$

- **Qualitatively:** $\varepsilon(T) \to 0$ when $T \to \infty$
- **Quantitatively:** $\mathcal{O}(\frac{1}{\sqrt{T}})$ / $\mathcal{O}(\frac{1}{T})$ / $\mathcal{O}(\frac{1}{T^2})$ / $\mathcal{O}(\frac{1}{e^T})$ / $\ldots$

Complexity:

- **Definition:** number of iterations required to achieve $f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^\star) \leq \varepsilon$.
- **Quantitatively:** $\mathcal{O}(\frac{1}{\varepsilon^2})$ / $\mathcal{O}(\frac{1}{\varepsilon})$ / $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ / $\mathcal{O}(\ln(\frac{1}{\varepsilon}))$ / $\ldots$

corresponds to $\mathcal{O}(\frac{1}{\sqrt{T}})$ / $\mathcal{O}(\frac{1}{T})$ / $\mathcal{O}(\frac{1}{T^2})$ / $\mathcal{O}(\frac{1}{e^T})$ / $\ldots$

GD Convergence Analysis

The First Gradient Descent Lemma

Lemma 1. *Suppose that f is proper, closed and convex; the feasible domain $\mathcal{X}$ is nonempty, closed and convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method, $\mathcal{X}^\star$ be the optimal set of the optimization problem and $f^\star$ be the optimal value. Then for any $\mathbf{x}^\star \in \mathcal{X}^\star$ and $t \geq 0$,*

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2.$$

Proof:

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 &= \|\Pi_{\mathcal{X}}[\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)] - \mathbf{x}^\star\|^2 \quad \text{(GD)} \\
&\leq \|\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t) - \mathbf{x}^\star\|^2 \quad \text{(Pythagoras Theorem)} \\
&= \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star \rangle + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2 \\
&\leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2
\end{aligned}$$

(convexity: $f(\mathbf{x}_t) - f^\star = f(\mathbf{x}_t) - f(\mathbf{x}^\star) \leq \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star \rangle$) $\square$

Part 2. Polyak Step Size

- Polyak Step Size
- Convergence
- Convergence Rate

Polyak Step Size

- GD method satisfies the following inequality:

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 \underbrace{- 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2}$$

$$h(\eta) \triangleq -2\eta(f(\mathbf{x}_t) - f^\star) + \eta^2\|\nabla f(\mathbf{x}_t)\|^2$$

A natural idea:

minimizing the right-hand side of the inequality

$$\Longrightarrow \quad \eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}$$

assume known $f^\star$ for a moment

Polyak Step Size

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 \underbrace{- 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2}$$

$$\Longrightarrow \quad \eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2} \qquad h(\eta) \triangleq -2\eta(f(\mathbf{x}_t) - f^\star) + \eta^2\|\nabla f(\mathbf{x}_t)\|^2$$

Cornercase: when $\nabla f(\mathbf{x}_t) = \mathbf{0}$

$\Longrightarrow$ actually a good news owing to convexity, $\nabla f(\mathbf{x}_t) = \mathbf{0}$ implies *optimality*

Polyak step size: $\eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$

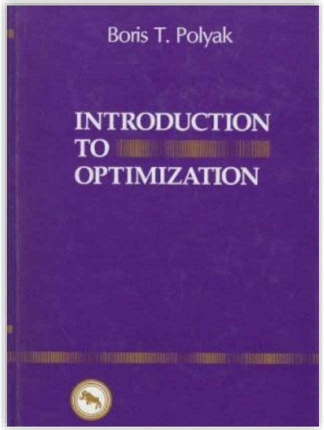
Without loss of generality, we assume $\nabla f(\mathbf{x}_t) \neq \mathbf{0}$ from now on.

Polyak Step Size

Polyak step size: $$\eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$$

assume known $f^\star$ for a moment.

Boris T. Polyak
1935-now



Introduction to optimization

Boris T. Polyak

Optimization Software, Inc., 1987

Convergence

- With Polyak step size, we obtain the convergence results:

Theorem 1. *Under the same assumptions with Lemma 1, assume the gradient of f is bounded by G, i.e., $\|\nabla f(\cdot)\| \leq G$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by the gradient descent method with Polyak step size and $f^\star$ be the optimal value. Then,*

(i) $\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2$.

(ii) $f(\mathbf{x}_t) \to f^\star$ *as* $t \to \infty$.

Note: recall that ***bounded gradients*** condition implies ***Lipschitz continuity***.

Convergence

Proof: $\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t(f(\mathbf{x}_t) - f^\star) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2$

(the first GD lemma)

- **Case 1:** $\nabla f(\mathbf{x}_t) = \mathbf{0}$. By convexity, $f(\mathbf{x}_t) = f^\star \Rightarrow \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 = \|\mathbf{x}_t - \mathbf{x}^\star\|^2$.
- **Case 2:** $\nabla f(\mathbf{x}_t) \neq \mathbf{0}$. Polyak's step size $\eta_t = \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}$

$$\Longrightarrow \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{\|\nabla f(\mathbf{x}_t)\|^2} \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2$$

(*i*) is proved.

Convergence

Proof: we can simply focus on the case of $\nabla f(\mathbf{x}_t) \neq \mathbf{0}$

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{\|\nabla f(\mathbf{x}_t)\|^2} \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{(f(\mathbf{x}_t) - f^\star)^2}{G^2}$$

$(\|\nabla f(\cdot)\| \leq G)$

$$\Longrightarrow \quad \frac{1}{G^2}\sum_{t=1}^{T}(f(\mathbf{x}_t) - f^\star)^2 \leq \|\mathbf{x}_1 - \mathbf{x}^\star\|^2 - \|\mathbf{x}_{T+1} - \mathbf{x}^\star\|^2$$

$$\Longrightarrow \quad \sum_{t=1}^{T}(f(\mathbf{x}_t) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2$$

Infinite summation is bounded by constants → **convergent** series.

***(ii)* is proved.** □

Convergence Rate

- We can also derive the convergence rate.

Theorem 2. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by the gradient descent method with Polyak step size and* $f^\star$ *be the optimal value. Define* $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$, *we have*

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right).$$

Proof:

$$\left.\begin{aligned} f(\bar{\mathbf{x}}_T) = \min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t) \leq f(\mathbf{x}_t) \\ \sum_{t=1}^T (f(\mathbf{x}_t) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 \end{aligned}\right\} T(f(\bar{\mathbf{x}}_T) - f^\star)^2 \leq G^2\|\mathbf{x}_1 - \mathbf{x}^\star\|^2$$

$\square$

Part 3. Convergence without Optimal Value

- The Second Gradient Descent Lemma

- Convergent Step Size

- Convergence without Optimal Value

Step Size without Optimal Value

- Note that Polyak step size requires the optimal value $f^\star$

Polyak step size: $\eta_t = \begin{cases} \frac{f(\mathbf{x}_t) - f^\star}{\|\nabla f(\mathbf{x}_t)\|^2}, & \nabla f(\mathbf{x}_t) \neq \mathbf{0} \\ 1, & \nabla f(\mathbf{x}_t) = \mathbf{0} \end{cases}$

assume known $f^\star$ for a moment

From now on, we try to design step sizes ***without*** the optimal value $f^\star$.

The Second Gradient Descent Lemma

- A second version of gradient descent lemma

Lemma 2. *Under the same assumptions as Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD. Then we have*

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \leq \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \frac{1}{2}\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2.$$

Proof: The statement can be derived directly from the gradient descent lemma:

$$\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 \leq \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - 2\eta_t (f(\mathbf{x}_t) - f^\star) + \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2$$

$$\Longrightarrow \quad \eta_t (f(\mathbf{x}_t) - f^\star) \leq \frac{1}{2}\left(\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{1}{2}\eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2 \qquad \square$$

Convergence Result

- GD lemma implies the following convergence result.

Lemma 3. *Under the same assumptions as Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD. Define* $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$, *we have*

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}.$$

Convergence Result

Proof:

- **Case 1:** $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$.

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \geq \left(\sum_{t=1}^{T} \eta_t\right) (f(\bar{\mathbf{x}}_T) - f^\star). \qquad (f(\bar{\mathbf{x}}_T) = \min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t) \leq f(\mathbf{x}_t))$$

Combining the above inequality with Lemma 2 (as restated below),

$$\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) \leq \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \frac{1}{2}\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2,$$

we have completed the proof of the desired result:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}.$$

Convergence Result

Proof:

- **Case 2:** $\bar{\mathbf{x}}_T = \sum_{t=1}^{T} \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^{T} \eta_t}$.

$$\begin{aligned}\sum_{t=1}^{T} \eta_t (f(\mathbf{x}_t) - f^\star) &= \left(\sum_{t=1}^{T} \eta_t\right) \left(\sum_{t=1}^{T} \underbrace{\frac{\eta_t}{\sum_{t=1}^{T} \eta_t}}_{\text{(distribution)}} f(\mathbf{x}_t) - f^\star\right) \\ &\geq \left(\sum_{t=1}^{T} \eta_t\right) \left(f\left(\sum_{t=1}^{T} \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^{T} \eta_t}\right) - f^\star\right)\end{aligned}$$

(Jensen's inequality)

Thus, we achieve the desired result:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^{T} \eta_t} + \frac{\sum_{t=1}^{T} \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^{T} \eta_t}. \qquad \square$$

Convergent Step Size

Theorem 3. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by the gradient descent method. If*

$$\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0 \text{ as } T \to \infty,$$

then $f(\bar{\mathbf{x}}_T) \to f^\star$ *as* $T \to \infty$.

Indeed, this structure appears in the second gradient descent lemma.

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t} \quad \|\nabla f(\mathbf{x}_t)\|^2 \leq G^2$$

Thus, the condition $\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0$ naturally implies $\sum_{t=1}^T \eta_t \to \infty$. □

Convergent Step Size

Theorem 3. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by the gradient descent method. If*

$$\frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \to 0 \text{ as } T \to \infty,$$

then $f(\bar{\mathbf{x}}_T) \to f^\star$ *as* $T \to \infty$.

Examples:

- *time-invariant* step size: $\eta_t = \eta = \frac{1}{\sqrt{T}} \Rightarrow \frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} = \frac{1}{\sqrt{T}} \to 0.$
- *time-varying* step sizes: $\eta_t = \frac{1}{\sqrt{t}} \Rightarrow \frac{\sum_{t=1}^T \eta_t^2}{\sum_{t=1}^T \eta_t} \approx \frac{\log T}{\sqrt{T}} \to 0.$

Convergence without Optimal Value

Theorem 4. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD with step size*

$$\eta_t = \frac{1}{\|\nabla f(\mathbf{x}_t)\|\sqrt{t}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$.

Convergence without Optimal Value

Proof:

$$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t} \quad \text{(the second GD lemma)}$$

$$\le \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t \|\nabla f(\mathbf{x}_t)\|} + \frac{G\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t \|\nabla f(\mathbf{x}_t)\|} \quad (\|\nabla f(\cdot)\| \le G)$$

$$\le \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \frac{1}{\sqrt{t}}} + \frac{G\sum_{t=1}^T \frac{1}{t}}{2\sum_{t=1}^T \frac{1}{\sqrt{t}}} \quad \begin{matrix}(\sum_{t=1}^T \frac{1}{t} \le \log T + 1) \\ (\sqrt{T} \le \sum_{t=1}^T \frac{1}{\sqrt{t}} \le 2\sqrt{T})\end{matrix}$$

Thus,

$$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right). \qquad \square$$

Part 4. Optimal in Convex and Lipschitz Case

- Optimal Result with Known T

- Optimal Result with Unknown T

Towards Optimal Resolutions

Theorem 4. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by GD with step size*

$$\eta_t = \frac{1}{\|\nabla f(\mathbf{x}_t)\|\sqrt{t}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\left(\|\mathbf{x}_1 - \mathbf{x}^\star\|^2 + \log T + 1\right)}{2\sqrt{T}} = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$.

Theorem 2. *Under the same assumptions with Theorem 1. Let* $\{\mathbf{x}_t\}_{t=1}^T$ *be the sequence generated by the gradient descent method with Polyak step size and* $f^\star$ *be the optimal value. Define* $\bar{\mathbf{x}}_T = \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$, *we have*

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{G\|\mathbf{x}_1 - \mathbf{x}^\star\|}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right).$$

with Polyak's step size (known $f^\star$***)***

Remark: The last theorem gives an $\mathcal{O}(\log T/\sqrt{T})$ convergence rate. However, this rate is ***worse*** than the $\mathcal{O}(1/\sqrt{T})$ with Polyak step size.

We show that this can be improved with an additional assumption of ***bounded domain***.

Optimal Result with Known T

Theorem 5. *Under the same assumptions with Theorem 1, assume the feasible domain $\mathcal{X}$ is bounded and convex with a diameter $D > 0$, that is, $\|\mathbf{x} - \mathbf{y}\|_2 \leq D$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{D}{G\sqrt{T}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$.

Optimal Result with Known T

step size $\eta_t = \frac{D}{G\sqrt{T}}$ $\Longrightarrow$ $$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$$

$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$

Proof: Plugging $\eta_t = \frac{D}{G\sqrt{T}}$ into

$$f(\bar{\mathbf{x}}_T) - f^\star \le \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}$$

$(\|\mathbf{x}_1 - \mathbf{x}^\star\| \le D)$

$(\|\nabla f(\cdot)\| \le G)$

Notice that $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t} = \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$. □

Optimal Result with Known T

step size $\eta_t = \frac{D}{G\sqrt{T}}$ $\Longrightarrow$ $f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)$

$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ or $\bar{\mathbf{x}}_T \triangleq \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t$

- $\frac{DG}{\sqrt{T}}$ convergence rate is equivalent to $T = \frac{D^2G^2}{\varepsilon^2}$ complexity result to achieve $f(\bar{\mathbf{x}}_T) - f^\star \leq \varepsilon$.
- $\frac{DG}{\sqrt{T}}$ is already minimax optimal for convex and Lispchitz functions.
- This result needs to know the total round number T in advance.

not desirable in practice

Optimal Result with Unknown T

Theorem 6. *Under the same assumptions with Theorem 1, assume the feasible domain $\mathcal{X}$ is bounded and convex with a diameter $D > 0$, that is, $\|\mathbf{x} - \mathbf{y}\|_2 \leq D$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{D}{G\sqrt{t}}.$$

Then

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=\lceil T/2 \rceil}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=\lceil T/2 \rceil}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=\lceil T/2 \rceil}^T \eta_t}$.

Intuition: bounded domain requires $\|\mathbf{x}_t - \mathbf{x}^\star\|$ (not just $\|\mathbf{x}_1 - \mathbf{x}^\star\|$) to be bounded to avoid $\mathcal{O}(\log T)$ in the analysis.

Optimal Result with Unknown T

Proof: It is easy to extend the second GD lemma from $t = 1, \ldots, T$ to $t = \lceil \frac{T}{2} \rceil, \ldots, T$:

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_1 - \mathbf{x}^\star\|^2}{2\sum_{t=1}^T \eta_t} + \frac{\sum_{t=1}^T \eta_t^2 \|\nabla f(\mathbf{x}_t)\|^2}{2\sum_{t=1}^T \eta_t}$$

$$\Longrightarrow f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{\|\mathbf{x}_{\lceil \frac{T}{2} \rceil} - \mathbf{x}^\star\|^2}{2\left(\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t\right)} + \frac{G^2}{2} \frac{\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t^2}{\sum_{t=\lceil \frac{T}{2} \rceil}^T \eta_t}$$

$$\left(\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}} \geq \frac{T}{2} \cdot \frac{1}{\sqrt{T}} = \frac{\sqrt{T}}{2}\right) \qquad \leq \frac{DG}{2} \frac{1}{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}}} + \frac{DG}{2} \frac{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{t}}{\sum_{t=\lceil \frac{T}{2} \rceil}^T \frac{1}{\sqrt{t}}} \approx \frac{1}{\sqrt{T}}$$

$$\Longrightarrow f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{DG}{\sqrt{T}} = \mathcal{O}\left(\frac{1}{\sqrt{T}}\right). \qquad \square$$

Part 5. Strongly Convex and Lipschitz

- Strong Convexity

- Convergence Result

Strongly Convex and Lipschitz

Theorem 7. *Under the same assumptions with Theorem 1, except that f is σ-strongly-convex. Let $\{\mathbf{x}_t\}_{t=1}^T$ be the sequence generated by GD with step size*

$$\eta_t = \frac{2}{\sigma(t+1)}.$$

Then (i)

$$f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{2G^2}{\sigma(T+1)} = \mathcal{O}\left(\frac{1}{T}\right),$$

where $\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ *or* $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{2t}{T(T+1)} \mathbf{x}_t$. *And (ii)*

$$\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\| \leq \frac{2G}{\sigma\sqrt{T+1}}.$$

Strongly Convex and Lipschitz

Proof: we start by extending *the first GD lemma* to strongly convex case.

Strongly convex case:

$$\begin{aligned}\|\mathbf{x}_{t+1}-\mathbf{x}^\star\|^2 &\leq \|\mathbf{x}_t-\mathbf{x}^\star\|^2 - 2\eta_t\langle\nabla f(\mathbf{x}_t),\mathbf{x}_t-\mathbf{x}^\star\rangle + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2 \\ &\leq \|\mathbf{x}_t-\mathbf{x}^\star\|^2 - 2\eta_t\left(f(\mathbf{x}_t)-f^\star+\frac{\sigma}{2}\|\mathbf{x}_t-\mathbf{x}^\star\|^2\right) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2\end{aligned}$$

(Strong convexity: $f(\mathbf{x}_t)-f(\mathbf{x}^\star)+\frac{\sigma}{2}\|\mathbf{x}_t-\mathbf{x}^\star\|^2 \leq \langle\nabla f(\mathbf{x}_t),\mathbf{x}_t-\mathbf{x}^\star\rangle$)

$$\leq (1-\sigma\eta_t)\|\mathbf{x}_t-\mathbf{x}^\star\|^2 - 2\eta_t\left(f(\mathbf{x}_t)-f^\star\right) + \eta_t^2\|\nabla f(\mathbf{x}_t)\|^2$$

$$\Longrightarrow f(\mathbf{x}_t)-f^\star \leq \frac{\eta_t^{-1}-\sigma}{2}\|\mathbf{x}_t-\mathbf{x}^\star\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1}-\mathbf{x}^\star\|^2 + \frac{\eta_t G^2}{2}$$

(rearranging)

Strongly Convex and Lipschitz

$$f(\mathbf{x}_t) - f^\star \le \frac{\eta_t^{-1} - \sigma}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{\eta_t^{-1}}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2 + \frac{\eta_t G^2}{2}$$
$$= \frac{\sigma}{4}\left((t-1)\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - (t+1)\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2}{\sigma(t+1)}$$

$$\Longrightarrow \quad t(f(\mathbf{x}_t) - f^\star) \le \frac{\sigma}{4}\left((t-1)t\|\mathbf{x}_t - \mathbf{x}^\star\|^2 - t(t+1)\|\mathbf{x}_{t+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2}{\sigma}$$

telescope now

$$\Longrightarrow \quad \sum_{t=1}^{T} t(f(\mathbf{x}_t) - f^\star) \le \frac{\sigma}{4}\left(0 \cdot 1 \cdot \|\mathbf{x}_1 - \mathbf{x}^\star\|^2 - T(T+1)\|\mathbf{x}_{T+1} - \mathbf{x}^\star\|^2\right) + \frac{G^2 T}{\sigma} = \frac{G^2 T}{\sigma}$$

Next step: relating $\sum_{t=1}^{T} t(f(\mathbf{x}_t) - f(\mathbf{x}^\star))$ to $f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^\star)$.

Strongly Convex and Lipschitz

Case 1: $\sum_{t=1}^{T} t(f(\mathbf{x}_t) - f^\star) \geq \left(\sum_{t=1}^{T} t\right)(f(\bar{\mathbf{x}}_T) - f^\star) = \frac{T(T+1)}{2}(f(\bar{\mathbf{x}}_T) - f^\star)$

(distribution)

Case 2: $\sum_{t=1}^{T} t(f(\mathbf{x}_t) - f^\star) = \sum_{t=1}^{T} tf(\mathbf{x}_t) - \frac{T(T+1)}{2}f^\star = \frac{T(T+1)}{2}\left(\sum_{t=1}^{T}\frac{2t}{T(T+1)}f(\mathbf{x}_t) - f^\star\right)$

$\geq \frac{T(T+1)}{2}(f(\bar{\mathbf{x}}_T) - f^\star)$

(Jensen's inequality)

(*i*) is proved. □

Strongly Convex and Lipschitz

Proof: *(ii)* can be derived directly from *(i)* and strong convexity.

$$\frac{\sigma}{2}\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\|^2 \leq \langle \nabla f(\mathbf{x}^\star), \mathbf{x}^\star - \bar{\mathbf{x}}_T \rangle + \frac{\sigma}{2}\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\|^2 \leq f(\bar{\mathbf{x}}_T) - f^\star \leq \frac{2G^2}{\sigma(T+1)}$$

(strong convexity) *(i)*

(first-order optimality condition: $\langle \nabla f(\mathbf{x}^\star), \mathbf{x}^\star - \mathbf{x} \rangle \geq 0$)

Thus, we prove that no matter for which constructions of $\bar{\mathbf{x}}_T$, it holds that

$$\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\| \leq \frac{2G}{\sigma\sqrt{T+1}}.$$

(ii) **is proved.** □

Summary

Table 1: A summary of convergence rates of GD method.

Function Family	Step Size	Output Sequence	Convergence Rate	Remark
convex and G-Lipschitz	$\eta_t = \frac{f(\mathbf{x}_t)-f^\star}{\\|\nabla f(\mathbf{x}_t)\\|^2}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$	$\mathcal{O}(1/\sqrt{T})$	optimal Polyak's step size require $f^\star, T$
	$\eta_t = \frac{1}{\\|\nabla f(\mathbf{x}_t)\\|\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(\log T/\sqrt{T})$	suboptimal
	$\eta_t = \frac{D}{G\sqrt{T}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain require T
	$\eta_t = \frac{D}{G\sqrt{t}}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=\lceil T/2\rceil}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=\lceil T/2\rceil}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=\lceil T/2\rceil}^T \eta_t}$	$\mathcal{O}(1/\sqrt{T})$	bounded domain
σ-strongly convex and G-Lipschitz	$\eta_t = \frac{2}{\sigma(t+1)}$	$\bar{\mathbf{x}}_T \triangleq \arg\min_{\{\mathbf{x}_t\}_{t=1}^T} f(\mathbf{x}_t)$ $\bar{\mathbf{x}}_T \triangleq \sum_{t=1}^T \frac{\eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t}$	$\mathcal{O}(1/T)$	$\\|\bar{\mathbf{x}}_T - \mathbf{x}^\star\\|$ is bounded

Summary

- **GRADIENT DESCENT**
  - Convex Optimization Problem
  - Gradient Descent
  - Performance Measure
  - The First Gradient Descent Lemma
- **POLYAK'S STEP SIZE**
  - Polyak's Step Size
  - Convergence
  - Convergence Rate
- **CONVERGENCE WITHOUT OPTIMAL VALUE**
  - The Second Gradient Descent Lemma
  - Convergent Step Size
  - Convergence without Optimal Value
- **OPTIMAL IN CONVEX AND LIPSCHITZ CASE**
  - Optimal Result with Known T
  - Optimal Result with Unknown T
- **STRONGLY CONVEX AND LIPSCHITZ**
  - Strong Convexity
  - Convergence Result