Non-stationary Online Learning with Memory and Non-stochastic Control

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Abstract

We study the problem of Online Convex Optimization (OCO) with memory, which allows loss functions to depend on past decisions and thus captures temporal effects of learning problems. In this paper, we introduce *dynamic policy regret* as the performance measure to design algorithms robust to non-stationary environments, which competes algorithms' decisions with a sequence of changing comparators. We propose a novel algorithm for OCO with memory that provably enjoys an optimal dynamic policy regret. The key technical challenge is how to control the *switching cost*, the cumulative movements of player's decisions, which is neatly addressed by a novel decomposition of dynamic policy regret and an appropriate meta-expert structure. Furthermore, we apply the results to the problem of online non-stochastic control, i.e., controlling a linear dynamical system with adversarial disturbance and convex loss functions. We derive a novel gradient-based controller with dynamic policy regret guarantees, which is the first controller competitive to a sequence of changing policies.

1. Introduction

Online Convex Optimization (OCO) (Shalev-Shwartz, 2012; Hazan, 2016) is a versatile model of learning in adversarial environments, which can be regarded as a sequential game between a player and an adversary (environments). At each round, the player makes a prediction from a convex set $\mathbf{w}_t \in \mathcal{W} \subseteq \mathbb{R}^d$, the adversary simultaneously selects a convex loss $f_t : \mathcal{W} \mapsto \mathbb{R}$, and the player incurs a loss $f_t(\mathbf{w}_t)$. The goal of the player is to minimize the cumulative loss. The framework is found useful in a variety of disciplines including learning theory, game theory, optimization, and time series analysis, etc (Cesa-Bianchi and Lugosi, 2006).

The standard OCO framework considers only *memoryless* adversary, in the sense that the resulting loss is only determined by the player's current prediction without involving past ones. In real-world applications, particularly those related to online decision making, it is often the case that past predictions/decisions would also contribute to the current loss, which makes the standard OCO framework not viable. To remedy this issue, Online Convex Optimization with Memory (OCO with Memory) was proposed as a simplified and elegant model to capture the temporal effects of learning problems (Merhav et al., 2002; Anava et al., 2015). Specifically, at each round, the player makes a prediction $\mathbf{w}_t \in \mathcal{W}$, the adversary chooses a loss function $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$, and the player will then suffer a loss $f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t)$. Notably, now the loss function depends on both current and past predictions. The parameter m is the memory length, and evidently the OCO with memory model reduces to the standard memoryless OCO when setting $m = 0$. The performance measure for OCO with memory is the *policy regret* (Dekel et al., 2012), defined as

$$\text{S-Regret}_T = \sum_{t=1}^{T} f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \min_{\mathbf{v} \in \mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{v}, \ldots, \mathbf{v}). \tag{1}$$

We start the index from 1 for convenience. Recent studies apply online learners with provably low policy regret to a variety of related problems (Chen et al., 2018; Agarwal et al., 2019; Daniely and Mansour, 2019; Chen et al., 2020). However, the policy regret (1) only measures the performance versus a *fixed* comparator and is thus not suitable for learning in non-stationary environments (Sugiyama and Kawanabe, 2012; Gama et al., 2014; Zhao et al., 2021). For instance, in the recommendation system, the users' interest may change when looking through the product pages; in the traffic flow scheduling, the traffic network pattern changes throughout the day. Therefore, it is necessary to design online decision-making algorithms with robustness to non-stationary environments. To this purpose, we introduce the *dynamic policy regret* to guide the algorithmic design, measuring the competitive performance against a arbitrary sequence of *time-varying* comparators $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{X}$ (shorthanded as $\mathbf{v}_{1:T}$),

$$\text{D-Regret}_T(\mathbf{v}_{1:T}) = \sum_{t=1}^{T} f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \sum_{t=1}^{T} f_t(\mathbf{v}_{t-m}, \ldots, \mathbf{v}_t). \tag{2}$$

The upper bound of $\text{D-Regret}_T(\mathbf{v}_{1:T})$ should be a function of the comparator sequence $\mathbf{v}_{1:T}$, while the algorithm is agnostic to the choice of comparators. The proposed measure is very general—it subsumes static policy regret (1) as a special case when comparators become the best predictor in hindsight, i.e., $\mathbf{v}_{1:T} = \mathbf{v}^* \in \arg\min_{\mathbf{v} \in \mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{v}, \ldots, \mathbf{v})$. Therefore, dynamic policy regret is a stringent measure and algorithms that optimize it are more adaptive to non-stationary environments.

The fundamental challenge of dynamic policy regret optimization is how to simultaneously compete with all comparator sequences with vastly different level of non-stationarity. Our approach builds upon recent advance of non-stationary online learning (Daniely et al., 2015; Zhang et al., 2018a) to hedge the uncertainty via meta-expert aggregation, along with several new ingredients specifically designed for the OCO with memory setting. In particular, it is essential to control the *switching cost* for OCO with memory, the cumulative movement of player's predictions. The amount is relatively easy to control in static policy regret (Anava et al., 2015), yet becomes much thorny in dynamic policy regret and could even scale linearly due to the meta-expert structure. Intuitively, for dynamic online algorithms, it is necessary to keep some probability of aggressive movement in order to catch up with the potential changes of the non-stationary environments, which results in tensions between the dynamic regret and switching cost. We elegantly address the difficulty by proposing a novel meta-expert decomposition and a switching-cost-regularized surrogate loss, which avoids explicitly handling switching cost all together and renders a unified design by online mirror descent for both meta- and expert-algorithms. We prove that our proposed algorithm enjoys an *optimal* $\mathcal{O}(\sqrt{T(1 + P_T)})$ dynamic policy regret, where $P_T = \sum_{t=2}^{T} \|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2$ denotes the unknown path-length of the comparators.

We further apply the method to the problem of *online non-stochastic control*, i.e., controlling a linear dynamical system with adversarial (non-stochastic) disturbance and adversarial convex cost functions (Agarwal et al., 2019; Hazan et al., 2020). As the disturbances and cost functions both change adversarially, the optimal controller of each round would also change over iterations. Therefore, it is natural and necessary to investigate the *dynamic policy regret* to compete the controller's performance with *time-varying* benchmark controllers. By adopting the "disturbance-action" policy parameterization (Agarwal et al., 2019), online non-stochastic control is reduced to OCO with memory, and thus its dynamic policy regret can be optimized by a similar meta-expert structure as developed before. Our designed controller attains an $\widetilde{\mathcal{O}}(\sqrt{T(1+P_T)})$ dynamic policy regret, where P_T measures the fluctuation of compared controllers and $\widetilde{\mathcal{O}}$-notation hides the logarithmic dependence on time horizon T. To the best of our knowledge, this is the first controller competitive to a sequence of changing "disturbance-action" policies.

This paper is organized as follows. In Section 3 problem setup and preliminaries are introduced. Section 2 discusses some related work. Section 4 and Section 5 present the results for OCO with memory and online non-stochastic control, respectively. Section 6 reports the empirical evaluations. Section 7 concludes the paper. All the proofs are deferred to the appendices.

2. Related Work

In this section we present more discussions on the related work, including OCO with memory, online non-stochastic control, as well as dynamic regret minimization for online learning.

Online Convex Optimization with Memory. OCO with memory is initiated by Merhav et al. (2002), who prove an $\mathcal{O}(T^{2/3})$ policy regret for convex and Lipschitz functions by a blocking technique. Later, for Lipschitz functions, Anava et al. (2015) propose a simple gradient-based algorithm that provably achieves $\mathcal{O}(\sqrt{T})$ and $\mathcal{O}(\log T)$ policy regret for convex and strongly convex functions, respectively. Recent study discloses that the policy regret of OCO with memory over exp-concave functions is at least $\Omega(T^{1/3})$ (Simchowit, 2020, Theorem 2.3). Online learning with memory is also studied in the prediction with expert advice setting (Geulen et al., 2010; György and Neu, 2014; Altschuler and Talwar, 2018; Cesa-Bianchi et al., 2013; Altschuler and Talwar, 2018) and bandit settings (Dekel et al., 2012, 2014; Altschuler and Talwar, 2018; Arora et al., 2019). One of the key concepts of OCO with memory is the *switching cost*, cumulative movement of the decisions, which is also concerned in smoothed online learning (Chen et al., 2018; Goel et al., 2019) and competitive online learning (Daniely and Mansour, 2019).

Online Non-stochastic Control. Recently, there is a surge of interest to apply modern statistical and algorithmic techniques to the control problem. We focus on the online non-stochastic control setting proposed by Agarwal et al. (2019), where the regret is chosen as the performance measure and the disturbance is allowed to be adversarially chosen. Under conditions of convex and Lipschitz cost functions as well as adversarial disturbance, Agarwal et al. (2019) obtain an $\mathcal{O}(\sqrt{T})$ policy regret for known linear dynamical system by introducing the DAC parameterization and reducing the problem to OCO with memory. Hazan et al. (2020) show an $\mathcal{O}(T^{2/3})$ policy regret for unknown system via system identification. In addition, Foster and Simchowitz (2020) propose the online learning with advantages techniques and obtain logarithmic regret for known system with quadratic cost and adversarial disturbance, and the results are strengthened by Simchowit (2020) to accommodate arbitrary, changing costs. All mentioned results are developed for fully observed system, and Simchowitz et al. (2020) present a clear picture for non-stochastic control with partially observed system. We are still witnessing a variety of recent advances, for example, non-stochastic control with bandit feedback (Gradu et al., 2020a; Cassel and Koren, 2020), adaptive regret minimization (Gradu et al., 2020b; Zhang et al., 2021), etc. At the end of this section, we will present more thorough discussions on the relationship between the two works for adaptive regret minimization (Gradu et al., 2020b; Zhang et al., 2021) and our work for dynamic regret minimization. There are other related works studying non-stationary online control from the lens of competitive ratio (Shi et al., 2020) and robust control (Goel and Hassibi).

Dynamic Regret. Benchmarking regret in term of changing comparators can date back to early development of prediction with expert advice (Herbster and Warmuth, 1998, 2001). In the OCO setting, Zinkevich (2003) pioneers the dynamic regret against any comparator sequence and shows that OGD can attain an $\mathcal{O}(\sqrt{T}(1+P_T))$ dynamic regret. It is revealed by Zhang et al. (2018b) that the result is not tight, who establish an $\Omega(\sqrt{T(1+P_T)})$ minimax lower bound for convex functions and close the gap by proposing an algorithm with optimal $\mathcal{O}(\sqrt{T(1+P_T)})$ rate. Recent improvement achieves problem-dependent dynamic regret by further exploiting the smoothness (Zhao et al., 2020b). Dynamic regret of bandit convex optimization is studies in (Zhao et al., 2020a). We finally emphasize that the dynamic regret measure studied in this paper is also called the *universal* dynamic regret, in that the guarantee holds universally against any comparator sequence in the domain. Another special form called the *worst-case* dynamic regret is also frequently investigated in the literature (Besbes et al., 2015; Jadbabaie et al., 2015; Mokhtari et al., 2016; Zhang et al., 2017; Baby and Wang, 2019; Zhang et al., 2020; Zhao and Zhang, 2021), which specifies comparators as the optimizers of online functions. The worst-case dynamic regret is less general than the universal one, and the reader is referred to the work of Zhang et al. (2018a) for more discussions.

More Discussions. In addition to OCO with memory, switching cost is also studies in smoothed OCO, particularly there are some recently efforts devoted to dynamic regret of smoothed OCO (Chen et al., 2018, Section 5). We remark that the settings of two problems are different: smoothed OCO requires to observe the cost f_t first and then choose the decision $\mathbf{w}_t \in \mathcal{W}$; while OCO with memory decides $\mathbf{w}_t$ without the knowledge of f_t. Besides, the dynamic regret bound of smoothed OCO (Chen et al., 2018, Corollary 11) needs prior knowledge of path-length, and our techniques might be useful for removing this undesired requirement.

Online non-stochastic control in non-stationary environments is also recently studied via the measure of *adaptive regret* (Hazan and Seshadhri, 2009) — the regret compared to the best policy on any interval in time horizon. Gradu et al. (2020b) propose the first controller with an $\widetilde{\mathcal{O}}(\sqrt{T})$ expected adaptive regret on any interval in the total horizon. The result is strengthened in a recent work (concurrent to our paper) (Zhang et al., 2021), which presents a strongly adaptive controller with an $\widetilde{\mathcal{O}}(\sqrt{|\mathcal{I}|})$ deterministic adaptive regret on any interval $\mathcal{I} \subseteq [T]$. The two papers and our work all study non-stationary online control, however, the concerned measures and used techniques are completely different. **(1)** Measures: dynamic regret studies the global behavior to ensure a competitive performance with time-varying compared polices, whereas adaptive regret focuses on the local behavior with respect to a fixed strategy. To the best of our knowledge, dynamic regret and adaptive regret reflect different perspectives of environments, and their relationship is still unclear even for the standard OCO setting (Zhang, 2020, Sec 5. Open Problems). **(2)** Techniques: optimizing either dynamic regret or adaptive regret requires the meta-expert structure to deal with uncertainty of the non-stationary environments. However, the specific techniques, especially the way to control switching cost, exhibit significant difference. Gradu et al. (2020b) follow the Follow-the-Leading History framework (Hazan and Seshadhri, 2009) with a shrinking technique (Geulen et al., 2010) to keep previous experts with a certain probability in order to reduce the switching cost, and thus their $\widetilde{\mathcal{O}}(\sqrt{T})$ adaptive regret guarantee holds in expectation only. The improved result of $\mathcal{O}(\sqrt{|\mathcal{I}|})$ deterministic bound (Zhang et al., 2021) is achieved by a very different framework (Cutkosky, 2020) drawn inspirations from parameter-free online learning. By contrast, the key ingredients of our approach are the novel meta-expert decomposition and the switching-cost-regularized loss, which avoid explicitly handling the switching cost of final decisions but directly control the switching cost of meta-algorithm and individual expert-algorithm. These mechanisms finally lead to a deterministic dynamic policy regret guarantee for our proposed controller.

3. Problem Setup and Preliminaries

In this section, we formalize the problem setup and introduce preliminaries for OCO with memory.

3.1 Problem Setup

Online Convex Optimization (OCO) with memory is a variant of standard OCO framework to capture the long-term effects of past decisions. The protocol is shown as follows.

1: **for** $t = m+1, \ldots, T$ **do**
2: the player chooses a decision $\mathbf{w}_t \in \mathcal{W}$;
3: the adversary reveals the loss $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$ that applies to last $m+1$ decisions;
4: the player suffers a loss of $f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t)$;
5: **end for**

In above, m is the memory length, and $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$ is convex in memory, which means its unary function $\widetilde{f}_t(\mathbf{w}) = f_t(\mathbf{w}, \ldots, \mathbf{w})$ is convex in $\mathbf{w}$. Clearly, OCO with memory recovers the standard memoryless OCO when $m = 0$. The standard measure in the literature is *policy regret* (Dekel et al., 2012) defined in (1). This paper investigates the *dynamic policy regret*, a strengthened measure to compete with changing comparators as defined in (2). As mentioned previously, algorithms that optimize the dynamic regret are more adaptive to non-stationary environments, whereas the gain is accompanied with challenge on how to tackle the uncertainty of the environmental non-stationarity.

We conclude this part by introducing several standard assumptions used in the analysis (Agarwal et al., 2019; Hazan et al., 2020). For simplicity we focus on the ℓ_2-norm and the extension to general primal-dual norms is straightforward.

Assumption 1 (coordinate-wise Lipschitzness)**.** The function $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$ is L-coordinate-wise Lipschitz, i.e., $|f_t(\mathbf{x}_0, \ldots, \mathbf{x}_m) - f_t(\mathbf{y}_0, \ldots, \mathbf{y}_m)| \leq L \sum_{i=0}^{m} \|\mathbf{x}_i - \mathbf{y}_i\|_2$.

Assumption 2 (bounded gradient)**.** The gradient norm of the unary loss is at most G, i.e., for all $\mathbf{w} \in \mathcal{W}$ and $t \in [T]$, $\|\nabla \widetilde{f}_t(\mathbf{w})\|_2 \leq G$.

Assumption 3 (bounded domain)**.** The domain $\mathcal{W}$ is convex, closed, and satisfies $\|\mathbf{w} - \mathbf{w}'\|_2 \leq D$ for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$. For convenience of analysis, we further assume $\mathbf{0} \in \mathcal{W}$.

3.2 Static Policy Regret of OCO with Memory

Before presenting dynamic policy regret of OCO with memory, we review the result of static policy regret. Anava et al. (2015) propose a simple approach based on the gradient descent, whose crucial observation is that when online functions are coordinate-wise Lipschitz, the policy regret can be upper bounded by the switching cost and the vanilla regret over the unary loss, formally,

$$\sum_{t=1}^{T} f_t(\mathbf{w}_{t-m:t}) - \min_{\mathbf{v}\in\mathcal{W}} \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}) \leq \lambda \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 + \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{w}_t) - \min_{\mathbf{v}\in\mathcal{W}} \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}), \tag{3}$$

where $\lambda = m^2 L$. The first term is the *switching cost* measuring the cumulative movement of decisions $\mathbf{w}_{1:T}$ and the remaining term is the standard regret of memoryless OCO. Consequently, it is natural to perform Online Gradient Descent (OGD) (Zinkevich, 2003) over the unary loss $\widetilde{f}_t$, i.e., $\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}[\mathbf{w}_t - \eta\nabla\widetilde{f}_t(\mathbf{w}_t)]$, where $\eta > 0$ is the step size and $\Pi_{\mathcal{W}}[\cdot]$ denotes the projection onto the nearest point in $\mathcal{W}$. It is well-known that with an appropriate step size OGD enjoys an $\mathcal{O}(\sqrt{T})$ regret in memoryless OCO. Further, Anava et al. (2015) show that the produced decisions move sufficiently slowly. Indeed, switching cost satisfies $\sum_{t=2}^{T}\|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \leq \mathcal{O}(\eta T)$, which will not affect the final regret order by choosing $\eta = \mathcal{O}(1/\sqrt{T})$. Combining both facts yields an $\mathcal{O}(\sqrt{T})$ static policy regret.

Theorem 1 (Theorem 3.1 of Anava et al. (2015))**.** *Under Assumptions 1–3, running OGD over the unary loss achieves* $\sum_{t=1}^{T} f_t(\mathbf{w}_{t-m:t}) - \min_{\mathbf{v}\in\mathcal{W}} \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}) \leq (G^2 + m^2LG)\eta T + \frac{2D^2}{\eta}$*. Setting the step size optimally as* $\eta = \eta^* = \sqrt{\frac{2D^2}{(G^2+m^2LG)T}}$*, we attain an* $\mathcal{O}(\sqrt{T})$ *static policy regret.*

4. Online Convex Optimization with Memory

This section presents dynamic policy regret of OCO with memory. We begin with the gentle case when the path-length is known, and then handle the general case when it is unknown. We elucidate the challenge of controlling switching cost in the non-stationary OCO with memory, next show how to resolve it by novel algorithmic ingredients, and finally present dynamic policy regret analysis.

4.1 A Gentle Start: known path-length P_T

We generalize Theorem 1 by showing that OGD also enjoys the dynamic policy regret.

Theorem 2. *Under Assumptions 1–3, running OGD over unary loss ensures* $\text{D-Regret}_T(\mathbf{v}_{1:T}) \leq (G^2 + m^2LG)\eta T + \frac{1}{2\eta}(D^2 + 2DP_T) + m^2LP_T$ *for any comparator sequence* $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{W}$*, where* $P_T = \sum_{t=2}^{T}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|_2$ *is the path-length that measures the fluctuation of comparators.*

Suppose the path-length P_T were known for a moment, we could obtain an $\mathcal{O}(\sqrt{T(1+P_T)})$ dynamic policy regret by simply setting the step size as $\eta = \sqrt{\frac{D^2+2DP_T}{(G^2+m^2LG)T}}$, which would then match the $\Omega(\sqrt{T(1+P_T)})$ lower bound of memoryless OCO (Zhang et al., 2018a). However, this step size tuning is not realistic in that it requires the knowledge of P_T a priori. In fact, the comparator sequence $\mathbf{v}_1, \ldots, \mathbf{v}_T$ can be arbitrarily selected by the environments, and thus $P_T = \sum_{t=2}^{T}\|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2$ reflects the environmental non-stationarity and is *unknown* to the player. The similar challenge also emerges in recent studies of memoryless OCO (Zhang et al., 2018a; Zhao et al., 2020b), inspired by which we employ the by-now-standard meta-expert framework to *hedge the non-stationarity*. In Section 4.2, we will elucidate the challenge of applying this framework to OCO with memory, mainly due to the tension between dynamic regret and switching cost. Section 4.3 demonstrates how to resolve the issue to have a good balance, by designing several novel and necessary algorithmic ingredients.

4.2 Challenge: switching cost of meta-expert structure

In the development of dynamic regret of memoryless OCO, the meta-expert framework is proposed to address the unknown path-length emerging in the optimal step size tuning (Zhang et al., 2018a; Zhao et al., 2020b). Below we briefly review the framework and clarify the challenge of its application in OCO with memory.

Meta-expert framework. The framework is essentially an online ensemble method consisting of three components: the pool of candidate step sizes, the expert-algorithm, and the meta-algorithm. We first need to design an appropriate pool of candidate step sizes $\mathcal{H} = \{\eta_1, \ldots, \eta_N\}$ and ensure the existence of a step size η_{i*} that approximates the optimal step size η_* well. Then, multiple experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ are maintained where each performs OGD with a step size $\eta_i \in \mathcal{H}$ and generates the decision sequence $\mathbf{w}_{1,i}, \mathbf{w}_{2,i}, \ldots, \mathbf{w}_{T,i}$ where $\mathbf{w}_{t+1,i} = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla \widetilde{f}_t(\mathbf{w}_{t,i})]$. Finally, a meta-algorithm, supposed to be able to track the best expert-algorithm, is used to combine all intermediate results of experts to produce the final decisions $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_T$, where $\mathbf{w}_t = \sum_{i=1}^N p_{t,i}\mathbf{w}_{t,i}$.

Similar to static policy regret analysis (3), dynamic policy regret (2) is also upper bounded by switching cost and dynamic regret of memoryless OCO over the unary loss $\widetilde{f}_t$, specifically,

$$\text{D-Regret}_T(\mathbf{v}_{1:T}) \leq \lambda \sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 + \lambda \sum_{t=2}^T \|\mathbf{v}_t - \mathbf{v}_{t-1}\|_2 + \sum_{t=1}^T \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \widetilde{f}_t(\mathbf{v}_t). \quad (4)$$

The second term is the path-length of the comparators $\mathbf{v}_1, \ldots, \mathbf{v}_T$, and is of order $\mathcal{O}(P_T)$. Moreover, applying existing dynamic regret bound of memoryless OCO (Zhang et al., 2018a) easily ensures $\sum_{t=1}^T \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \widetilde{f}_t(\mathbf{v}_t) \leq \mathcal{O}(\sqrt{T(1+P_T)})$. Thus, the key is the first term, switching cost of final decisions. However, below we show that the existing meta-expert method (Zhang et al., 2018a) may move too fast to achieve a sublinear switching cost, which necessitates some novel algorithmic ingredients to better balance the dynamic regret and switching cost.

Switching cost. The switching cost is the pivot of the analysis for OCO with memory. Anava et al. (2015) demonstrate that many popular OCO algorithms for static regret minimization naturally produce slow-moving decisions, however, it becomes much thorny in dynamic regret. Intuitively, for dynamic online algorithms, it is necessary to keep some probability of aggressive movement in order to catch up with the potential changes of non-stationary environments, which results in tensions between dynamic regret and switching cost. Formally, suppose we adopt the meta-expert structure to yield the decision $\mathbf{w}_t = \sum_{i=1}^N p_{t,i}\mathbf{w}_{t,i}$, then the switching cost can be bounded by (proof is in Appendix B.2)

$$\sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \leq D \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \sum_{t=2}^T \sum_{i=1}^N p_{t,i} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2. \quad (5)$$

The first term is the switching cost of the meta-algorithm, which is at most $\mathcal{O}(\sqrt{T})$. However, the second term becomes the main barrier as it could be very large and even grow linearly with iterations. Specifically, the switching cost of expert-algorithm $\mathcal{E}_i$ (OGD with step size η_i) is $\mathcal{O}(\eta_i T)$; additionally, to ensure a coverage of the optimal step size, the pool of candidate step sizes is usually set as $\mathcal{H} = \{\eta_i = \mathcal{O}(2^i \cdot T^{-\frac{1}{2}}), i \in [N]\}$ such that $\eta_1 = \mathcal{O}(T^{-\frac{1}{2}})$ and $\eta_N = \mathcal{O}(1)$. Therefore, experts with larger step sizes would incur unacceptable switching cost, for instance, the switching cost of expert $\mathcal{E}_N$ could grow linearly, of order $\mathcal{O}(T)$. As a result, the second term, a weighted combination of experts' switching cost, could be enlarged by experts whose step sizes are too large and therefore become difficult to control. In Section 6, empirical evaluations also valid that the standard meta-expert method will incur large switching cost, almost growing linearly with iterations.

4.3 Algorithmically Enforcing Low Switching Cost: a new meta-expert decomposition

As indicated in the last part, it is actually challenging to control the switching cost in dynamic regret. Our idea is to avoid directly controlling the switching cost of the final predictions, instead, we propose the following new meta-expert decomposition for the dynamic policy regret.

$$\begin{aligned}
&\sum_{t=1}^{T} \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}_t) + \lambda \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \\
\overset{(5)}{\leq}\ & \sum_{t=1}^{T} \langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle + \lambda D \sum_{t=2}^{T} \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \lambda \sum_{t=2}^{T}\sum_{i=1}^{N} p_{t,i} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \\
=\ & \underbrace{\sum_{t=1}^{T} \big(\langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \ell_{t,i}\big) + \lambda D \sum_{t=2}^{T} \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1}_{\texttt{meta-regret}} + \underbrace{\sum_{t=1}^{T} \big(g_t(\mathbf{w}_{t,i}) - g_t(\mathbf{v}_t)\big) + \lambda \sum_{t=2}^{T} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2}_{\texttt{expert-regret}}.
\end{aligned}$$

The first inequality follows from the convexity of the unary function and the switching cost decomposition (5), and for convenience we introduce the notation of linearized loss $g_t(\mathbf{w}) = \langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w} \rangle$. The second equation is the crucial step, in which the key ingredient is the introduced *switching-cost-regularized surrogate loss* $\boldsymbol{\ell}_t \in \mathbb{R}^N$ for the meta-algorithm, defined as

$$\ell_{t,i} = g_t(\mathbf{w}_{t,i}) + \lambda \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2. \tag{6}$$

Indeed, the equation essentially decomposes the dynamic policy regret in a novel way by incorporating a *switching-cost regularizer* into the meta-expert decomposition. As a result, we have the following observations on the requirements of the meta- and expert- regret optimization:

- expert-algorithm needs to achieve low dynamic regret over unary functions and meanwhile tolerate the switching cost of its own local decisions $\sum_{t=2}^{T} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2$;
- meta-algorithm needs to optimize the switching-cost-regularized loss to impose more penalty on experts with larger switching cost, and tolerate the switching cost $\sum_{t=2}^{T} \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1$.

Consequently, it is *not* necessary to explicitly handle switching cost all together (i.e., $\sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2$). Instead, we only need to tackle the switching cost of each individual expert-algorithm (i.e., $\sum_{t=2}^{T} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2$) and that of meta-algorithm (i.e., $\sum_{t=2}^{T} \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1$), which turns out to be much easier to control. In the following, we specify the expert-algorithm and meta-algorithm.

We first consider the expert-algorithm. Actually, Theorem 2 proves that OGD over unary functions enjoys the dynamic policy regret guarantee, so we can simply choose the expert-algorithm as OGD over linearized loss $\{g_t\}_{t=1:T}$ and then the switching cost of its local decisions can be safely controlled. More specifically, there are N experts denoted by $\mathcal{E}_1, \ldots, \mathcal{E}_N$ and the expert $\mathcal{E}_i$ performs

$$\mathbf{w}_{t+1,i} = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla g_t(\mathbf{w}_{t,i})] = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla \widetilde{f}_t(\mathbf{w}_t)], \tag{7}$$

where η_i is a certain step size selected from the pool of candidate step sizes $\mathcal{H}$. Note that the second equation in the above update exhibits the advantage of using the linearized loss: although multiple experts are performed simultaneously, all of them use the same gradient and thus we require only one gradient per iteration, rather than N gradients as was anticipated (N is the number of experts).

On the other hand, noting that the meta-regret $\sum_{t=1}^T \left(\langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \ell_{t,i}\right) + \lambda D \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1$ is essentially the static regret with switching cost, we can thus simply choose the meta-algorithm as Hedge (Freund and Schapire, 1997), which can optimize the regret and naturally produce a sequence of slow-moving decisions. Concretely, the meta-algorithm performs the multiplicative update over the switching-cost-regularized loss, i.e., updating the weight $\boldsymbol{p}_{t+1} \in \Delta_N$ according to $p_{t+1,i} \propto p_{t,i} \exp(-\varepsilon \ell_{t,i})$, where $\boldsymbol{\ell}_t \in \mathbb{R}^N$ is the surrogate loss defined in (6) and the learning rate is set as $\varepsilon = \mathcal{O}(\sqrt{1/T})$. For technical reasons, we adopt a non-uniform initialization by setting $\boldsymbol{p}_1 \in \Delta_N$ with $p_{1,i} \propto 1/(i^2 + i)$. Note that the dependence of learning rate on T can be removed by either a time-varying tuning or the doubling trick (Cesa-Bianchi et al., 1997).

We finally remark that expert-algorithm (OGD) and meta-algorithm (Hedge) can be understood in a unified view from the aspect of Online Mirror Descent (OMD) (Shalev-Shwartz, 2012; Srebro et al., 2011). OMD is a powerful online method accommodating general geometries and both OGD and Hedge are its special instances. We can generalize the dynamic policy regret of Theorem 2 from OGD to OMD actually, and then the meta-regret (static regret and switching cost of Hedge) can be easily derived when setting a fixed comparator and negative-entropy regularizer. More descriptions are supplied in Appendix B.3.

Overall Algorithm. Combining all above ingredients, we propose the Switching-Cost-Regularized meta-Expert Aggregation for OCO with Memory (SCREAM) algorithm, which is based on the online mirror descent algorithm and admits a structure of meta-expert aggregation. Specifically, we initiate $N = \lceil \frac{1}{2} \log_2(1+T) \rceil + 1 = \mathcal{O}(\log T)$ experts, with the step size pool set as

$$\mathcal{H} = \left\{ \eta_i \;\middle|\; \eta_i = 2^{i-1} \cdot \sqrt{\frac{D^2}{(\lambda G + G^2)T}}, i \in [N] \right\}. \tag{8}$$

Algorithm 1 presents the overall procedures. Each expert performs OGD with its corresponding step size as shown in Line 8; the meta-algorithm combines local decisions and updates the weight according to the switching-cost-regularized loss as described in Lines 3–7. Our algorithm provably enjoys an optimal dynamic policy regret, striking a good balance between regret and switching cost.

Theorem 3. *Under Assumptions 1–3, by setting the learning rate optimally of meta-algorithm as* $\varepsilon = \sqrt{2/((2m^2L+G)(m^2L+G)D^2T)}$ *and the step size pool* $\mathcal{H}$ *as (8), the proposed* SCREAM *algorithm ensures* $\sum_{t=1}^T f_t(\mathbf{w}_{t-m:t}) - \sum_{t=1}^T f_t(\mathbf{v}_{t-m:t}) \leq \mathcal{O}\big(\sqrt{T(1+P_T)}\big)$ *for any comparator sequence* $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{W}$*, where* $P_T = \sum_{t=2}^T \|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2$ *is the path-length of the comparators.*

Remark 1. First, since the dynamic policy regret holds for any comparator sequence, by simply setting comparators as the fixed best decision in hindsight (and now $P_T = 0$), our dynamic policy regret implies the $\mathcal{O}(\sqrt{T})$ static regret in Theorem 1. Second, the attained dynamic policy regret is minimax optimal in terms of the dependence on time horizon T and the path-length P_T, because an $\Omega(\sqrt{T(1+P_T)})$ lower bound has been established for the dynamic regret of memoryless OCO (Zhang et al., 2018a).

Algorithm 1 SCREAM: Switching-Cost-Regularized meta-Expert Aggregation for OCOwMemory

Input: time horizon T, step size pool $\mathcal{H} = \{\eta_1, \ldots, \eta_N\}$, learning rate of meta-algorithm ε

1: Initialization: $\mathbf{w}_{1:m} \in \mathcal{W}$, $\mathbf{w}_{m,i} \in \mathcal{W}$, $\forall i \in [N]$; $\boldsymbol{p}_m \in \Delta_N$ with $p_{m,i} \propto 1/(i^2 + i)$, $\forall i \in [N]$
2: **for** $t = m + 1$ **to** T **do**
3: Receive $\mathbf{w}_{t,i}$ from expert $\mathcal{E}_i$ for $i \in [N]$
4: Submit the decision $\mathbf{w}_t = \sum_{i=1}^{N} p_{t,i}\mathbf{w}_{t,i}$
5: Observe the online function $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$ that applies to last $m + 1$ decisions
6: Suffer a loss of $f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t)$
7: Construct the switching-cost-regularized loss $\boldsymbol{\ell}_t \in \mathbb{R}^N$: $\ell_{t,i} = g_t(\mathbf{w}_{t,i}) + \lambda\|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2$
8: Update the weight $\boldsymbol{p}_{t+1} \in \Delta_N$ according to $p_{t+1,i} \propto p_{t,i} \exp(-\varepsilon\ell_{t,i})$
9: Expert-algorithm $\mathcal{E}_i$ updates local decision by $\mathbf{w}_{t+1,i} = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla \widetilde{f}_t(\mathbf{w}_t)]$, $\forall i \in [N]$
10: **end for**

Remark 2. The regret order does not highlight the memory dependence, in that the memory length is typically chosen in the order of $m = \mathcal{O}(\log T)$ (for instance, when it is applied to online non-stochastic control). However, if we scrutinize this dependence, our attained dynamic policy regret exhibits a squared memory dependence, whereas the static policy regret has a linear dependence only (Anava et al., 2015) (note that their paper presents an $\mathcal{O}(m^{3/4}\sqrt{T})$ static regret actually (Anava et al., 2015, Theorem 3.1), since their Lipschitzness assumption is slightly stronger than ours; and actually their bound would become linear in m under Assumption 1 as used in this paper). The difficulty mainly arises from the range of the meta surrogate loss (6), which scales quadratically with the memory length. It remains unclear whether it is possible to and how to improve this dependence in dynamic policy regret, which is left as future work for investigation. We present more discussions in Appendix B.5.

5. Online Non-stochastic Control

In this section we present the results on dynamic policy regret of online non-stochastic control.

5.1 Problem Statement and Performance Measure

Problem Setting. We study the online control of the linear dynamical system (LDS) governed by

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{9}$$

where at iteration t, the controller provides the control u_t upon the observed dynamical state x_t and suffers a cost $c_t(x_t, u_t)$ with convex function $c_t : \mathbb{R}^{d_x} \times \mathbb{R}^{d_u} \mapsto \mathbb{R}$. Following the notational convention of previous works, throughout the section we will use unbold fonts to denote vectors (including control signal, state, disturbance, etc.). In this paper, we focus on the online *non-stochastic* control setting (Agarwal et al., 2019; Hazan et al., 2020). Specifically, the disturbance can be generated arbitrarily and no statistical assumption is imposed on its distribution; additionally, cost functions can be chosen adversarially. The adversarial nature of the disturbance hinders an a priori computation of the optimal policy as in settings of classical control theory (Kalman, 1960). We will leverage recent advance in online control (Agarwal et al., 2019; Hazan et al., 2020) and results of OCO with memory presented in the last section to address the issue.

Policy Regret. The standard measure for online non-stochastic control is the *policy regret* (Agarwal et al., 2019; Hazan et al., 2020),

$$\text{Regret}_T = J_T(\mathcal{A}) - \min_{\pi \in \Pi} J_T(\pi) = \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi), \tag{10}$$

the difference between cumulative loss of the designed controller $\mathcal{A}$ and that of the compared controller $\pi \in \Pi$, in which the comparator could be chosen with complete foreknowledge of the disturbance and loss functions within the compared policy class Π. A variety of control algorithms have been proposed to optimize the measure under different settings (Agarwal et al., 2019; Hazan et al., 2020; Simchowitz et al., 2020; Cassel and Koren, 2020; Gradu et al., 2020a; Foster and Simchowitz, 2020). However, we argue that competing with a fixed controller may be not appropriate, especially because the unknown disturbance and cost functions can change arbitrarily in the non-stochastic control setting so that the optimal controller of each round would also change accordingly. Therefore, it is necessary to facilitate the online controller with capability of competing with *time-varying* controllers to adapt to those changes. To this end, we generalize the standard measure (10) to the *dynamic policy regret*,

$$\text{D-Regret}_T = J_T(\mathcal{A}) - J_T(\pi_1, \ldots, \pi_T) = \sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}), \tag{11}$$

to benchmark the algorithm with a sequence of *time-varying* controllers $\pi_1, \ldots, \pi_T$ from a certain controller class Π. The measure subsumes static policy regret (10) when choosing compared controllers as a fixed one. In this work, the benchmark set Π is chosen as the class of the disturbance-action controllers (DAC) (cf. Definition 1), which encompasses many controllers of interest.

5.2 Reduction to OCO with Memory

Following the pioneering work (Agarwal et al., 2019), we will work on the *Disturbance-Action Controller (DAC)* class, which parametrizes the executed action as a linear function of the past disturbances. By doing so, we can reduce the online non-stochastic control to OCO with memory so that the results of Section 4 can be leveraged to design robust controllers with provably dynamic policy regret guarantees.

Definition 1 (Disturbance-Action Controller, DAC)**.** A disturbance-action controller $\pi(K, M)$ with memory H is specified by a fixed matrix K (required to be strongly stable) and parameters $M = (M^{[1]}, \ldots, M^{[H]})$. At each iteration t, $\pi(K, M)$ chooses the action as a linear map of past disturbances with an offset linear controller, formally, $u_t = -Kx_t + \sum_{i-1}^{H} M^{[i]} w_{t-i}$.

For convenience, we define $w_i = 0$ for $i < 0$. The DAC policy can be implemented because the disturbance can be perfectly recovered by $w_t = x_{t+1} - Ax_t - Bu_t$ as system dynamics A and B are supposed to be known. Moreover, the dynamical state obtained by executing any DAC controller can be represented by a linear function of the parameters of the policy (Agarwal et al., 2019).

Proposition 4. *Suppose the initial state is $x_0 = 0$ and one chooses the DAC controller $\pi(K, M_t)$ at iteration t, the reaching state and the corresponding DAC control are*

$$x_t^K(M_{0:t-1}) = \sum_{i=0}^{H+t-1} \Psi_{t-1,i}^{K,t-1}(M_{0:t-1}) w_{t-1-i},$$

$$u_t^K(M_{0:t}) = \; -Kx_t^K(M_{0:t-1}) + \sum_{i=1}^{H} M_t^{[i-1]} w_{t-i},$$

where $\widetilde{A}_K = A - BK$ *and*

$$\Psi_{t,i}^{K,h}(M_{t-h:t}) = \widetilde{A}_K^i \mathbf{1}_{i \le h} + \sum_{j=0}^{h} \widetilde{A}_K^j B M_{t-j}^{[i-j-1]} \mathbf{1}_{1 \le i-j \le H}.$$

Evidently, both state x_t and control signal u_t are linear functions of DAC parameters $M_0, \ldots, M_t$, so the cost $c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t}))$ as a function of $M_{0:t}$ is convex. The problem is reminiscent of online convex optimization with memory (Anava et al., 2015). However, there is one big caveat in applying the technique—the current memory length is not fixed but growing with time, which is not feasible in the OCO with memory setting. A truncated method is proposed by Agarwal et al. (2019) to address the issue, which truncates the state with a fixed memory length H and thereby defines the truncated loss.

Definition 2 (Truncated Loss)**.** The truncated loss $f_t : \mathcal{M}^{H+2} \mapsto \mathbb{R}$ is defined as

$$f_t(M_{t-1-H:t}) = c_t(y_t^K(M_{t-1-H:t-1}), v_t^K(M_{t-1-H:t})), \tag{12}$$

where truncated state y_t and truncated DAC control v_t are

$$y_{t+1}^K(M_{t-H:t}) = \sum_{i=0}^{2H} \Psi_{t,i}^{K,H}(M_{t-H:t}) w_{t-i},$$

$$v_{t+1}^K(M_{t-H:t+1}) = \; -Ky_{t+1}(M_{t-H:t}) + \sum_{i=1}^{H} M_{t+1}^{[i-1]} w_{t+1-i}.$$

The truncated loss f_t is then fed to the OCO with memory framework with a memory length of $H+2$. Besides, the error introduced by the truncation (the gap between f_t and c_t) can be precisely controlled. As a result, we finish the reduction from online non-stochastic control to OCO with memory.

5.3 Dynamic Policy Regret of Online Non-stochastic Control

The above reduction enables us to leverage results in Section 4 to design online controllers competitive with time-varying compared policies. Our SCREAM.CONTROL algorithm combines the following two ideas:

(1) DAC parameterization for reduction: using DAC control $u_t = \pi(K, M_t)$ to parametrize the space and define the unary loss of the truncated loss, i.e., $\widetilde{f}_t : \mathcal{M} \mapsto \mathbb{R}$ with $\widetilde{f}_t(M) = f_t(M, \ldots, M)$, defined in Definition 2.

(2) Meta-expert aggregation for OCO with memory: performing SCREAM algorithm of Section 4 over the unary loss $\widetilde{f}_t$, and combining intermediate parameters $M_{t,1}, \ldots, M_{t,N}$ from all experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ to produce the final parameter M_t by the meta-algorithm.

Algorithm 2 SCREAM.CONTROL

Input: time horizon T, step size pool $\mathcal{H} = \{\eta_1, \ldots, \eta_N\}$; learning rate of meta-algorithm ε; memory length H; linear controller K; feasible set $\mathcal{M}$

Initialization: $M_1, M_2, \ldots, M_m \in \mathcal{M}$ and $M_{m,i} \in \mathcal{M}, \forall i \in [N]$; non-uniform weight $\boldsymbol{p}_{m+1} \in \Delta_N$ with $p_{m+1,i} \propto 1/(i^2+i), \forall i \in [N]$
for $t = H+1$ **to** T **do**
  Receive $M_{t,i}$ from expert $\mathcal{E}_i$ for $i \in [N]$
  Obtain the parameter $M_t = \sum_{i=1}^N p_{t,i} M_{t,i}$
  Output the DAC control $u_t = -Kx_t + \sum_{i=1}^H M_t^{[i-1]} w_{t-i}$
  Observe the cost function $c_t : \mathbb{R}^{d_x} \times \mathbb{R}^{d_u} \mapsto \mathbb{R}$ and suffer a loss of $c_t(x_t, u_t)$
  Construct the truncated state, truncated DAC control, and truncated loss via (12)
  Compute the switching-cost-regularized loss $\boldsymbol{\ell}_t \in \mathbb{R}^N$: $\ell_{t,i} = \lambda \|M_{t,i} - M_{t-1,i}\|_{\mathrm{F}} + g_t(M_{t,i})$
  Update the weight $\boldsymbol{p}_{t+1} \in \Delta_N$ according to $p_{t+1,i} \propto p_{t,i} \exp(-\varepsilon \ell_{t,i})$
  Expert-algorithm $\mathcal{E}_i$ updates the local parameter by $M_{t+1,i} = \Pi_{\mathcal{M}}[M_{t,i} - \eta_i \nabla \widetilde{f}_t(M_t)]$, where $\Pi_{\mathcal{M}}[\cdot]$ denotes the Euclidean projection
  Observe the new state x_{t+1} and calculate the disturbance $w_t = x_{t+1} - Ax_t - Bu_t$
end for

The descriptions of the SCREAM.CONTROL algorithm are summarized in Algorithm 2. Next, we introduce several common assumptions used in the literature (Agarwal et al., 2019; Hazan et al., 2020; Gradu et al., 2020a) and then present the dynamic policy regret guarantee of our proposed algorithm as well as several corollaries.

Assumption 4. The system matrices are bounded, i.e., $\|A\|_{\mathrm{op}} \leq \kappa_A$ and $\|B\|_{\mathrm{op}} \leq \kappa_B$. Besides, the disturbance w_t is bounded by W, i.e., $\|w_t\| \leq W$ holds for any $t \in [T]$.

Assumption 5. The cost $c_t(x, u)$ is convex. Further, as long as it is guaranteed that $\|x\|, \|u\| \leq D$, it holds that $|c_t(x,u)| \leq \beta D^2$, and $\|\nabla_x c_t(x,u)\|, \|\nabla_u c_t(x,u)\| \leq G_c D$.

Assumption 6. The DAC controller $\pi(K, M)$ satisfies:

(1) K is (κ, γ)-strongly stable, whose precise definition is in Definition 3 of Appendix A.2;

(2) $M \in \mathcal{M}$, $\mathcal{M} = \{M = (M^{[1]}, \ldots, M^{[H]}) \mid \|M^{[i]}\|_{\mathrm{op}} \leq \kappa_B \kappa^3 (1-\gamma)^i\}$.

Theorem 5. *Under Assumptions 4–6, we set learning rate optimally and the step size pool $\mathcal{H}$ as*

$$\mathcal{H} = \left\{ \eta_i \;\middle|\; \eta_i = 2^{i-1} \cdot \sqrt{\frac{D_f^2}{(\lambda G_f + G_f^2) T}}, i \in [N] \right\}, \tag{13}$$

where $N = \lceil \frac{1}{2} \log_2(1+T) \rceil + 1 = \mathcal{O}(\log T)$ is the number of experts, and $\lambda = (H+2)^2 L_f$. The parameters L_f, G_f, D_f are defined in Lemma 26 and only depend on the natural parameters of the linear dynamical system and the hyperparameter H. By choosing the truncated memory length $H = \Theta(\log T)$, our SCREAM.CONTROL *algorithm enjoys*

$$\sum_{t=1}^T c_t(x_t, u_t) - \sum_{t=1}^T c_t(x_t^{\pi_t}, u_t^{\pi_t}) \leq \widetilde{\mathcal{O}}\big(\sqrt{T(1+P_T)}\big), \tag{14}$$

where $\pi_1, \ldots, \pi_T \in \Pi$ is any comparator sequence from the compared DAC policy class $\Pi = \{\pi(K, M) \mid M \in \mathcal{M}\}$. The path-length P_T is the cumulative variation of compared policies, defined as $P_T = \sum_{t=2}^{T} \|M_{t-1} - M_t\|_{\mathrm{F}}$. The $\widetilde{\mathcal{O}}(\cdot)$-notation hides poly-logarithmic factors in T.

By using the system identification via random inputs developed by Hazan et al. (2020), the result can be extended to the case of unknown systems that are strongly controllable (cf. Definition 4 in Appendix C.3), with an $\widetilde{\mathcal{O}}(T^{2/3})$ regret overhead due to the identification.

Corollary 6. *Under the same assumptions of Theorem 5 except that system matrices A and B are now unknown, and suppose the systems are strongly controllable and the time horizon T is sufficiently large,* SCREAM.CONTROL *with system identification (Hazan et al., 2020, Algorithm 2) ensures that $\sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \le \widetilde{\mathcal{O}}(\sqrt{T(1 + P_T)} + T^{2/3})$ with high probability, where $\pi_1, \ldots, \pi_T \in \Pi$ is any comparator sequence from the compared DAC policy class.*

Finally, we note that our obtained dynamic policy regret bounds can recover the $\widetilde{\mathcal{O}}(\sqrt{T})$ static policy regret guarantee for non-stochastic control with known systems (Agarwal et al., 2019) as well as the $\widetilde{\mathcal{O}}(T^{2/3})$ high-probability static policy regret for non-stochastic control with unknown systems (Hazan et al., 2020).

Corollary 7. *For known systems, under the same assumptions of Theorem 5,* SCREAM.CONTROL *ensures that the static policy regret is at most $\sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^{\pi}, u_t^{\pi}) \le \widetilde{\mathcal{O}}(\sqrt{T})$. For unknown systems, under the same assumptions of Corollary 6,* SCREAM.CONTROL *with system identification ensures an $\widetilde{\mathcal{O}}(T^{2/3})$ high-probability guarantee. In above, the comparator set Π can be chosen as either the set of DAC policies or the set of strongly linear controllers.*

6. Empirical Studies

Although our paper mostly focuses on the theoretical investigation, in this section we further present some empirical studies to support our proposed algorithm. We focus on the OCO with memory setting. The standard way to tackle OCO with memory is by optimizing the upper bound of the policy regret, which consists of the vanilla regret over the unary functions and the switching cost, as explained in (3) for static policy regret and (4) for dynamic policy regret. In the empirical studies, we directly investigate the performance of different algorithms in optimizing this upper bound, i.e., the unary regret with switching cost. More specifically, we consider the following OCO with switching cost problem: at each round, the player makes a prediction $\mathbf{w}_t \in \mathcal{W}$ and the environments choose the loss function $f_t : \mathcal{W} \mapsto \mathbb{R}$. The player will then suffer a loss of $f_t(\mathbf{w}_t)$ as well as switching cost of $\|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2$, and thus the overall loss is $f_t(\mathbf{w}_t) + \lambda\|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2$ with some $\lambda > 0$ as the trade-off parameter.

Settings. We simulate the online learning scenario by the following setting: the player sequentially receives the feature of data item and then predicts its label. The data item of each round is denoted by $(\mathbf{x}_t, y_t) \in \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ is a d-dimensional ball with diameter Γ and $\mathcal{Y} \in \mathbb{R}$ is the space of real values. The time horizon is set as $T = 20000$ and the dimension is set as $d = 10$. To simulate the distribution changes, we generate the output according to $\mathbf{y}_t = \mathbf{x}_t^{\mathrm{T}} \mathbf{w}_t^* + \varepsilon_t$, where $\mathbf{w}_t^* \in \mathbb{R}^d$ is the underlying model and $\varepsilon_t \in [0, 0.1]$ is the random noise. The underlying model $\mathbf{w}_t^*$ will change every 2000 rounds, randomly sampled from a d-dimensional ball with diameter $D/2$, so there are in total $S = 10$ changes. We choose the loss function as the square loss, defined as

$f_t(\mathbf{w}) = \frac{1}{2}(\mathbf{w}^{\mathrm{T}}\mathbf{x}_t - y_t)^2$ and thus the gradient is $\nabla f_t(\mathbf{w}) = (\mathbf{w}^{\mathrm{T}}\mathbf{x}_t - y_t) \cdot \mathbf{x}_t$. The feasible set $\mathcal{W}$ is also set as d-dimensional ball with diameter $D/2$, and thus from all above settings, we know that $\|\mathbf{x}_t\|_2 \leq \Gamma$, $\|\mathbf{w}\|_2 \leq D/2$, and $\|\nabla f_t(\mathbf{w})\|_2 \leq D\Gamma^2$. We set $\Gamma = 1$ and $D = 2$, so the gradient norm is upper bounded by $G = D\Gamma^2 = 2$.

Contenders and Measure. We benchmark our proposed SCREAM algorithm with the following two algorithms: (1) OGD (Zinkevich, 2003), is the online gradient descent algorithm. The work of Anava et al. (2015) proves that this simple *static* regret minimization algorithm also enjoys a low switching cost when choosing the step size as $\eta = \mathcal{O}(1/\sqrt{T})$. (2) Ader (Zhang et al., 2018a), is the online algorithm designed in non-stationary online convex optimization. Ader is also in a meta-expert structure to optimize the dynamic regret, but the algorithm does not consider the switching cost and thus its switching cost might be very large (as analyzed in Section 4.2). We examine the performance via three measures: the overall cost $\sum_{t=1}^{T} f_t(\mathbf{w}_t) + \lambda \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2$, the cumulative loss $\sum_{t=1}^{T} f_t(\mathbf{w}_t)$, and the switching cost $\lambda \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2$. Here, we set the regularizer coefficient $\lambda = \alpha G$, where G is the gradient norm upper bound, with the purpose of matching the magnitude of cumulative loss and the switching cost. We consider three situations with different regularizer coefficients:

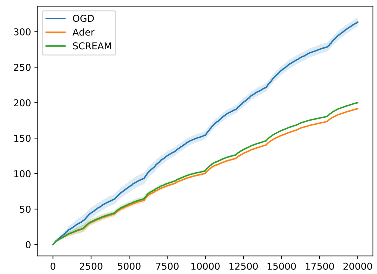
(i) small regularizer ($\alpha = 0.1$): in this case the switching cost is small so that optimizing the dynamic regret would dominate the performance;

(ii) medium regularizer ($\alpha = 0.5$): in this case the algorithm needs to have a good balance of dynamic regret and switching cost in order to behave well;

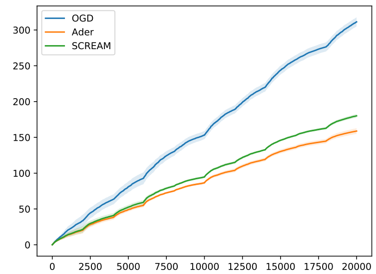
(iii) large regularizer ($\alpha = 1$): in this case dynamic regret is small so that optimizing the switching cost would dominate the performance.

We conduct experiments for five times and report mean and standard variance of different algorithms with respect to three performance measures (overall loss, cumulative loss, and switching cost).

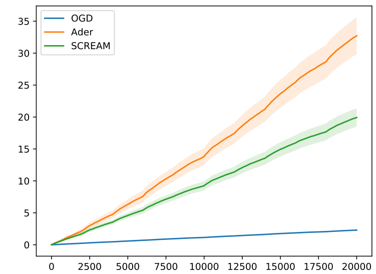
Results. Figure 1 plots performance comparisons of three algorithms (OGD, Ader, SCREAM) under different regularizer coefficients. There are in total nine sub-figures, where each row presents the performance under a particular setting of regularizer coefficient ($\alpha = 0.1, 0.5, 1$) and each column reports the performance in terms of a specific measure (overall loss, cumulative loss, and switching cost). For instance, Figure 1(d) plots the overall loss under the setting of $\lambda = \alpha G$ with $\alpha = 0.5$. Let us first focus on the measure of overall loss. From the results of overall loss (Figures 1(a), 1(d), 1(g)), we can see that under the case of small regularizer ($\alpha = 0.1$), Ader achieves the best, and SCREAM is comparable, while the performance of OGD is not desired; under the case of medium regularizer ($\alpha = 0.5$), SCREAM evidently ranks the first, whereas Ader and OGD are not well-behaved; under the case of large regularizer ($\alpha = 1$), OGD performs surprisingly well, and SCREAM is comparable, whereas the performance of Ader is not desired. The results actually accord to our theory well, especially after a further examination of corresponding cumulative loss (Figures 1(b), 1(e), 1(h)) and switching cost (Figures 1(c), 1(f), 1(i)). Indeed, we can observe that Ader focuses on optimizing the dynamic regret (i.e., cumulative loss) but fails to control the switching cost; and OGD indeed yields a sequence slow-moving decisions but it fails to optimize the dynamic regret. Consequently, under the case of small regularizer, one can optimize the overall loss by simply forgetting about the switching cost, and this is why Ader could behave well in this setting.



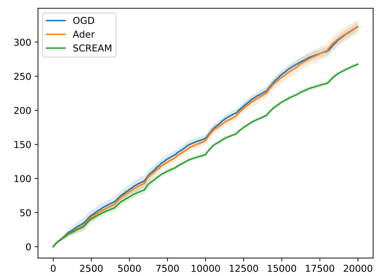
(a) overall ($\alpha = 0.1$)



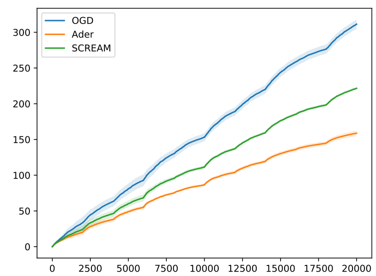
(b) cumulative loss ($\alpha = 0.1$)



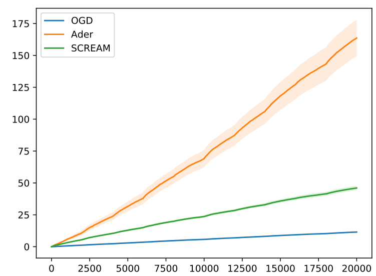
(c) switching cost ($\alpha = 0.1$)



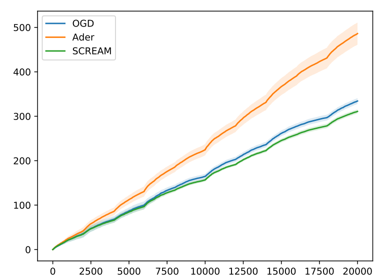
(d) overall ($\alpha = 0.5$)



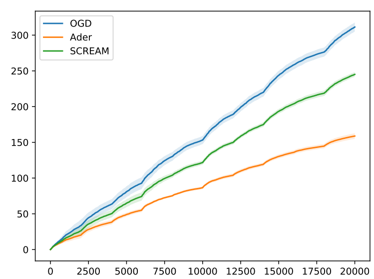
(e) cumulative loss ($\alpha = 0.5$)



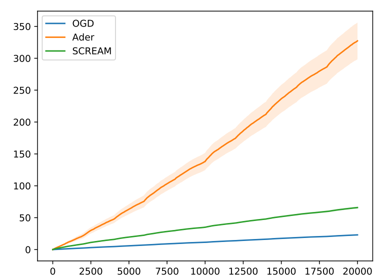
(f) switching cost ($\alpha = 0.5$)



(g) overall ($\alpha = 1$)



(h) cumulative loss ($\alpha = 1$)



(i) switching cost ($\alpha = 1$)

Figure 1: Performance comparisons of OGD, Ader, SCREAM, under different regularizer coefficients ($\lambda = \alpha G$, G is the gradient norm upper bound). The performance is evaluated by three different measures: overall loss, cumulative loss, and switching cost.

Moreover, under the case of large regularizer, the switching cost plays a more important role in the overall loss, therefore, the algorithm can optimize the overall loss by simply producing a sequence of slow-moving decisions regardless of the regret minimization, and this is why OGD could achieve a surprisingly good performance in this setting. However, under the non-degenerate settings (for example, the setting of medium regularizer in our experiments), the two compared methods behave worse and SCREAM achieves the best one. This is due to the fact that our SCREAM algorithm strikes a good balance between minimizing the dynamic regret and controlling the switching cost, owing to the novel meta-expert structure via the introduced switching-cost-regularized loss.

7. Conclusion

In this paper, we investigate the dynamic policy regret of online convex optimization with memory and online non-stochastic control. For OCO with memory, we propose the SCREAM algorithm and prove an optimal $\mathcal{O}(\sqrt{T(1+P_T)})$ dynamic policy regret, where P_T is the path-length of comparators that reflects the environmental non-stationarity. Our approach admits a structure of meta-expert aggregation to deal with the unknown environments, and introduces a novel meta-expert decomposition via switching-cost regularized surrogate loss to algorithmically address the tension between dynamic regret and switching cost. The approach is further used to design robust controllers for online non-stochastic control, where the underlying disturbance and cost functions could be chosen adversarially. We adopt the DAC parameterization and design the SCREAM.CONTROL controller that provably achieves an $\widetilde{\mathcal{O}}(\sqrt{T(1+P_T)})$ dynamic policy regret, where P_T is the path-length of compared controllers. Minimizing dynamic policy regret facilitates our controller with more robustness, since it can compete with any sequence of time-varying controllers instead of a fixed one.

In the future, we will explore the possibility of extension to *bandit* feedback, where the only feedback to the controller is the loss value (Cassel and Koren, 2020; Gradu et al., 2020a). Moreover, it would be also intriguing to investigate whether dynamic policy regret can be improved when the cost functions are *strongly convex* (Foster and Simchowitz, 2020; Baby and Wang, 2021).

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A. Preliminaries

In this section, we present the preliminaries, including the dynamic regret results of memoryless online convex optimization, additional notions, and some technical lemmas.

A.1 Dynamic Regret of Memoryless OCO

In this part we present the dynamic regret analysis of the online gradient descent (OGD) algorithm for memoryless online convex optimization (Zinkevich, 2003; Zhang et al., 2018a).

We first specify the problem settings and notations of memoryless online convex optimization. Specifically, the player iteratively selects a decision $\mathbf{w} \in \mathcal{W}$ from a convex set $\mathcal{W} \subseteq \mathbb{R}^d$ and then suffers a loss of $f_t(\mathbf{w}_t)$, in which the loss function $f_t : \mathcal{W} \mapsto \mathbb{R}$ is assumed to be convex and chosen adversarially by the environments. The performance measure we are concerned with is the *dynamic regret*, defined as

$$\text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) = \sum_{t=1}^T f_t(\mathbf{w}_t) - \sum_{t=1}^T f_t(\mathbf{v}_t),$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{W}$ is the comparator sequence arbitrarily chosen in the domain by the environments. The critical advantage of the above measure is that it supports to compete with a sequence of *time-varying* comparators, instead of a fixed one as specified in the standard (static) regret.

In the development of dynamic regret of memoryless OCO, one of the most crucial building blocks is the well-known Online Gradient Descent (OGD) algorithm (Zinkevich, 2003), which starts from any $\mathbf{w}_1 \in \mathcal{W}$ and performs the following update,

$$\mathbf{w}_{t+1} = \Pi_{\mathcal{W}}[\mathbf{w}_t - \eta \nabla f_t(\mathbf{w}_t)]. \tag{15}$$

Here, $\eta > 0$ is the step size and $\Pi_{\mathcal{W}}[\cdot]$ denotes the Euclidean projection onto the nearest point in the feasible domain $\mathcal{W}$. The standard textbooks of online convex optimization (Shalev-Shwartz, 2012; Hazan, 2016) show that OGD can achieves an optimal $\mathcal{O}(\sqrt{T})$ static regret for convex functions, providing with appropriate step size settings. Furthermore, such a simple algorithm actually also enjoys the following dynamic regret guarantee (Zinkevich, 2003, Theorem 2), and we supply the proof for self-containedness.

Theorem 8. *Let $\mathcal{W} \in \mathbb{R}^d$ be a bounded convex and compact set in Euclidean space, and we denote by D an upper bound of the diameter of the domain, i.e., $\|\mathbf{w} - \mathbf{w}'\|_2 \leq D$ holds for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$. Suppose the gradient norm of f_t over $\mathcal{W}$ is bounded by G, i.e., $\|\nabla f_t(\mathbf{w})\|_2 \leq G$ holds for any $\mathbf{w} \in \mathcal{W}$ and $t \in [T]$. Then, OGD (15) enjoys the following dynamic regret,*

$$\text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) \leq \frac{\eta}{2} G^2 T + \frac{1}{2\eta}(D^2 + 2DP_T),$$

which holds for any comparator sequence $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{W}$, and $P_T = \sum_{t=2}^T \|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2$ is the path-length that measures the cumulative movements of the comparator sequence.

Proof [of Theorem 8] Since the online functions are convex, we have

$$\text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) = \sum_{t=1}^T f_t(\mathbf{w}_t) - \sum_{t=1}^T f_t(\mathbf{v}_t) \leq \sum_{t=1}^T \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle.$$

Thus, it suffices to bound the sum of $\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle$ over iterations. Note that from the update rule in (42),

$$\begin{aligned}\|\mathbf{w}_{t+1} - \mathbf{v}_t\|_2^2 &= \|\Pi_{\mathcal{X}}[\mathbf{w}_t - \eta \nabla f_t(\mathbf{w}_t)] - \mathbf{v}_t\|_2^2 \\ &\leq \|\mathbf{w}_t - \eta \nabla f_t(\mathbf{w}_t) - \mathbf{v}_t\|_2^2 \\ &= \eta^2 \|\nabla f_t(\mathbf{w}_t)\|_2^2 - 2\eta \langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle + \|\mathbf{w}_t - \mathbf{v}_t\|_2^2\end{aligned}$$

The inequality holds due to Pythagorean theorem (Hazan, 2016, Theorem 2.1). After rearranging, we obtain

$$\langle \nabla f_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle \leq \frac{\eta}{2} \|\nabla f_t(\mathbf{w}_t)\|_2^2 + \frac{1}{2\eta} \left(\|\mathbf{w}_t - \mathbf{v}_t\|_2^2 - \|\mathbf{w}_{t+1} - \mathbf{v}_t\|_2^2 \right).$$

Summing the above inequality from $t = 1$ to T yields,

$$\text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) \leq \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(\mathbf{w}_t)\|_2^2 + \frac{1}{2\eta} \sum_{t=1}^{T} \left(\|\mathbf{w}_t - \mathbf{v}_t\|_2^2 - \|\mathbf{w}_{t+1} - \mathbf{v}_t\|_2^2 \right).$$

We further provide an upper bound for the second term in the right hand side. Indeed,

$$\begin{aligned}\sum_{t=1}^{T} \left(\|\mathbf{w}_t - \mathbf{v}_t\|_2^2 - \|\mathbf{w}_{t+1} - \mathbf{v}_t\|_2^2 \right) &\leq \sum_{t=1}^{T} \|\mathbf{w}_t - \mathbf{v}_t\|_2^2 - \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{v}_{t-1}\|_2^2 \\ &\leq \|\mathbf{w}_1 - \mathbf{v}_1\|_2^2 + \sum_{t=2}^{T} \left(\|\mathbf{w}_t - \mathbf{v}_t\|_2^2 - \|\mathbf{w}_t - \mathbf{v}_{t-1}\|_2^2 \right) \\ &= \|\mathbf{w}_1 - \mathbf{v}_1\|_2^2 + \sum_{t=2}^{T} \langle \mathbf{v}_{t-1} - \mathbf{v}_t, 2\mathbf{w}_t - \mathbf{v}_{t-1} - \mathbf{v}_t \rangle \\ &\leq D^2 + 2D \sum_{t=2}^{T} \|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2.\end{aligned}$$

Combining all above inequalities, we have

$$\begin{aligned}\text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) &\leq \frac{\eta}{2} \sum_{t=1}^{T} \|\nabla f_t(\mathbf{w}_t)\|_2^2 + \frac{1}{2\eta} \left(D^2 + 2D \sum_{t=2}^{T} \|\mathbf{v}_{t-1} - \mathbf{v}_t\|_2 \right) \\ &\leq \frac{\eta}{2} G^2 T + \frac{1}{2\eta} (D^2 + 2DP_T).\end{aligned}$$

Hence, we complete the proof. ■

A.2 Additional Notions

We introduce the formal definition of strongly stable linear controllers (Cohen et al., 2018; Agarwal et al., 2019). Indeed, the stable condition can guarantee the convergence, but nothing can be ensured about the rate of convergence. While working on the class of strongly stable controllers, we can establish the non-asymptotic convergence rate.

Definition 3. A linear controller K is (κ, γ)-strongly stable if there exist matrices L, H satisfying $A - BK = HLH^{-1}$, such that the following two conditions are satisfied:

(1) The spectral norm of L satisfies $\|L\| \leq 1 - \gamma$.

(2) The controller and transforming matrices are bounded, i.e., $\|K\| \leq \kappa$ and $\|H\|, \|H^{-1}\| \leq \kappa$.

A.3 Technical Lemmas

The following lemma plays an important role in analyzing algorithms based on the mirror descent.

Lemma 9 (Lemma 3.2 of Chen and Teboulle (1993))**.** *Let $\mathcal{X}$ be a convex set in a Banach space $\mathcal{B}$. Let $f : \mathcal{X} \mapsto \mathbb{R}$ be a closed proper convex function on $\mathcal{X}$. Given a convex regularizer $\psi : \mathcal{X} \mapsto \mathbb{R}$, we denote its induced Bregman divergence by $\mathcal{D}_\psi(\cdot, \cdot)$. Then, any update of the form*

$$\mathbf{x}_k = \arg\min_{\mathbf{x} \in \mathcal{X}} \{f(\mathbf{x}) + \mathcal{D}_\psi(\mathbf{x}, \mathbf{x}_{k-1})\}$$

satisfies the following inequality

$$f(\mathbf{x}_k) - f(\mathbf{u}) \leq \mathcal{D}_\psi(\mathbf{u}, \mathbf{x}_{k-1}) - \mathcal{D}_\psi(\mathbf{u}, \mathbf{x}_k) - \mathcal{D}_\psi(\mathbf{x}_k, \mathbf{x}_{k-1})$$

for any $\mathbf{u} \in \mathcal{X}$.

Lemma 10. *If the regularizer $\psi : \mathcal{X} \mapsto \mathbb{R}$ is λ-strongly convex with respect to a norm $\|\cdot\|$, then we have the following lower bound for the induced Bregman divergence: $\mathcal{D}_\psi(\mathbf{x}, \mathbf{y}) \geq \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|$.*

The following concentration inequality is used in analyzing dynamic policy regret for non-stochastic control with unknown systems.

Lemma 11 (Azuma-Hoeffding's Inequality for Vectors (Hayes, 2005, Theorem 1.8))**.** *Suppose that $S_m = \sum_{t=1}^m X_t$ is a martingale where $X_1, \ldots, X_m$ take values in $\mathbb{R}^n$ and are such that $\mathbb{E}[X_t] = \mathbf{0}$ and $\|X_t\|_2 \leq D$ for all t, for $t > 0$. Then for every $\varepsilon > 0$,*

$$\Pr[\|S_m\|_2 \geq \varepsilon] \leq 2e^2 e^{-\frac{\varepsilon^2}{2mD^2}}.$$

B. Omitted Details for Section 4 (OCO with Memory)

In this section, we present omitted details for Section 4 OCO with memory, including proofs of Theorem 2 (in Section B.1) and Theorem 3 (in Section B.4). Moreover, we provide the proof of the switching cost decomposition (5) in Section B.2 and supply more details for the online mirror descent in Section B.3.

B.1 Proof of Theorem 2

Proof The coordinate-Lipschitz continuity of f_t (Assumption 1) implies that

$$|f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \widetilde{f}_t(\mathbf{w}_t)| \leq L \cdot \sum_{i=1}^m \|\mathbf{w}_t - \mathbf{w}_{t-i}\|_2 \leq mL \sum_{i=1}^m \|\mathbf{w}_{t-i+1} - \mathbf{w}_{t-i}\|_2$$

Therefore, we have

$$\sum_{t=m}^{T} f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \sum_{t=m}^{T} \widetilde{f}_t(\mathbf{w}_t) \le m^2 L \sum_{t=m}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2, \tag{16}$$

and the dynamic policy regret can be thus upper bounded by

$$\begin{aligned} \text{D-Regret}_T(\mathbf{v}_1, \ldots, \mathbf{v}_T) &= \sum_{t=1}^{T} f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \sum_{t=1}^{T} f_t(\mathbf{v}_{t-m}, \ldots, \mathbf{v}_t) \\ &\overset{(16)}{\le} \underbrace{\sum_{t=1}^{T} \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}_t)}_{\texttt{dynamic regret over unary loss}} + \underbrace{\lambda \sum_{t=1}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2}_{\texttt{switching cost of decisions}} + \underbrace{\lambda \sum_{t=1}^{T} \|\mathbf{v}_t - \mathbf{v}_{t-1}\|_2}_{\texttt{switching cost of comparators}}, \end{aligned} \tag{17}$$

where we define $\lambda := m^2 L$ for notational convenience. Note that the first term is the dynamic regret over the unary loss, which is optimized by OGD over the unary loss. Since the sequence of unary loss $\{\widetilde{f}_t\}_{t=1}^T$ is convex and *memoryless*, from the standard dynamic regret analysis (Zinkevich, 2003; Zhang et al., 2018a), as shown in Theorem 8, we know that

$$\sum_{t=1}^{T} \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}_t) \le \frac{\eta}{2} G^2 T + \frac{1}{2\eta}(D^2 + 2DP_T), \tag{18}$$

where $P_T = \sum_{t=2}^T \|\mathbf{v}_t - \mathbf{v}_{t-1}\|_2$ is the path-length measuring the fluctuation of the comparator sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_T$. Next, the last term of (17) is the switching cost of the comparators, which is exactly the path-length λP_T.

So we only need to further examine the switching cost of the decisions, i.e., $\sum_{t=2}^T \|\mathbf{w}_{t-1} - \mathbf{w}_t\|_2$, as well as the dynamic regret over the unary loss, i.e., $\sum_{t=1}^T \widetilde{f}_t(\mathbf{w}_t) - \widetilde{f}_t(\mathbf{v}_t)$. By the non-expansive property of the projection operator, we can derive an upper bound for the switching cost:

$$\sum_{t=1}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 = \sum_{t=1}^{T} \|\Pi_{\mathcal{W}}[\mathbf{w}_{t-1} - \eta \mathbf{g}_{t-1}] - \mathbf{w}_{t-1}\|_2 \le \eta \sum_{t=1}^{T} \|\mathbf{g}_{t-1}\|_2 \le \eta G T. \tag{19}$$

Combining above two inequalities (19) and (18) yields

$$\sum_{t=1}^{T} f_t(\mathbf{w}_{t-m}, \ldots, \mathbf{w}_t) - \sum_{t=1}^{T} f_t(\mathbf{v}_{t-m}, \ldots, \mathbf{v}_t) \le \frac{\eta}{2}(G^2 + 2\lambda G)T + \frac{1}{2\eta}(D^2 + 2DP_T) + \lambda P_T,$$

with $\lambda = m^2 L$. We thus compete the proof. ■

B.2 Proof of Switching Cost Decomposition

The following lemma restates the switching cost decomposition presented in (5) of the main paper.

Lemma 12. *The switching cost of meta-expert outputs can be upper bounded in the following way:*

$$\sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \le D \sum_{t=2}^{T} \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \sum_{t=2}^{T} \sum_{i=1}^{N} p_{t,i} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2.$$

Proof [of Lemma 12] By the meta-expert structure, the final decision of each round is $\mathbf{w}_t = \sum_{i=1}^N p_{t,i}\mathbf{w}_{t,i}$. Therefore, we can expand the switching cost of the final prediction sequence as

$$\begin{aligned}
\|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 &= \left\| \sum_{i=1}^N p_{t,i}\mathbf{w}_{t,i} - \sum_{i=1}^N p_{t-1,i}\mathbf{w}_{t-1,i} \right\|_2 \\
&\leq \left\| \sum_{i=1}^N p_{t,i}\mathbf{w}_{t,i} - \sum_{i=1}^N p_{t,i}\mathbf{w}_{t-1,i} \right\|_2 + \left\| \sum_{i=1}^N p_{t,i}\mathbf{w}_{t-1,i} - \sum_{i=1}^N p_{t-1,i}\mathbf{w}_{t-1,i} \right\|_2 \\
&\leq \sum_{i=1}^N p_{t,i}\|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 + D\sum_{i=1}^N |p_{t,i} - p_{t-1,i}| \\
&= \sum_{i=1}^N p_{t,i}\|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 + D\|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1, \qquad (20)
\end{aligned}$$

where the first inequality holds due to the triangle inequality and the second inequality is true owing to the boundedness of the feasible domain (Assumption 3). Hence, we complete the proof. ■

B.3 Additional Results for Online Mirror Descent

In this section, we present additional results and descriptions for Online Mirror Descent (OMD), which enables a unified view for algorithmic design of both meta-algorithm and expert-algorithm.

Consider the standard online convex optimization setting, and the sequence of online convex functions are $\{h_t\}_{t=1,\dots,T}$ with $h_t : \mathcal{W} \mapsto \mathbb{R}$. Online mirror descent starts from any $\mathbf{w}_1 \in \mathcal{W}$, and at iteration t, the algorithm performs the following update:

$$\mathbf{w}_{t+1} = \arg\min_{\mathbf{w}\in\mathcal{W}} \eta\langle \nabla h_t(\mathbf{w}_t), \mathbf{w}\rangle + \mathcal{D}_\psi(\mathbf{w}, \mathbf{w}_t), \qquad (21)$$

where $\eta > 0$ is the step size. The regularizer $\psi : \mathcal{W} \mapsto \mathbb{R}$ is a differentiable convex function defined on $\mathcal{W}$ and is assumed (without loss of generality) to be 1-strongly convex w.r.t. some norm $\|\cdot\|$ over $\mathcal{W}$. The induced Bregman divergence $\mathcal{D}_\psi$ is defined by $\mathcal{D}_\psi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}) - \psi(\mathbf{y}) - \langle \nabla\psi(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle$.

The following generic result gives an upper bound of dynamic regret with switching cost of OMD, which can be regarded as a generalization of Theorem 2 from gradient descent (for Euclidean norm) to mirror descent (for general primal-dual norm).

Theorem 13. *Online Mirror Descent (21) satisfies that*

$$\sum_{t=1}^T h_t(\mathbf{w}_t) - \sum_{t=1}^T h_t(\mathbf{v}_t) + \lambda\sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \leq \frac{1}{\eta}\left(R^2 + \gamma P_T\right) + \eta(\lambda G + G^2)T, \qquad (22)$$

provided that $\mathcal{D}_\psi(\mathbf{x}, \mathbf{z}) - \mathcal{D}_\psi(\mathbf{y}, \mathbf{z}) \leq \gamma\|\mathbf{x} - \mathbf{y}\|$ *holds for any* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{W}$. *In above,* $R^2 = \sup_{\mathbf{x},\mathbf{y}\in\mathcal{W}} \mathcal{D}_\psi(\mathbf{x}, \mathbf{y})$*, and* $G = \sup_{\mathbf{w}\in\mathcal{W}, t\in[T]} \|\nabla h_t(\mathbf{w})\|_*$. *Note that the above result holds for any comparator sequence* $\mathbf{v}_1, \dots, \mathbf{v}_T \in \mathcal{W}$.

Remark 3. The dynamic regret of Theorem 13 holds against *any* comparator sequence in the domain, in particular, we can set comparators as the best fixed decision in hindsight and thus obtain static regret with switching cost, $\sum_{t=1}^T h_t(\mathbf{w}_t) - \sum_{t=1}^T h_t(\mathbf{w}^*) + \lambda\sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\| \leq$

$R^2/\eta + \eta(\lambda G + G^2)T$, that holds for any $\mathbf{w}^* \in \mathcal{W}$. A technical caveat is that when deriving the static regret, the Bregman divergence is not required to satisfy the Lipschitz condition.

Theorem 13 exhibits general dynamic regret analysis for OMD algorithm. By flexibly choosing the regularizer ψ and comparator sequence $\mathbf{v}_1, \dots, \mathbf{v}_T$, we can obtain the following two implications, which corresponds to expert-regret (dynamic regret with switching cost of OGD) and meta-regret (static regret with switching cost of Hedge).

Before presenting the proof of Theorem 13, we first analyze the switching cost of the online mirror descent, as demonstrated in the following stability lemma.

Lemma 14. *For Online Mirror Descent (21), the instantaneous switching cost is at most*

$$\|\mathbf{w}_t - \mathbf{w}_{t+1}\| \leq \eta \|\nabla h_t(\mathbf{w}_t)\|_*. \tag{23}$$

Proof [Proof of Lemma 14] From the update procedure of OMD (21) and Lemma 9, we know that

$$\langle \mathbf{w}_{t+1} - \mathbf{w}_t, \eta \nabla h_t(\mathbf{w}_t) \rangle \leq \mathcal{D}_\psi(\mathbf{w}_t, \mathbf{w}_t) - \mathcal{D}_\psi(\mathbf{w}_t, \mathbf{w}_{t+1}) - \mathcal{D}_\psi(\mathbf{w}_{t+1}, \mathbf{w}_t),$$

which implies

$$\mathcal{D}_\psi(\mathbf{w}_t, \mathbf{w}_{t+1}) + \mathcal{D}_\psi(\mathbf{w}_{t+1}, \mathbf{w}_t) \leq \langle \mathbf{w}_t - \mathbf{w}_{t+1}, \eta \nabla h_t(\mathbf{w}_t) \rangle.$$

Since the regularizer ψ is chosen as a 1-strongly convex function with respect to the norm $\|\cdot\|$, by Lemma 10 we have

$$\mathcal{D}_\psi(\mathbf{w}_t, \mathbf{w}_{t+1}) + \mathcal{D}_\psi(\mathbf{w}_{t+1}, \mathbf{w}_t) \geq \|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2.$$

Combining above two inequalities and further applying the Hölder's inequality, we obtain that

$$\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 \leq \langle \mathbf{w}_t - \mathbf{w}_{t+1}, \eta \nabla h_t(\mathbf{w}_t) \rangle \leq \|\mathbf{w}_t - \mathbf{w}_{t+1}\| \|\eta \nabla h_t(\mathbf{w}_t)\|_*.$$

Therefore, we conclude that $\|\mathbf{w}_t - \mathbf{w}_{t+1}\| \leq \eta \|\nabla h_t(\mathbf{w}_t)\|_*$ and finish the proof. ∎

Based on the above stability lemma, we can now prove Theorem 13 regarding dynamic regret with switching cost for OMD.

Proof [of Theorem 13] Notice that the dynamic regret can be decomposed in the following way:

$$\begin{aligned}\sum_{t=1}^T h_t(\mathbf{w}_t) - \sum_{t=1}^T h_t(\mathbf{v}_t) &\leq \sum_{t=1}^T \langle \nabla h_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle \\ &= \underbrace{\sum_{t=1}^T \langle \nabla h_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{w}_{t+1} \rangle}_{\texttt{term (a)}} + \underbrace{\sum_{t=1}^T \langle \nabla h_t(\mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{v}_t \rangle}_{\texttt{term (b)}}.\end{aligned}$$

From Lemma 14 and Hölder's inequality, we have

$$\texttt{term (a)} \leq \sum_{t=1}^T \|\nabla h_t(\mathbf{w}_t)\|_* \|\mathbf{w}_t - \mathbf{w}_{t+1}\| \leq \eta \sum_{t=1}^T \|\nabla h_t(\mathbf{w}_t)\|_*^2. \tag{24}$$

Next, we investigate the term (b):

$$
\begin{aligned}
\texttt{term (b)} &\le \frac{1}{\eta}\sum_{t=1}^{T}\left(\mathcal{D}_{\psi}(\mathbf{v}_t,\mathbf{w}_t)-\mathcal{D}_{\psi}(\mathbf{v}_t,\mathbf{w}_{t+1})-\mathcal{D}_{\psi}(\mathbf{w}_{t+1},\mathbf{w}_t)\right)\\
&\le \frac{1}{\eta}\sum_{t=2}^{T}\left(\mathcal{D}_{\psi}(\mathbf{v}_t,\mathbf{w}_t)-\mathcal{D}_{\psi}(\mathbf{v}_{t-1},\mathbf{w}_t)\right)+\mathcal{D}_{\psi}(\mathbf{v}_1,\mathbf{w}_1)\\
&\le \frac{\gamma}{\eta}\sum_{t=2}^{T}\|\mathbf{v}_t-\mathbf{v}_{t-1}\|+\frac{1}{\eta}R^2, \qquad (25)
\end{aligned}
$$

where the first inequality holds due to Lemma 9, and the second inequality makes uses of the non-negativity of the Bregman divergence. The last inequality holds due to the assumption of Lipschitz property that $\mathcal{D}_{\psi}(\mathbf{x},\mathbf{z})-\mathcal{D}_{\psi}(\mathbf{y},\mathbf{z})\le\gamma\|\mathbf{x}-\mathbf{y}\|$ holds for any $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{W}$.

Furthermore, the switching cost can be bounded by Lemma 14,

$$
\sum_{t=2}^{T}\|\mathbf{w}_t-\mathbf{w}_{t-1}\|\le\eta\sum_{t=2}^{T}\|\nabla h_{t-1}(\mathbf{w}_{t-1})\|_*. \qquad (26)
$$

Combining (24), (25), and (26), we can attain that

$$
\begin{aligned}
&\lambda\sum_{t=2}^{T}\|\mathbf{w}_t-\mathbf{w}_{t-1}\|+\sum_{t=1}^{T}h_t(\mathbf{w}_t)-\sum_{t=1}^{T}h_t(\mathbf{v}_t)\\
\le{}&\frac{1}{\eta}(R^2+\gamma P_T)+\eta\sum_{t=1}^{T}(\lambda\|\nabla h_t(\mathbf{w}_t)\|_*+\|\nabla h_{t-1}(\mathbf{w}_{t-1})\|_*^2)\\
\le{}&\frac{1}{\eta}(R^2+\gamma P_T)+\eta(\lambda G+G^2)T,
\end{aligned}
$$

which finishes the proof. ■

As we mentioned earlier, Theorem 2 can be regarded as a corollary of Theorem 13, by specifying the Euclidean norm and $\psi(\mathbf{w})=\frac{1}{2}\|\mathbf{w}\|_2^2$. We give a formal statement in the following corollary.

Corollary 15. *Setting the ℓ_2 regularizer $\psi(\mathbf{w})=\frac{1}{2}\|\mathbf{w}\|_2^2$ and step size $\eta>0$ for OMD, suppose $\|\nabla\widetilde{f}_t(\mathbf{w})\|_2\le G$ and $\|\mathbf{w}-\mathbf{w}'\|_2\le D$ hold for all $\mathbf{w} in \mathcal{W}$ and $t\in[T]$, then we have*

$$
\lambda\sum_{t=2}^{T}\|\mathbf{w}_t-\mathbf{w}_{t-1}\|_2+\sum_{t=1}^{T}\widetilde{f}_t(\mathbf{w}_t)-\sum_{t=1}^{T}\widetilde{f}_t(\mathbf{v}_t)\le(G^2+\lambda G)\eta T+\frac{1}{2\eta}(D^2+2DP_T), \qquad (27)
$$

which holds for any comparator sequence $\mathbf{v}_1,\dots,\mathbf{v}_T\in\mathcal{W}$, and $P_T=\sum_{t=2}^{T}\|\mathbf{v}_{t-1}-\mathbf{v}_t\|_2$ is the path-length that measures the cumulative movements of the comparator sequence.

Further, we present a corollary regarding the static regret with switching cost for the meta-algorithm, which is essentially a specialization of OMD algorithm by setting the negative-entropy regularizer.

Corollary 16. *Setting the negative-entropy regularizer* $\psi(\boldsymbol{p}) = \sum_{i=1}^N p_i \log p_i$ *and learning rate* $\varepsilon > 0$ *for OMD, suppose* $\|\boldsymbol{\ell}_t\|_\infty \le G$ *holds for any* $t \in [T]$ *and the algorithm starts from the initial weight* $p_1 \in \Delta_N$*, then we have*

$$\lambda \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \sum_{t=1}^T \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \sum_{t=1}^T \ell_{t,i} \le \frac{\ln(1/p_{1,i})}{\varepsilon} + \varepsilon(\lambda G + G^2)T. \tag{28}$$

Proof [Proof of Corollary 16] From the proof of Theorem 13, we can easily obtain that

$$\lambda \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \sum_{t=1}^T \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \sum_{t=1}^T \ell_{t,i} \le \frac{\mathcal{D}_\psi(\mathbf{e}_i, \boldsymbol{p}_1)}{\varepsilon} + \varepsilon(\lambda G + G^2)T.$$

When choosing the negative-entropy regularizer, the induced Bregman divergence becomes Kullback-Leibler divergence, i.e., $\mathcal{D}_\psi(\boldsymbol{q}, \boldsymbol{p}) = \mathrm{KL}(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^N q_i \ln(q_i/p_i)$. Therefore, $\mathcal{D}_\psi(\boldsymbol{e}_i, \boldsymbol{p}_1) = \ln(1/p_{1,i})$, which implies the desired result. ∎

B.4 Proof of Theorem 3

Proof As indicated in (17), the dynamic policy regret can be upper bounded by three terms, including dynamic regret over the unary regret, switching cost of decisions, and switching cost of comparators. The third term is essentially the path-length of the comparators, and we focus on the first two terms.

$$\begin{aligned}
&\sum_{t=1}^T \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \widetilde{f}_t(\mathbf{v}_t) + \lambda \sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \\
\overset{(5)}{\le} &\sum_{t=1}^T \langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w}_t - \mathbf{v}_t \rangle + \lambda D \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \lambda \sum_{t=2}^T \sum_{i=1}^N p_{t,i} \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \\
= &\sum_{t=1}^T \sum_{i=1}^N p_{t,i} \Big(\langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w}_{t,i} \rangle + \lambda \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \Big) - \sum_{t=1}^T \Big(\langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w}_{t,i} \rangle + \lambda \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \Big) \\
&\quad + \lambda D \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 + \sum_{t=1}^T \Big(\langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w}_{t,i} \rangle - \langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{v}_t \rangle \Big) + \lambda \sum_{t=2}^T \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \\
= &\underbrace{\sum_{t=1}^T \big(\langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \ell_{t,i} \big) + \lambda D \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1}_{\texttt{meta-regret}} + \underbrace{\sum_{t=1}^T \big(g_t(\mathbf{w}_{t,i}) - g_t(\mathbf{v}_t) \big) + \lambda \sum_{t=2}^T \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2}_{\texttt{expert-regret}}.
\end{aligned}$$

where the last step uses the convexity of $\widetilde{f}_t$ and the definition of linearized loss $g_t(\mathbf{w}) = \langle \nabla \widetilde{f}_t(\mathbf{w}_t), \mathbf{w} \rangle$. We will formally prove that our proposed algorithm optimizes the right hand side.

Bounding meta-regret. Denote by $\mathbf{e}_i$ the i-th standard basis of $\mathbb{R}^N$-space and by $\lambda' = \lambda D$ for simplicity. Since the meta-algorithm actually performs Hedge over the switching-cost-regularized loss $\boldsymbol{\ell}_t \in \mathbb{R}^N$, Corollary 16 implies that for any $i \in [N]$,

$$\sum_{t=1}^T \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \sum_{t=1}^T \ell_{t,i} + \lambda' \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 \le \varepsilon(\lambda' G_{\text{meta}} + G_{\text{meta}}^2)T + \frac{\mathcal{D}_\psi(\mathbf{e}_i, \boldsymbol{p}_1)}{\varepsilon}$$

$$
\begin{aligned}
&= \varepsilon(2\lambda+G)(\lambda+G)D^2T + \frac{\ln(1/p_{1,i})}{\varepsilon} \\
&\le \varepsilon(2\lambda+G)(\lambda+G)D^2T + \frac{2\ln(i+1)}{\varepsilon}.
\end{aligned}
$$

It can be verified that $G_{\text{meta}} = \max_{t\in[T]}\|\boldsymbol{\ell}_t\|_\infty \le (\lambda+G)D$. Moreover, the last step holds because we adopt a *non-uniform* weight initialization with the initial weight $\boldsymbol{p}_1 \in \Delta_N$ set as $p_{1,i} = \frac{1}{i(i+1)} \cdot \frac{N+1}{N}$ for any $i \in [N]$. By choosing the learning rate as $\varepsilon = \varepsilon^* = \sqrt{\frac{2}{(2\lambda+G)(\lambda+G)D^2T}}$, we can obtain the following upper bound for the meta-regret,

$$
\sum_{t=1}^T \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t\rangle - \sum_{t=1}^T \ell_{t,i} + \lambda' \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1 \le D\sqrt{2(2\lambda+G)(\lambda+G)T}(1+\ln(i+1)). \quad (29)
$$

Note that the dependence of learning rate tuning on T can be removed by either a time-varying tuning or doubling trick.

Bounding expert-regret. As specified by our algorithm, there are multiple experts, each performing OGD over the linearized loss with a particular step size $\eta_i \in \mathcal{H}$ for expert $\mathcal{E}_i$:

$$
\mathbf{w}_{t+1,i} = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla g_t(\mathbf{w}_{t,i})] = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_i \nabla \widetilde{f}_t(\mathbf{w}_t)].
$$

As a result, Theorem 13 implies that the expert-regret satisfies that

$$
\sum_{t=1}^T \big(g_t(\mathbf{w}_{t,i}) - g_t(\mathbf{v}_t)\big) + \lambda \sum_{t=2}^T \|\mathbf{w}_{t,i} - \mathbf{w}_{t-1,i}\|_2 \le (G^2+\lambda G)\eta_i T + \frac{1}{2\eta_i}(D^2 + 2DP_T), \quad (30)
$$

which holds for any comparator sequence $\mathbf{v}_1, \ldots, \mathbf{v}_T \in \mathcal{W}$ as well as any expert $i \in [N]$.

Bounding overall dynamic regret. Due to the boundedness of the path-length, we know that the optimal step size η_* provably lies in the range of $[\eta_1, \eta_N]$. Furthermore, by the construction of the pool of candidate step sizes, we can confirm that there exists an index $i^* \in [N]$ ensuring $\eta_{i^*} \le \eta_* \le \eta_{i^*+1} = 2\eta_{i^*}$. Therefore, we know that

$$
i^* \le \left\lceil \frac{1}{2}\log_2\big(1 + \frac{2P_T}{D}\big)\right\rceil + 1. \quad (31)
$$

Notice that the meta-expert decomposition at the beginning of the proof holds for any expert index $i \in [N]$. Thus, in particular, we can choose the index i^* and achieve the following result by using the upper bounds of meta-regret (29) and expert-regret (30).

$$
\begin{aligned}
&\sum_{t=1}^T \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \widetilde{f}_t(\mathbf{v}_t) + \lambda\sum_{t=2}^T \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 \\
\le\ & \underbrace{\sum_{t=1}^T \big(\langle \boldsymbol{p}_t, \boldsymbol{\ell}_t\rangle - \ell_{t,i^*}\big) + \lambda D\sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1}_{\texttt{meta-regret}} + \underbrace{\sum_{t=1}^T \big(g_t(\mathbf{w}_{t,i^*}) - g_t(\mathbf{v}_t)\big) + \lambda\sum_{t=2}^T \|\mathbf{w}_{t,i^*} - \mathbf{w}_{t-1,i}\|_2}_{\texttt{expert-regret}} \\
\le\ & D\sqrt{2(2\lambda+G)(\lambda+G)T}(1+\ln(i^*+1)) + (G^2+\lambda G)\eta_{i^*}T + \frac{1}{2\eta_{i^*}}(D^2+2DP_T)
\end{aligned}
$$

$$\leq D\sqrt{2(2\lambda+G)(\lambda+G)T}(1+\ln(i^*+1)) + (G^2+\lambda G)\eta_* T + \frac{1}{\eta_*}(D^2+2DP_T)$$

$$\leq \underbrace{2D(\lambda+G)\sqrt{T}\left(1+\ln\left(\lceil \log_2(1+2P_T/D)\rceil + 2\right)\right)}_{\leq \mathcal{O}(\sqrt{T}(1+\log\log P_T))} + \underbrace{2\sqrt{2}\sqrt{(G^2+\lambda G)(D^2+2DP_T)T}}_{\leq \mathcal{O}(\sqrt{T(1+P_T)})}$$

$$\leq \mathcal{O}(\sqrt{T(1+P_T)}).$$

Combining the upper bound of the dynamic policy regret exhibited in (4), we can achieve that

$$\begin{aligned}\text{D-Regret}_T(\mathbf{v}_{1:T}) &\leq \lambda \sum_{t=2}^{T} \|\mathbf{w}_t - \mathbf{w}_{t-1}\|_2 + \lambda \sum_{t=2}^{T} \|\mathbf{v}_t - \mathbf{v}_{t-1}\|_2 + \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}_t) \\ &\leq \mathcal{O}(\sqrt{T(1+P_T)}) + \mathcal{O}(P_T) = \mathcal{O}(\sqrt{T(1+P_T)}),\end{aligned}$$

where the last step holds as $P_T \leq DT$ due to the boundedness of the domain. We hence complete the proof of Theorem 3. ∎

B.5 Discussion on Memory Dependence

In this part, we supplement the comparison of static regret bounds of Anava et al. (2015) and ours (Theorem 1), particularly in terms of the memory dependence.

In the work of Anava et al. (2015), the Lipschitz continuity assumption is different from ours (Assumption 1), whose formal statement is presented below.

Assumption 7 (Lipschitzness of Anava et al. (2015)). The function $f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R}$ is $\bar{L}$-Lipschitz, i.e., $|f_t(\mathbf{x}_0, \ldots, \mathbf{x}_m) - f_t(\mathbf{y}_0, \ldots, \mathbf{y}_m)| \leq \bar{L}\|(\mathbf{x}_0, \ldots, \mathbf{x}_m) - (\mathbf{y}_0, \ldots, \mathbf{y}_m)\|_2 = \bar{L}\sqrt{\sum_{i=0}^{m}\|\mathbf{x}_i - \mathbf{y}_i\|_2^2}$.

We compare this definition of Lipschitzness with the version used in our paper, namely, the coordinate-wise Lipschitzness defined in Assumption 1. Actually, their definition imposes a stronger requirement on the function than ours. Clearly, when the online function f_t satisfies $\bar{L}$-Lipschitz assumption as specified in Assumption 7, it is also $\bar{L}$-coordinate-wise Lipschitz due to the simple fact that $\sqrt{\sum_{i=0}^{m}\|\mathbf{x}_i - \mathbf{y}_i\|_2^2} \leq \sum_{i=0}^{m}\|\mathbf{x}_i - \mathbf{y}_i\|_2$. On the other hand, when the online function f_t is L-coordinate-wise Lipschitz as required by Assumption 1, we can conclude that it is Lipschitz in the sense of Assumption 7 with the Lipschitz coefficient $\bar{L} = \sqrt{m}L$, due to the following inequality (by Cauchy-Schwarz inequality) $L\sum_{i=0}^{m}\|\mathbf{x}_i - \mathbf{y}_i\|_2 \leq L\sqrt{m}\sqrt{\sum_{i=0}^{m}\|\mathbf{x}_i - \mathbf{y}_i\|_2}$.

In the following, we restate the static regret bound of Anava et al. (2015) under Assumption 7. We adapt their results to our notations to ease the understanding.

Theorem 17 (Theorem 3.1 of Anava et al. (2015)). *Under Assumptions 2, 3, and the assumption that the online functions are $\bar{L}$-Lipschitz (Assumption 7), running OGD over the unary loss achieves*

$$\sum_{t=1}^{T} f_t(\mathbf{w}_{t-m:t}) - \min_{\mathbf{v}\in\mathcal{W}} \sum_{t=1}^{T} \widetilde{f}_t(\mathbf{v}) \leq 2\eta G^2 T + \frac{2D^2}{\eta} + 2\bar{L}m^{\frac{3}{2}}\eta G T. \tag{32}$$

Setting the step size optimally yields an $\mathcal{O}(\bar{L}^{1/2}m^{3/4}\sqrt{T})$ static policy regret.

Therefore, when the online functions are only L-coordinate-wise Lipschitz as considered in this paper, applying above theorem immediately obtains an $\mathcal{O}(\bar{L}^{1/2}m^{3/4}\sqrt{T}) = \mathcal{O}((\sqrt{m}L)^{1/2}m^{3/4}\sqrt{T}) = \mathcal{O}(L^{1/2}m\sqrt{T})$, which is exactly the same with the static regret result presented in Theorem 1 even in terms of the dependence in the memory length.

C. Omitted Details for Section 5 (Online Non-stochastic Control)

In this section, we present omitted details for Section 5 online non-stochastic control, including the proofs of Proposition 4, Theorem 5, and Corollary 7.

C.1 Proof of Proposition 4 (DAC Parametrization)

We will prove the following statement that gives the state recurrence for any $h \leq t$, which is essentially a strengthened result of Proposition 4.

Proposition 18. *Suppose one chooses the DAC controller $\pi(M_t, K)$ at iteration t, the reaching state is*

$$x_{t+1} = \widetilde{A}_K^{h+1} x_{t-h} + \sum_{i=0}^{H+h} \Psi_{t,i}^{K,h}(M_{t-h:t}) w_{t-i}, \tag{33}$$

where $\widetilde{A}_K = A - BK$, and $\Psi_{t,i}^{K,h}(M_{t-h:t})$ is the transfer matrix defined as

$$\Psi_{t,i}^{K,h}(M_{t-h:t}) = \widetilde{A}_K^i \mathbf{1}_{i \leq h} + \sum_{j=0}^{h} \widetilde{A}_K^j B M_{t-j}^{[i-j-1]} \mathbf{1}_{1 \leq i-j \leq H}. \tag{34}$$

The evolving equation holds for any $h \in \{0, \ldots, t\}$.

Proof [of Proposition 18] First, by substituting the DAC policy into the dynamics equation, we have

$$\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + w_t = (A - BK)x_t + \sum_{i=1}^{H} BM_t^{[i-1]} w_{t-i} + w_t \\
&= \widetilde{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \widetilde{A}_K^j \left(\sum_{i=1}^{H} BM_{t-j}^{[i-1]} w_{t-j-i} + w_{t-j} \right) \\
&= \widetilde{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \sum_{i=1}^{H} \widetilde{A}_K^j BM_{t-j}^{[i-1]} w_{t-j-i} + \sum_{j=0}^{h} \widetilde{A}_K^j w_{t-j}.
\end{aligned}$$

Exchanging the summation index yields,

$$\begin{aligned}
\sum_{j=0}^{h} \sum_{i=1}^{H} \widetilde{A}_K^j BM_{t-j}^{[i-1]} w_{t-j-i} &= \sum_{i=1}^{H} \sum_{k=i}^{i+h} \widetilde{A}_K^{k-i} BM_{t-k+i}^{[i-1]} w_{t-k} \qquad (35) \\
&= \sum_{k=1}^{H+h} \sum_{i=k-h}^{k} \widetilde{A}_K^{k-i} BM_{t-k+i}^{[i-1]} w_{t-k} \mathbf{1}_{\{1 \leq i \leq H\}} \qquad (36)
\end{aligned}$$

$$= \sum_{k=1}^{H+h} \sum_{l=0}^{h} \widetilde{A}_K^{h-l} B M_{t+l-h}^{[l+k-h-1]} w_{t-k} \mathbf{1}_{\{1 \le l+(k-h) \le H\}} \tag{37}$$

$$= \sum_{k=1}^{H+h} \sum_{m=0}^{h} \widetilde{A}_K^{m} B M_{t-m}^{[k-m-1]} w_{t-k} \mathbf{1}_{\{1 \le k-m \le H\}} \tag{38}$$

$$= \sum_{i=1}^{H+h} \sum_{j=0}^{h} \widetilde{A}_K^{j} B M_{t-j}^{[i-j-1]} w_{t-i} \mathbf{1}_{\{1 \le i-j \le H\}}, \tag{39}$$

where (35) holds by defining a third variable $k = j + i$, and (36) is obtained by exchanging the summation index i and k and the new range of i is from inequality $i \le k \le i + h$. Moreover, (37) is obtained by another change of variable $l = i - k + h$, (38) is obtained by replacing l by $h - m$, and (39) is true by setting $i = k, j = m$. Therefore, we can obtain that

$$\begin{aligned} x_{t+1} &= \widetilde{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \sum_{i=1}^{H} \widetilde{A}_K^{j} B M_{t-j}^{[i-1]} w_{t-j-i} + \sum_{j=0}^{h} \widetilde{A}_K^{j} w_{t-j} \\ &= \widetilde{A}_K^{h+1} x_{t-h} + \sum_{i=0}^{H+h} \sum_{j=0}^{h} \widetilde{A}_K^{j} B M_{t-j}^{[i-j-1]} w_{t-i} \mathbf{1}_{\{1 \le i-j \le H\}} + \sum_{i=0}^{h} \widetilde{A}_K^{i} w_{t-i} \\ &= \widetilde{A}_K^{h+1} x_{t-h} + \sum_{i=0}^{H+h} \left(\widetilde{A}_K^{i} \mathbf{1}_{\{i \le h\}} + \sum_{j=0}^{h} \widetilde{A}_K^{j} B M_{t-j}^{[i-j-1]} \mathbf{1}_{\{1 \le i-j \le H\}} \right) w_{t-i} \end{aligned}$$

and hence complete the proof. ∎

C.2 Proof of Theorem 5

To prove the dynamic policy regret of online non-stochastic control (Theorem 5), we will first present theoretical analysis of the reduction to OCO with memory in Section C.2.1, then give the dynamic regret analysis over the $\mathcal{M}$-space in Section C.2.2, and finally present the overall proof of Theorem 5 in Section C.2.3.

C.2.1 Reduction to OCO with Memory & Approximation Theorems

In Section 5.2 of the main paper, we have presented how to reduce from online non-stochastic control to OCO with memory, by employing the DAC parameterization and introducing the truncated loss functions. In this part, we introduce the following approximation theorem that discloses that the truncation loss f_t approximates the original cost function c_t well.

Theorem 19 (Theorem 5.3 of Agarwal et al. (2019)). *Suppose the disturbance are bounded by W. For any (κ, γ)-strongly stable linear controller K, and any $\tau > 0$ such that the sequence of $M_1, \ldots, M_T$ satisfies $\|M_t^{[i]}\|_{\text{op}} \le \tau(1-\gamma)^i$, the approximation error between original loss and truncated loss is at most*

$$\left| \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} f_t(M_{t-1-H:t}) \right| \le 2TG_c D^2 \kappa^3 (1-\gamma)^{H+1}, \tag{40}$$

where

$$D := \frac{W\kappa^3(1+H\kappa_B\tau)}{\gamma(1-\kappa^2(1-\gamma)^{H+1})} + \frac{W\tau}{\gamma}. \tag{41}$$

Proof [Proof of Theorem 19] By Lipschitzness and definition of the truncated loss, we get that

$$\begin{aligned}
&c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - f_t(M_{t-H-1:t}) \\
&= c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - c_t(y_t^K(M_{t-H-1:t-1}), v_t^K(M_{t-H-1:t})) \\
&\leq G_c D \left(\|x_t^K(M_{0:t-1}) - y_t^K(M_{t-H-1:t-1})\| + \|u_t^K(M_{0:t}) - v_t^K(M_{t-H-1:t})\|\right) \\
&\leq G_c D(\kappa^2(1-\gamma)^{H+1}D + \kappa^3(1-\gamma)^{H+1}D) \\
&\leq 2G_c D^2\kappa^3(1-\gamma)^{H+1},
\end{aligned}$$

where the last two inequalities use the Lipschitzness and the boundedness presented in Lemma 25. We complete the proof by summing over the iterations from $t = 1, \ldots, T$. ■

C.2.2 Dynamic Regret Analysis over $\mathcal{M}$-space

In previous sections, we have analyzed the dynamic regret of OGD over the $\mathbb{R}^d$-space. However, after reducing online non-stochastic control to OCO with memory, we need to apply their results to the $\mathcal{M}$-space and thus require to generalize the arguments of previous sections from Euclidean norm for $\mathbb{R}^d$-space to Frobenius norm for $\mathcal{M}$-space. For completeness, we present the proof here.

At the first place, we analyze the dynamic regret of the online gradient descent (OGD) algorithm over the $\mathbb{R}^d$-space. OGD begins with any $M_1 \in \mathcal{M}$ and performs the following update procedure,

$$M_{t+1} = \Pi_{\mathcal{M}}[M_t - \eta\nabla_M \widetilde{f}_t(M_t)] \tag{42}$$

where $\eta > 0$ is the step size and $\Pi_{\mathcal{M}}[\cdot]$ denotes the projection onto the nearest point in the feasible set $\mathcal{M}$. We have the following dynamic regret regarding its dynamic regret.

Theorem 20. *Suppose the function $\widetilde{f} : \mathcal{M} \mapsto \mathbb{R}$ is convex; the gradient norm $\|\nabla_M \widetilde{f}_t(M)\|_{\mathrm{F}} \leq G_f$ holds for any $M \in \mathcal{M}$ and $t \in [T]$; and the Euclidean diameter of $\mathcal{M}$ is at most D_f, i.e., $\sup_{M,M'\in\mathcal{M}}\|M - M'\|_{\mathrm{F}} \leq D_f$. Then, OGD with a step size $\eta > 0$ as shown in (42) satisfies that*

$$\lambda\sum_{t=2}^{T}\|M_{t-1} - M_t\|_{\mathrm{F}} + \sum_{t=1}^{T}\widetilde{f}_t(M_t) - \sum_{t=1}^{T}\widetilde{f}_t(M_t^*) \leq \frac{\eta}{2}(G_f^2 + 2\lambda G_f)T + \frac{1}{2\eta}(D_f^2 + 2D_f P_T), \tag{43}$$

which holds for any comparator sequence $M_1, \ldots, M_T \in \mathcal{M}$. Besides, the path-length $P_T = \sum_{t=2}^{T}\|M_{t-1}^* - M_t^*\|_{\mathrm{F}}$ measures the non-stationarity of the comparator sequence.

Proof [of Theorem 20] Denote the gradient by $G_t = \nabla_M \widetilde{f}_t(M_t)$. The convexity of online surrogate loss functions implies that

$$\sum_{t=1}^{T}\widetilde{f}_t(M_t) - \sum_{t=1}^{T}\widetilde{f}_t(M_t^*) \leq \sum_{t=1}^{T}\langle G_t, M_t - M_t^*\rangle.$$

Thus, it suffices to bound the sum of $\langle G_t, M_t - M_t^* \rangle$. From the OGD update rule and the non-expensive property, we have

$$\begin{aligned}\|M_{t+1} - M_t^*\|_{\mathrm{F}}^2 = &\ \|\Pi_{\mathcal{M}}[M_t - \eta G_t] - M_t^*\|_{\mathrm{F}}^2 \leq \|M_t - \eta G_t - M_t^*\|_{\mathrm{F}}^2 \\ &= \eta^2 \|G_t\|_{\mathrm{F}}^2 - 2\eta \langle G_t, M_t - M_t^* \rangle + \|M_t - M_t^*\|_{\mathrm{F}}^2\end{aligned}$$

After rearranging, we obtain

$$\langle G_t, M_t - M_t^* \rangle \leq \frac{\eta}{2} \|G_t\|_{\mathrm{F}}^2 + \frac{1}{2\eta} \left(\|M_t - M_t^*\|_{\mathrm{F}}^2 - \|M_{t+1} - M_t^*\|_{\mathrm{F}}^2 \right).$$

Next, we turn to analyze the second term in the right hand side. Indeed,

$$\begin{aligned}&\sum_{t=1}^{T} \left(\|M_t - M_t^*\|_{\mathrm{F}}^2 - \|M_{t+1} - M_t^*\|_{\mathrm{F}}^2 \right) \\ \leq &\ \sum_{t=1}^{T} \|M_t - M_t^*\|_{\mathrm{F}}^2 - \sum_{t=2}^{T} \|M_t - M_{t-1}^*\|_{\mathrm{F}}^2 \\ \leq &\ \|M_1 - M_1^*\|_{\mathrm{F}}^2 + \sum_{t=2}^{T} \left(\|M_t - M_t^*\|_{\mathrm{F}}^2 - \|M_t - M_{t-1}^*\|_{\mathrm{F}}^2 \right) \\ = &\ \|M_1 - M_1^*\|_{\mathrm{F}}^2 + \sum_{t=2}^{T} \langle M_{t-1}^* - M_t^*, 2M_t - M_{t-1}^* - M_t^* \rangle \\ \leq &\ D_f^2 + 2D_f \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_{\mathrm{F}}.\end{aligned}$$

Hence, combining all above inequalities, we have

$$\begin{aligned}\sum_{t=1}^{T} \widetilde{f}_t(M_t) - \sum_{t=1}^{T} \widetilde{f}_t(M_t^*) &\leq \frac{\eta}{2} \sum_{t=1}^{T} \|G_t\|_{\mathrm{F}}^2 + \frac{1}{2\eta} \left(D_f^2 + 2D_f \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_{\mathrm{F}} \right) \\ &\leq \frac{\eta}{2} G_f^2 T + \frac{1}{2\eta} (D_f^2 + 2D_f P_T).\end{aligned}$$

On the other hand, the switching cost can be bounded by

$$\begin{aligned}\sum_{t=2}^{T} \|M_t - M_{t-1}\|_{\mathrm{F}} = &\ \|\Pi_{\mathcal{M}}[M_{t-1} - \eta G_{t-1}] - M_{t-1}\|_{\mathrm{F}}^2 \\ &\leq \|M_{t-1} - \eta G_{t-1} - M_{t-1}\|_{\mathrm{F}} \leq \eta G_f T,\end{aligned}$$

which together with the previous dynamic regret bound yields the desired result. ■

C.2.3 Proof of Theorem 5

Proof We begin with the following dynamic policy regret decomposition,

$$\begin{aligned}
&\sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \\
&= \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}^*), u_t^K(M_{0:t}^*)) \\
&= \underbrace{\sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} f_t(M_{t-1-H:t})}_{:=A_T} \\
&\quad + \underbrace{\sum_{t=1}^{T} f_t(M_{t-1-H:t}) - \sum_{t=1}^{T} f_t(M_{t-1-H:t}^*)}_{:=B_T} + \underbrace{\sum_{t=1}^{T} f_t(M_{t-1-H:t}^*) - \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}^*), u_t^K(M_{0:t}^*))}_{:=C_T}.
\end{aligned} \tag{44}$$

Notice that both A_T and C_T essentially represent the approximation error introduced by the truncated loss, so we can apply Theorem 19 and obtain that

$$A_T + C_T \leq 4TG_cD^2\kappa^3(1-\gamma)^{H+1}. \tag{45}$$

We now focus on the quantity B_T, which is the dynamic policy regret over the truncated loss functions $\{f_t\}_{t=1,\ldots,T}$. Indeed,

$$\begin{aligned}
B_T &= \sum_{t=1}^{T} f_t(M_{t-1-H:t}) - \sum_{t=1}^{T} f_t(M_{t-1-H:t}^*) \\
&\leq \sum_{t=1}^{T} \widetilde{f}_t(M_t) - \sum_{t=1}^{T} \widetilde{f}_t(M_t^*) + \lambda\sum_{t=2}^{T}\|M_{t-1} - M_t\|_{\mathrm{F}} + \lambda\sum_{t=2}^{T}\|M_{t-1}^* - M_t^*\|_{\mathrm{F}} \\
&\leq \sum_{t=1}^{T}\langle\nabla_M \widetilde{f}_t(M_t), M_t - M_t^*\rangle + \lambda\sum_{t=2}^{T}\|M_{t-1} - M_t\|_{\mathrm{F}} + \lambda\sum_{t=2}^{T}\|M_{t-1}^* - M_t^*\|_{\mathrm{F}} \\
&= \sum_{t=1}^{T} g_t(M_t) - \sum_{t=1}^{T} g_t(M_t^*) + \lambda\sum_{t=2}^{T}\|M_{t-1} - M_t\|_{\mathrm{F}} + \lambda\sum_{t=2}^{T}\|M_{t-1}^* - M_t^*\|_{\mathrm{F}},
\end{aligned} \tag{46}$$

where $\lambda = (H+2)^2 L_f$ and $g_t(M) = \langle\nabla_M \widetilde{f}_t(M_t), M\rangle$ is the surrogate linearized loss. As a consequence, we are reduced to proving an dynamic regret over the sequence of functions $\{g_t\}_{t=1,\ldots,T}$ with switching cost, namely, the first three terms in the right hand side. We thus make use of the techniques developed in Section B.4 (dynamic policy regret minimization for OCO with memory) to decompose the terms into meta-regret and expert-regret:

$$\sum_{t=1}^{T} g_t(M_t) - \sum_{t=1}^{T} g_t(M_t^*) + \lambda\sum_{t=2}^{T}\|M_{t-1} - M_t\|_{\mathrm{F}}$$

$$= \underbrace{\left(\lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_{\mathrm{F}} + \sum_{t=1}^{T} g_t(M_t)\right) - \left(\lambda \sum_{t=2}^{T} \|M_{t-1,i} - M_{t,i}\|_{\mathrm{F}} + \sum_{t=1}^{T} g_t(M_{t,i})\right)}_{\texttt{meta-regret}}$$

$$+ \underbrace{\left(\lambda \sum_{t=2}^{T} \|M_{t-1,i} - M_{t,i}\|_{\mathrm{F}} + \sum_{t=1}^{T} g_t(M_{t,i}) - \sum_{t=1}^{T} g_t(M_t^*)\right)}_{\texttt{expert-regret}}.$$

We remark that the regret decomposition holds for any expert index $i \in [N]$. We now provide the upper bounds for the meta-regret and expert-regret, respectively. First, Theorem 20 ensures the expert-regret satisfies that

$$\texttt{expert-regret} \leq \frac{\eta_i}{2}(G_f^2 + 2\lambda G_f)T + \frac{1}{2\eta_i}(D_f^2 + 2D_f P_T),$$

where $P_T = \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_{\mathrm{F}}$ is the path-length of the comparator sequence. On the other hand, similar to Lemma 12 of Section B.2, we can show that the meta-regret satisfies that

$$\texttt{meta-regret} \leq \lambda' \sum_{t=2}^{T} \|\boldsymbol{p}_{t-1} - \boldsymbol{p}_t\|_1 + \sum_{t=1}^{T} \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \sum_{t=1}^{T} \ell_{t,i},$$

where the surrogate loss vector $\boldsymbol{\ell}_t \in \Delta_N$ of the meta-algorithm is defined as

$$\ell_{t,i} = \lambda \|M_{t-1,i} - M_{t,i}\|_{\mathrm{F}} + g_t(M_{t,i}), \text{ for } i \in [N].$$

Then, we can make use the static regret with switching cost of online mirror descent for the prediction with expert advice setting (c.f. Corollary 16 in Section B.3) and obtain that

$$\begin{aligned} \texttt{meta-regret} &\leq \varepsilon(2\lambda + G_f)(\lambda_f + G_f)D_f^2 T + \frac{\ln(1/p_{1,i})}{\varepsilon} \\ &= D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f)T}\big(1 + \ln(1+i)\big), \end{aligned}$$

where the equation can be obtained by an appropriate setting of the learning rate ε.

Since the above decomposition and the upper bounds of meta-regret and expert-regret all hold for any expert index $i \in [N]$, we will choose the best index denoted by i^* to make the regret bound tightest possible. Specifically, from the construction of the step size pool, we can ensure that there exists a step size η_{i^*} such that the optimal step size provably satisfies $\eta_{i^*} \leq \eta_* \leq 2\eta_{i^*}$. As a result, we have

$$\begin{aligned} &\sum_{t=1}^{T} g_t(M_t) - \sum_{t=1}^{T} g_t(M_t^*) + \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_{\mathrm{F}} \\ &\leq \frac{\eta_{i^*}}{2}(G_f^2 + 2\lambda G_f)T + \frac{1}{2\eta_{i^*}}(D_f^2 + 2D_f P_T) + D_f\sqrt{2(2\lambda + G_f)(\lambda + G_f)T}\big(1 + \ln(1+i)\big) \\ &\leq \frac{\eta_*}{2}(G_f^2 + 2\lambda G_f)T + \frac{1}{\eta_*}(D_f^2 + 2D_f P_T) + D_f\sqrt{2(2\lambda + G_f)(\lambda + G_f)T}\big(1 + \ln(1+i)\big) \\ &\leq \frac{3}{2}\sqrt{(G_f^2 + 2\lambda G_f)(D_f^2 + 2D_f P_T)T} + D_f\sqrt{2(2\lambda + G_f)(\lambda + G_f)T}\left(1 + \ln(\lceil \log_2(1 + 2P_T/D) \rceil + 2)\right). \end{aligned}$$

Algorithm 3 System Identification via Random Inputs (Hazan et al., 2020)

Input: rounds of exploration T_0.

1: **for** $t = 1, \ldots, T_0$ **do**

2: Execute the control $u_t = -Kx_t + \widetilde{u}_t$ with $\widetilde{u}_t \sim_{i.i.d.} \{\pm 1\}^{d_u}$

3: Record the observed state x_{t+1}

4: **end for**

5: Declare $N_j = \frac{1}{T_0-k} \sum_{t=0}^{T_0-k-1} x_{t+j+1} \widetilde{u}_t^{\mathrm{T}}$, for all $j \in [k]$

6: Define $\widehat{C}_0 = [N_0, \ldots, N_{k-1}], \widehat{C}_1 = [N_1, \ldots, N_k]$ and return estimation $\widehat{A}, \widehat{B}$ as

$$\widehat{B} = N_0, \quad \widehat{A}_K := \widehat{C}_1 \widehat{C}_0^{\mathrm{T}} \left(\widehat{C}_0 \widehat{C}_0^{\mathrm{T}}\right)^{-1}, \quad \widehat{A} = \widehat{A}_K + \widehat{B}K.$$

Combining this result with the regret decomposition (44) and the upper bounds (45), (46), we have

$$\begin{aligned}
&\sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \\
&\leq 4TG_c D^2 \kappa^3 (1-\gamma)^{H+1} + \frac{3}{2}\sqrt{(G_f^2 + 2\lambda G_f)(D_f^2 + 2D_f P_T)T} \\
&\qquad + D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f)T}\,(1 + \ln(\lceil \log_2(1 + 2P_T/D) \rceil + 2)) + \lambda P_T.
\end{aligned}$$

The specific values of D, L_f, G_f, D_f can be found in Lemma 26. By setting $H = \mathcal{O}(\log T)$, we can ensure the final dynamic policy regret is at most $\widetilde{\mathcal{O}}(\sqrt{T(1+P_T)})$ and hence complete the proof. ■

C.3 Proofs of Corollary 6 and Corollary 7

This part we present the proofs of two corollaries of the main results, that is, Corollary 6 (dynamic regret guarantees for unknown systems), and Corollary 7 (implications for static regret).

C.3.1 Proof of Corollary 6

In this part, we first add the omitted algorithmic details for non-stochastic control with unknown systems, and then present the proof of Corollary 6.

When the system is unknown, i.e., A and B are not known in advance, we follow the explore-then-commit method of Hazan et al. (2020) to identify the underlying dynamics and then deploy the control algorithm based on the estimated system dynamics. The algorithmic descriptions are summarized in Algorithm 3. In the exploration phase, the identification algorithm (Hazan et al., 2020, Algorithm 2) uses some random inputs to approximately recover the system dynamics. Specifically, given an estimation budget $T_0 < T$, in the first T_0 rounds, we input the control signal $u_t = -Kx_t + \widetilde{u}_t$ with the random inputs $\widetilde{u}_t \sim \{\pm 1\}^{d_u}$ and then observe the corresponding state x_{t+1}. Then, by the estimation method presented in Line 6 of Algorithm 3, we can show that the estimation regret overhead is $\widetilde{\mathcal{O}}(T^{2/3})$ when choosing $T_0 = \Theta(T^{2/3})$.

To give the formal regret analysis, we need the following definitions and notations.

Definition 4 (Strong Controllability)**.** For a linear dynamical system $(A, B, \{\mathbf{w}\})$ and a strongly stable linear controller K, for $k \geq 1$, define a matrix $C_k \in \mathbb{R}^{d_x \times k d_u}$ as

$$C_k = \left[B, \widetilde{A}_K B, \ldots, \widetilde{A}_K^{k-1} B\right], \tag{47}$$

where $\widetilde{A}_K = A - BK$. A linear dynamical system $(A, B, \{\mathbf{w}\})$ is controllable with controllability index k if C_k has full row-rank. In addition, such a system is also (k, κ_c)-strongly controllable if $\|\left(C_k C_k^{\mathrm{T}}\right)^{-1}\| \leq \kappa_c$.

To ensure finite-sample convergence rate, we need the following assumption of strong controllability following the work of Hazan et al. (2020).

Assumption 8 (Strong Controllability)**.** The dynamical system (9) is (k, κ_c)-strongly controllable.

Notations. We further define some notations for convenience. Define ε_w an upper bound for the gap between the true disturbance w_t and the estimated one $\widehat{w}_t$, i.e., $\|w_t - \widehat{w}_t\|_2 \leq \varepsilon_w$, and define a universal upper bound W_0 for ε_w and disturbance bound W (cf. Assumption 4) as $W, \varepsilon_w \leq W_0$. We also define $d_{\min} = \min\{d_x, d_u\}$, $\widetilde{A}_K = A - BK$, $\widehat{A}_K = \widehat{A} - \widehat{B}K$ for notational convenience.

Proof [of Corollary 6] The overall dynamic regret is at most

$$\text{D-Regret}_T \leq \underbrace{\sum_{t=1}^{T_0} c_t(x_t, u_t)}_{\texttt{term (A)}} + \underbrace{\sum_{t=T_0+1}^{T} c_t(x_t, u_t) - \sum_{t=T_0+1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t})}_{\texttt{term (B)}},$$

where term (A) is the cumulative cost during the system identification procedure and term (B) is the dynamic regret caused by SCREAM.CONTROL algorithm over the rest rounds. Note that term (A) enjoys a trivial upper bound of $\mathcal{O}(T_0)$, and term (B) can be decomposed into two parts:

$$\begin{aligned}\texttt{term (B)} = {} & \underbrace{\sum_{t=T_0+1}^{T} c_t(x_t, u_t) - \sum_{t=T_0+1}^{T} c_t(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S}))}_{\texttt{term (b-1)}} \\ & + \underbrace{\sum_{t=T_0+1}^{T} c_t(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S})) - \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(S), u_t^{\pi_t}(S)\right)}_{\texttt{term (b-2)}}.\end{aligned}$$

Here, $(x_t^{\pi_t}(S), u_t^{\pi_t}(S))$ is the state-action pair produced by the policy π_t on the true system $S = (A, B, \{w\})$, whereas $(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S}))$ is the state-action pair produced by the policy π_t on the estimated system $\widehat{S} = (\widehat{A}, \widehat{B}, \{\widehat{w}\})$. Summarizing, term (b-1) is the dynamic regret on the estimated system and term (b-2) is the gap between the cumulative cost of the true system and that of the estimated system. From Theorem 5, it holds that $\texttt{term (b-1)} \leq \widetilde{\mathcal{O}}(\sqrt{T(1+P_T)})$. From Lemma 22, we can bound term (b-2) as $\texttt{term (b-2)} \leq \mathcal{O}(\varepsilon_{A,B} T)$. Overall, *with probability* at least $1 - \delta$, the total dynamic regret is at most

$$\text{D-Regret}_T \leq \mathcal{O}(T_0) + \widetilde{\mathcal{O}}(\sqrt{T(1+P_T)}) + \mathcal{O}(\varepsilon_{A,B} T)$$

$$
\begin{aligned}
&= \mathcal{O}(\varepsilon_{A,B}^{-2} + \varepsilon_{A,B}T) + \widetilde{\mathcal{O}}(\sqrt{T(1+P_T)}) \\
&\leq \mathcal{O}(T^{2/3}) + \widetilde{\mathcal{O}}(\sqrt{T(1+P_T)}).
\end{aligned}
$$

The second step makes use of the relationship between the system identification rounds T_0 and the estimation error $\|\widehat{A}-A\|_{\text{op}}, \|\widehat{B}-B\|_{\text{op}} \leq \varepsilon_{A,B}$, as demonstrated in Lemma 21. The last step holds by setting the rounds of exploration to ensure $\varepsilon_{A,B} = \min\{10^{-3}\kappa^{-10}\gamma^2, T^{-1/3}\}$, which is realized when total time horizon is large enough, i.e., $T \geq 10^9\kappa^{30}\gamma^{-6}$. ■

The above proof relies on the two key lemmas (Lemma 21 and Lemma 22). In the following, we provide the formal statements and corresponding proofs.

Lemma 21 establishes the relationship between the estimation accuracy $\varepsilon_{A,B}$ and the number of estimation rounds T_0. This lemma is firstly due to Hazan et al. (2020) and is restated here for self-containedness.

Lemma 21 (System Recovery)**.** *Under Assumptions 4, 6, 8, when Algorithm 3 runs for T_0 rounds, if the output pair $(\widehat{A}, \widehat{B})$ satisfies, with probability at least $1-\delta$, that $\|\widehat{A}-A\|_{\text{op}}, \|\widehat{B}-B\|_{\text{op}} \leq \varepsilon_{A,B}$, then it holds that $T_0 = \mathcal{O}(\varepsilon_{A,B}^{-2})$.*

Proof [of Lemma 21] Based on the observation, we have the following two equations:

$$
\widetilde{A}_K C_k = (\widetilde{A}_K C_k), \quad \widehat{A}_K \widehat{C}_0 = \widehat{C}_1.
$$

Using Lemma 33, it holds that

$$
\|\widetilde{A}_K - \widehat{A}_K\|_{\text{op}} \leq \frac{\|\widetilde{A}_K C_k - \widehat{C}_1\|_{\text{op}} + \|C_k - \widehat{C}_0\|_{\text{op}}\|\widetilde{A}_K\|_{\text{op}}}{\sigma_{\min}(C_k) - \|C_k - \widehat{C}_0\|_{\text{op}}}. \tag{48}
$$

By Lemma 23, we know that with probability at least $1-\delta$, it holds that $\|N_j - \widetilde{A}_K^j B\|_{\text{F}} \leq \varepsilon$, where

$$
\varepsilon := 3\kappa_B\kappa^2 d_u W\gamma^{-1}\sqrt{\frac{2d_{\min}\log(2e^2k\delta^{-1})}{T_0-k}}. \tag{49}
$$

Owing to the benign high-probability guarantee, we only need to focus on the successful event, that is, under the case when $\|N_j - \widetilde{A}_K^j B\|_{\text{F}} \leq \varepsilon$ is true. We then try to bound $\|C_k - C_0\|_{\text{op}}, \|\widetilde{A}_K C_k - C_1\|_{\text{op}}$,

$$
\begin{aligned}
\|C_k - \widehat{C}_0\|_{\text{op}} \leq \|C_k - \widehat{C}_0\|_{\text{F}} &= \left\|\left[N_0 - B, \ldots, N_{k-1} - \widetilde{A}_K^{k-1}B\right]\right\|_{\text{F}} \\
&= \sqrt{\sum_{i=0}^{k-1}\|N_i - \widetilde{A}_K^i B\|_{\text{F}}^2} \leq \sqrt{k\varepsilon^2} = \varepsilon\sqrt{k},
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\|\widetilde{A}_K C_k - \widehat{C}_1\|_{\text{op}} \leq \|\widetilde{A}_K C_k - \widehat{C}_1\|_{\text{F}} &= \left\|\left[N_1 - \widetilde{A}_K B, \ldots, N_k - \widetilde{A}_K^k B\right]\right\|_{\text{F}} \\
&= \sqrt{\sum_{i=1}^{k}\|N_i - \widetilde{A}_K^i B\|_{\text{F}}^2} \leq \sqrt{k\varepsilon^2} = \varepsilon\sqrt{k}.
\end{aligned} \tag{51}
$$

Using Lemma 31 to give an upper bound of $\sigma_{\min}(C_k)$, and plugging (50) and (51) into (48), we have

$$\|\widetilde{A}_K - \widehat{A}_K\|_{\mathrm{op}} \leq \frac{\varepsilon\sqrt{k} + \varepsilon\sqrt{k}\cdot\kappa^2(1-\gamma)}{1/\sqrt{\kappa_c} - \varepsilon\sqrt{k}}.$$

The gap between A and $\widehat{A}$ can be bounded as

$$\begin{aligned}
\|A - \widehat{A}\|_{\mathrm{op}} &= \|\widetilde{A}_K + BK - \widehat{A}_K - \widehat{B}K\|_{\mathrm{op}} \\
&\leq \|\widetilde{A}_K - \widehat{A}_K\|_{\mathrm{op}} + \|K\|_{\mathrm{op}}\|B - \widehat{B}\|_{\mathrm{op}} \\
&\leq \frac{\varepsilon\sqrt{k} + \varepsilon\sqrt{k}\cdot\kappa^2(1-\gamma)}{1/\sqrt{\kappa_c} - \varepsilon\sqrt{k}} + \kappa\varepsilon \leq \frac{3\varepsilon\kappa^{5/2}}{\sqrt{1/\kappa_c} - \varepsilon\sqrt{\kappa}}.
\end{aligned}$$

If we want $\|\widehat{A} - A\|_{\mathrm{F}}, \|\widehat{B} - B\|_{\mathrm{F}} \leq \varepsilon_{A,B}$, the following equations should hold:

$$\begin{aligned}
&\|\widehat{A} - A\|_{\mathrm{F}} \leq \sqrt{d_x}\|\widehat{A} - A\|_{\mathrm{op}} \leq \sqrt{d_x}\left(\frac{3\varepsilon\kappa^{5/2}}{\sqrt{1/\kappa_c} - \varepsilon\sqrt{\kappa}}\right) := \varepsilon_A \leq \varepsilon_{A,B}, \\
&\|\widehat{B} - B\|_{\mathrm{F}} \leq \sqrt{d_{\min}}\|\widehat{B} - B\|_{\mathrm{op}} \leq \sqrt{d_{\min}}\varepsilon := \varepsilon_B \leq \varepsilon_{A,B}.
\end{aligned} \tag{52}$$

Besides, it is easy to see that $\varepsilon_B = \sqrt{d_{\min}}\varepsilon \leq \sqrt{d_x}\varepsilon \leq \varepsilon_A$, thus conditions in (52) can be simplified as $\varepsilon_A \leq \varepsilon_{A,B}$. Finally, combining the above inequality with the value of ε (c.f. (49)), we can obtain that $T_0 = \mathcal{O}(\varepsilon_{A,B}^{-2})$. ∎

Lemma 22 measures the difference of the cumulative costs of a policy between the true system and the estimated one. This result holds for both strongly stable linear controllers and non-stationary DAC policy and here we only give a proof of the latter, for the former result, we refer readers to Hazan et al. (2020, Lemma 16).

Lemma 22 (Identification Accuracy). *Under Assumptions 4-6, suppose* $\|\widehat{A} - A\|_{\mathrm{op}}, \|\widehat{B} - B\|_{\mathrm{op}} \leq \varepsilon_{A,B} \leq 0.25\kappa^{-3}\gamma$ *and let* K *be any* (κ, γ)*-strongly stable linear controller with respect to* (A, B)*. Then for any non-stationary DAC policy* $\pi_{1:T}$ *parameterized via* $M_{1:T}$*, it holds that*

$$\left|\sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S})\right) - \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(S), u_t^{\pi_t}(S)\right)\right| \leq \mathcal{O}\left(\varepsilon_{A,B}T + \varepsilon_{A,B}^2 T\right),$$

where $(x_t^{\pi_t}(S), u_t^{\pi_t}(S))$ *is the state-action pair produced by policy* π_t *on the true system* $S = (A, B, \{w\})$ *and* $(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S}))$ *is produced on the estimated system* $\widehat{S} = (\widehat{A}, \widehat{B}, \{\widehat{w}\})$.

Proof [of Lemma 22] If the policy is a non-stationary DAC policy parameterized via $M_{1:T}$, in system $(A, B, \{w\})$, it holds that

$$\begin{aligned}
\|x_{t+1}^{\pi_t}(S)\|_2 &\leq W\sum_{i=0}^{H+t}\|\Psi_{t,i}^{K,t}(M_{0:t})\|_{\mathrm{op}} \\
&= W\sum_{i=0}^{H+t}\|\widetilde{A}_K^i\mathbf{1}_{\{i\leq t\}} + \sum_{j=0}^{t}\widetilde{A}_K^j B M_{t-j}^{[i-j-1]}\mathbf{1}_{\{1\leq i-j\leq H\}}\|_{\mathrm{op}}
\end{aligned}$$

$$
\begin{aligned}
&\leq W\left(\kappa^2\sum_{i=0}^{H+t}(1-\gamma)^i+\kappa_B^2\kappa^3\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^j\mathbf{1}_{\{1\leq i-j\leq H\}}\|_{\mathrm{op}}\right)\\
&\leq W\left(\kappa^2\gamma^{-1}+\kappa_B^2\kappa^3\sum_{i=0}^{H+t}\sum_{j=i-H}^{i-1}\|\widetilde{A}_K^j\|_{\mathrm{op}}\mathbf{1}_{\{0\leq j\leq t\}}\right)\\
&\leq W\left(\kappa^2\gamma^{-1}+\kappa_B^2\kappa^5\sum_{i=0}^{H+t}\sum_{j=i-H}^{i-1}(1-\gamma)^j\mathbf{1}_{\{0\leq j\leq t\}}\right)\\
&\leq W\left(\kappa^2\gamma^{-1}+\kappa_B^2\kappa^5 H\sum_{i=0}^{t}(1-\gamma)^i\right)\\
&\leq W\left(\kappa^2\gamma^{-1}+\kappa_B^2\kappa^5 H\gamma^{-1}\right)\\
&\leq 2W\kappa_B^2\kappa^5\gamma^{-1}H.
\end{aligned}
$$

By Lemma 29, a linear controller K is $\left(\kappa,\gamma-2\kappa^3\varepsilon_{A,B}\right)$-strongly stable with respect to the estimated system $\widehat{S}=(\widehat{A},\widehat{B},\{\widehat{w}\})$ if it is (κ,γ)-strongly stable for the true system $S=(A,B,\{w\})$. Thus it can be easily verified that

$$
1-\gamma+2\kappa^3\varepsilon_{A,B}\leq 1-\gamma+2\kappa^3\cdot 0.25\kappa^{-3}\gamma=1-\gamma/2.
$$

For simplicity, we can say that linear controller K is $(\kappa,\gamma/2)$-strongly stable for the estimated system $\widehat{S}$. Further, let $\|\widehat{B}\|_{\mathrm{op}}\leq\kappa_{\widehat{B}}$, it holds that

$$
\kappa_{\widehat{B}}=\|\widehat{B}\|_{\mathrm{op}}=\|(\widehat{B}-B)+B\|_{\mathrm{op}}\leq\varepsilon_{A,B}+\kappa_B\leq 2\kappa_B.
$$

As a result, we can bound $\|x_{t+1}^{\pi_t}(\widehat{S})\|_2$ as

$$
\|x_{t+1}^{\pi_t}(\widehat{S})\|_2\leq 2(\varepsilon_w+W)(2\kappa_B)^2\kappa^5(\gamma/2)^{-1}H=32W_0\kappa_B^2\kappa^5\gamma^{-1}H.
$$

As for the action $u_t^{\pi_t}(\widehat{S})$, we can bound it as

$$
\begin{aligned}
\|u_t^{\pi_t}(\widehat{S})\|_2&\leq\|-Kx_t^{\pi_t}(\widehat{S})\|_2+\left\|\sum_{i=1}^{H}M_t^{[i-1]}\widehat{w}_{t-i}\right\|_2\\
&\leq 32W_0\kappa_B^2\kappa^6\gamma^{-1}H+2W_0\kappa_B\kappa^3\gamma^{-1}\\
&\leq 34W_0\kappa_B^2\kappa^6\gamma^{-1}H.
\end{aligned}
$$

Thus, the diameter of the state-action domain in the estimated system, denoted as $\widehat{D}$, is at most $\widehat{D}:=\max_{t\in[T]}\max\{\|x_t(\widehat{S})\|_2,\|u_t(\widehat{S})\|_2\}=34W_0\kappa_B^2\kappa^6\gamma^{-1}H$. The gap of the cumulative costs between the true system and the estimated system can be bounded as

$$
\begin{aligned}
&\left|\sum_{t=T_0+1}^{T}c_t\left(x_t^{\pi_t}(\widehat{S}),u_t^{\pi_t}(\widehat{S})\right)-\sum_{t=T_0+1}^{T}c_t\left(x_t^{\pi_t}(S),u_t^{\pi_t}(S)\right)\right|\\
&\leq G_c\widehat{D}\sum_{t=1}^{T}\|x_t^{\pi_t}(\widehat{S})-x_t^{\pi_t}(S)\|_2+G_c\widehat{D}\sum_{t=1}^{T}\|u_t^{\pi_t}(\widehat{S})-u_t^{\pi_t}(S)\|_2.
\end{aligned}\tag{53}
$$

We start by analyzing $\|u_t^{\pi_t}(\widehat{S}) - u_t^{\pi_t}(S)\|_2$:

$$\begin{aligned}
\|u_t^{\pi_t}(\widehat{S}) - u_t^{\pi_t}(S)\|_2 &= \left\| \left(-K x_t^{\pi_t}(\widehat{S}) + \sum_{i=1}^{H} M_t^{[i-1]} \widehat{w}_{t-i} \right) - \left(-K x_t^{\pi_t}(S) + \sum_{i=1}^{H} M_t^{[i-1]} w_{t-i} \right) \right\|_2 \\
&\le \kappa \|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 + \sum_{i=1}^{H} \|M_t^{[i-1]}(\widehat{w}_{t-i} - w_{t-i})\| \\
&\le \kappa \|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 + \varepsilon_w \kappa_B \kappa^3 \sum_{i=1}^{H} (1-\gamma)^i \\
&\le \kappa \|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 + \varepsilon_w \kappa_B \kappa^3 \gamma^{-1}.
\end{aligned} \tag{54}$$

Plugging (54) into (53), it holds that

$$\begin{aligned}
&\left| \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S})\right) - \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(S), u_t^{\pi_t}(S)\right) \right| \\
&\le 2\kappa G_c \widehat{D} \sum_{t=1}^{T} \|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 + G_c \widehat{D} \varepsilon_w \kappa_B \kappa^3 \gamma^{-1} T.
\end{aligned} \tag{55}$$

This motivates the need to analyze $\|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2$. To begin with, we define $\widehat{\Psi}_{t,i}^{K,h}(M_{t-h:t}) = \widehat{A}_K^i \mathbf{1}_{i\le h} + \sum_{j=0}^{h} \widehat{A}_K^j \widehat{B} M_{t-j}^{[i-j-1]} \mathbf{1}_{1\le i-j\le H}$, where $\widehat{A}_K := \widehat{A} - \widehat{B}K$. Expanding $x_t^{\pi_t}(\widehat{S})$ and $x_t^{\pi_t}(S)$ using Proposition 4, it holds that

$$\begin{aligned}
&\|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 \\
&= \|\sum_{i=0}^{H+t} \Psi_{t,i}^{K,t}(M_{1:t}) w_{t-i} - \sum_{i=0}^{H+t} \widehat{\Psi}_{t,i}^{K,t}(M_{1:t}) \widehat{w}_{t-i}\|_2 \\
&\le \underbrace{\|\sum_{i=0}^{H+t} \Psi_{t,i}^{K,t}(M_{1:t}) w_{t-i} - \sum_{i=0}^{H+t} \Psi_{t,i}^{K,t}(M_{1:t}) \widehat{w}_{t-i}\|_2}_{\texttt{term (i)}} + \underbrace{\|\sum_{i=0}^{H+t} \Psi_{t,i}^{K,t}(M_{1:t}) \widehat{w}_{t-i} - \sum_{i=0}^{H+t} \widehat{\Psi}_{t,i}^{K,t}(M_{1:t}) \widehat{w}_{t-i}\|_2}_{\texttt{term (ii)}}.
\end{aligned} \tag{56}$$

First, we analyze term (i):

$$\texttt{term (i)} \le \varepsilon_w \sum_{i=0}^{H+t} \|\Psi_{t,i}^{K,t}(M_{1:t})\|_{\text{op}} \le 2\varepsilon_w \kappa_B^2 \kappa^5 \gamma^{-1} H. \tag{57}$$

Second, we investigate term (ii):

$$
\begin{aligned}
\texttt{term (ii)} &\leq (W+\varepsilon_w)\sum_{i=0}^{H+t}\left\|\Psi_{t,i}^{K,t}(M_{1:t})-\widehat{\Psi}_{t,i}^{K,t}(M_{1:t})\right\|_{\mathrm{op}} \\
&\leq 2W_0\sum_{i=0}^{H+t}\left(\left\|\left(\widetilde{A}_K^i-\widehat{A}_K^i\right)\mathbf{1}_{\{i\leq t\}}\right\|_{\mathrm{op}}+\kappa_B\kappa^3\sum_{j=0}^{t}\|\widetilde{A}_K^jB-\widehat{A}_K^j\widehat{B}\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}}\right) \\
&\leq \underbrace{2W_0\kappa^2\sum_{i=0}^{t}\|L^i-\widehat{L}^i\|_{\mathrm{op}}}_{\texttt{term (a)}}+\underbrace{2W_0\kappa_B\kappa^3\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^jB-\widehat{A}_K^j\widehat{B}\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}}}_{\texttt{term (b)}}.
\end{aligned}
\tag{58}
$$

For term (a), using Lemma 32, it holds that

$$
\sum_{i=0}^{t}\|L^i-\widehat{L}^i\|_{\mathrm{op}}\leq 3\gamma^{-2}\|L-\widehat{L}\|_{\mathrm{op}}\leq 3\gamma^{-2}\cdot 2\kappa^3\varepsilon_{A,B}=6\kappa^3\gamma^{-2}\varepsilon_{A,B}.
$$

For term (b), by inserting an intermediate term, we have

$$
\begin{aligned}
&\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^jB-\widehat{A}_K^j\widehat{B}\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}} \\
&\leq \sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^jB-\widetilde{A}_K^j\widehat{B}\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}}+\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^j\widehat{B}-\widehat{A}_K^j\widehat{B}\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}} \\
&\leq \varepsilon_{A,B}\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^j\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}}+\kappa_{\widehat{B}}\sum_{i=0}^{H+t}\sum_{j=0}^{t}\|\widetilde{A}_K^j-\widehat{A}_K^j\|_{\mathrm{op}}\mathbf{1}_{\{1\leq i-j\leq H\}} \\
&\leq \varepsilon_{A,B}H\gamma^{-1}+2\kappa_B\kappa^2H\sum_{i=0}^{t}\|L^i-\widehat{L}^i\|_{\mathrm{op}} \\
&\leq \varepsilon_{A,B}H\gamma^{-1}+2\kappa_B\kappa^2H\cdot 6\kappa^3\gamma^{-2}\varepsilon_{A,B}.
\end{aligned}
$$

Plugging term (a) and term (b) into (58), we have

$$
\begin{aligned}
\texttt{term (ii)} &\leq 2W_0\kappa^2\cdot 6\kappa^3\gamma^{-2}\varepsilon_{A,B}+2W_0\kappa_B\kappa^3\cdot(\varepsilon_{A,B}H\gamma^{-1}+2\kappa_B\kappa^2H\cdot 6\kappa^3\gamma^{-2}\varepsilon_{A,B}) \\
&\leq 38W_0\kappa_B^2\kappa^8\gamma^{-2}H\varepsilon_{A,B}.
\end{aligned}
$$

Plugging the bounds of (57) and (58) into (56), we have

$$
\|x_t^{\pi_t}(\widehat{S})-x_t^{\pi_t}(S)\|_2\leq 2\varepsilon_w\kappa_B^2\kappa^5\gamma^{-1}H+38W_0\kappa_B^2\kappa^8\gamma^{-2}H\varepsilon_{A,B}. \tag{59}
$$

Furthermore, by Lemma 30, we have

$$
W_0\leq 2\sqrt{d_u}\kappa^3\gamma^{-1}W,\quad \varepsilon_w\leq 42\sqrt{d_u}\kappa^{12}\gamma^{-3}W\varepsilon_{A,B}
$$

Plugging W_0 and ε_w into (59), it holds that

$$\|x_t^{\pi_t}(\widehat{S}) - x_t^{\pi_t}(S)\|_2 \leq \mathcal{O}(\varepsilon_{A,B} + \varepsilon_{A,B}^2).$$

Plugging the above bound into (55), we have

$$\left| \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(\widehat{S}), u_t^{\pi_t}(\widehat{S})\right) - \sum_{t=T_0+1}^{T} c_t\left(x_t^{\pi_t}(S), u_t^{\pi_t}(S)\right) \right| \leq \mathcal{O}\left(\varepsilon_{A,B}T + \varepsilon_{A,B}^2 T\right),$$

which finishes the proof. ■

Lemma 23 gives the high-probability bound, which guarantees the accuracy of our estimation of the quantity $\widetilde{A}_K^j = (A - BK)^j$ for $j \geq 0$.

Lemma 23 (Moment Recovery)**.** *Under Assumption 6, Algorithm 3 satisfies for all $j \in [k]$, with probability at least $1 - \delta$, it holds that*

$$\|N_j - \widetilde{A}_K^j B\|_{\mathrm{F}} \leq 3\kappa_B \kappa^2 d_u W \gamma^{-1} \sqrt{\frac{2 d_{\min} \log\left(2e^2 k \delta^{-1}\right)}{T_0 - k}}. \tag{60}$$

Proof [of Lemma 23] When the control inputs are chosen as $u_t = -Kx_t + \widetilde{u}_t$, using the transition equation of linear dynamical systems, it holds that

$$\begin{aligned}
x_{t+1} &= Ax_t + Bu_t + w_t \\
&= Ax_t + B\left(-Kx_t + \widetilde{u}_t\right) + w_t \\
&= \widetilde{A}_K x_t + B\widetilde{u}_t + w_t \\
&= \widetilde{A}_K\left(Ax_{t-1} + Bu_{t-1} + w_{t-1}\right) \\
&= \widetilde{A}_K\left(\widetilde{A}_K x_{t-1} + B\widetilde{u}_{t-1} + w_{t-1}\right) + B\widetilde{u}_t + w_t \\
&= \widetilde{A}_K^2 x_{t-1} + \widetilde{A}_K\left(B\widetilde{u}_{t-1} + w_{t-1}\right) + \left(B\widetilde{u}_t + w_t\right) \\
&= \ldots \\
&= \sum_{i=0}^{t} \widetilde{A}_K^{t-i}\left(B\widetilde{u}_i + w_i\right).
\end{aligned}$$

Let $N_{j,t} = x_{t+j+1}\widetilde{u}_t^{\mathrm{T}}$, we can prove that

$$\begin{aligned}
\mathbb{E}\left[N_{j,t}\right] = \mathbb{E}\left[x_{t+j+1}\widetilde{u}_t^{\mathrm{T}}\right] &= \mathbb{E}\left[\sum_{i=0}^{t+j} \widetilde{A}_K^{t+j-i}\left(B\widetilde{u}_i + w_i\right)\widetilde{u}_t^{\mathrm{T}}\right] \\
&= \sum_{i=0}^{t+j} \widetilde{A}_K^{t+j-i} \cdot \mathbb{E}\left[\left(B\widetilde{u}_i + w_i\right)\widetilde{u}_t^{\mathrm{T}}\right] \\
&= \widetilde{A}_K^j \cdot \mathbb{E}\left[\left(B\widetilde{u}_t + w_t\right)\widetilde{u}_t^{\mathrm{T}}\right] \\
&= \widetilde{A}_K^j B \cdot \mathbb{E}\left[\widetilde{u}_t\widetilde{u}_t^{\mathrm{T}}\right] + \widetilde{A}_K^j w_t \cdot \mathbb{E}\left[\widetilde{u}_t^{\mathrm{T}}\right] = \widetilde{A}_K^j B,
\end{aligned}$$

where the second last equation is due to the fact that $\widetilde{u}_i$ and $\widetilde{u}_j$ are independent when $i \neq j$, and the last step is true because $\mathbb{E}_{\widetilde{u}_t}\left[\widetilde{u}_t\widetilde{u}_t^{\mathrm{T}}\right] = I, \mathbb{E}_{\widetilde{u}_t}\left[\widetilde{u}_t\right] = \mathbf{0}$. Consequently, we can prove that $\mathbb{E}[N_j] = \frac{1}{T_0-k}\sum_{t=0}^{T_0-k-1}\mathbb{E}\left[N_{j,t}\right] = \widetilde{A}_K^j B$. Note that for $0 \leq t_1, t_2 \leq T_0 - k - 1$ and $t_1 \neq t_2$, N_{j,t_1} and N_{j,t_2} are not independent because they contains the same random variables η, so we cannot use Hoeffding's inequality here.

For each index $j \in [k]$, we can define a sequence of variables $\widetilde{N}_{j,t} := N_{j,t} - \widetilde{A}_K^j B$, we can prove that $\{\widetilde{N}_{j,t}\}_{t=0}^{T_0-k-1}$ is a *martingale difference sequence* w.r.t. the sequence $\{\widetilde{u}_t\}_{t=0}^{T_0-k-1}$:

$$
\begin{aligned}
&\mathbb{E}\left[\widetilde{N}_{j,t}|\widetilde{u}_{0:t-1}\right] = \mathbb{E}\left[N_{j,t}|\widetilde{u}_{0:t-1}\right] - \widetilde{A}_K^j B \\
&= \mathbb{E}\left[\sum_{i=0}^{t+j}\widetilde{A}_K^{t+j-i}\left(B\widetilde{u}_i + w_i\right)\widetilde{u}_t^{\mathrm{T}}|\widetilde{u}_{0:t-1}\right] - \widetilde{A}_K^j B \\
&= \mathbb{E}\left[\sum_{i=0}^{t-1}\widetilde{A}_K^{t+j-i}\left(B\widetilde{u}_i + w_i\right)\widetilde{u}_t^{\mathrm{T}}|\widetilde{u}_{0:t-1}\right] + \mathbb{E}\left[\sum_{i=t}^{t+j}\widetilde{A}_K^{t+j-i}\left(B\widetilde{u}_i + w_i\right)\widetilde{u}_t^{\mathrm{T}}\right] - \widetilde{A}_K^j B \\
&= \mathbb{E}\left[\widetilde{A}_K^j\left(B\widetilde{u}_t + w_t\right)\widetilde{u}_t^{\mathrm{T}}\right] - \widetilde{A}_K^j B = \mathbf{0}.
\end{aligned}
$$

For all $j \in [k], t = 0, \ldots, T_0 - k - 1$, the operator norm of $N_{j,t}$ can be bounded by

$$
\|N_{j,t}\|_{\mathrm{op}} \leq \|x_{t+j+1}\|_{\mathrm{op}}\|\widetilde{u}_t\|_{\mathrm{op}} \leq \|x_{t+j+1}\|_2\|\widetilde{u}_t\|_2 \leq 2\kappa_B\kappa^2\sqrt{d_u}W\gamma^{-1}\cdot\sqrt{d_u} = 2\kappa_B\kappa^2 d_u W\gamma^{-1}.
$$

Also, for $\widetilde{N}_{j,t}$, we can prove that

$$
\|\widetilde{N}_{j,t}\|_{\mathrm{op}} \leq \|N_{j,t}\|_{\mathrm{op}} + \|\widetilde{A}_K^j B\|_{\mathrm{op}} \leq 2\kappa_B\kappa^2 d_u W\gamma^{-1} + \kappa_B\kappa^2(1-\gamma)^j \leq 3\kappa_B\kappa^2 d_u W\gamma^{-1}.
$$

$$
\|\widetilde{N}_{j,t}\|_{\mathrm{F}} \leq \sqrt{d_{\min}}\|\widetilde{N}_{j,t}\|_{\mathrm{op}} \leq 3\sqrt{d_{\min}}\kappa_B\kappa^2 d_u W\gamma^{-1} := D_N
$$

Using Lemma 11, we have $\Pr\left[\|\sum_{t=0}^{T_0-k}\widetilde{N}_{j,t}\|_{\mathrm{F}} \geq x\right] \leq 2e^2\exp\left(\frac{-x^2}{2(T_0-k)D_N^2}\right)$. By substituting $\widetilde{N}_{j,t}$ by $N_{j,t} - \widetilde{A}_K^j B$, it holds that $\Pr\left[\|N_j - \widetilde{A}_K^j B\|_{\mathrm{F}} \geq \frac{x}{T_0-k}\right] \leq 2e^2\exp\left(\frac{-x^2}{2(T_0-k)D_N^2}\right)$. Finally, let $\varepsilon = \frac{x}{T_0-k}$, we have

$$
\Pr\left[\|N_j - \widetilde{A}_K^j B\|_{\mathrm{F}} \geq \varepsilon\right] \leq 2e^2\exp\left(\frac{-(T_0-k)\varepsilon^2}{2D_N^2}\right)
$$

We set $2e^2\exp\left(\frac{-(T_0-k)\varepsilon^2}{2D_N^2}\right) = \frac{\delta}{k}$ to make above concentration inequality holds for each $j \in [k]$ with probability at least $1-\delta$, which implies that

$$
\varepsilon = 3\kappa_B\kappa^2 d_u W\gamma^{-1}\sqrt{\frac{2d_{\min}\log\left(2e^2k\delta^{-1}\right)}{T_0-k}}.
$$

Hence, we complete the proof. ■

C.3.2 Proof of Corollary 7

We now present the proof of Corollary 7, i.e., the static policy regret of the controller. Corollary 7 states that when the system dynamics are known, Scream.Control enjoys the following static policy regret,

$$\sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \leq \widetilde{\mathcal{O}}(\sqrt{T}), \tag{61}$$

where the comparator set Π can be chosen as either the set of DAC policies or the set of strongly linear controllers. Let us denote the two comparator sets as Π_{DAC} and Π_{SLC}, respectively. Moreover, when the system dynamics are unknown, using the identification algorithm of Hazan et al. (2020), we can achieve an $\widetilde{\mathcal{O}}(T^{2/3})$ static regret, which also holds for either the set of DAC policies or the set of strongly linear controllers. Therefore, in the following we will prove the statement for two comparator sets separately.

Proof [of Corollary 7] When the comparator set Π is chosen as the set of DAC policies, i.e., $\pi \in \Pi_{\text{DAC}} = \{\pi(K, M) \mid M \in \mathcal{M}\}$, the result of (61) can be easily obtained from Theorem 5 by setting $\pi_1 = \ldots = \pi_T = \pi_* \in \arg\min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi)$. Under such a case, the path-length $P_T = \sum_{t=2}^{T} \|M_{t-1} - M_t\|_{\text{F}} = 0$, and thus

$$\sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{\text{DAC}}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \leq \widetilde{\mathcal{O}}(\sqrt{T}).$$

On the other hand, when choosing the comparator set Π as Π_{SL}, i.e., $\pi = K \in \Pi_{\text{SL}} = \{K \mid K \text{ is } (\kappa, \gamma)\text{-strongly stable}\}$, we will need some efforts to prove the statement.

We show that the statement can be obtained by further incorporating Lemma 28, which demonstrates that minimizing static policy regret over the DAC class is sufficient to deliver a policy regret competing with the strongly linear controller class (Agarwal et al., 2019, Lemma 5.2). In fact, denote by $\pi^* = K^\star = \arg\min_{K \in \Pi_{\text{SL}}} \sum_{t=1}^{T} c_t(x_t^K, u_t^K)$, and we have

$$\begin{aligned}
&\sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{\text{SLC}}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \\
&= \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{\text{DAC}}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) + \min_{\pi \in \Pi_{\text{DAC}}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) - \sum_{t=1}^{T} c_t(x_t^{K^*}, u_t^{K^*}) \\
&\leq \widetilde{\mathcal{O}}(\sqrt{T}) + \sum_{t=1}^{T} c_t(x_t^{\pi(M_\Delta, K)}, u_t^{\pi(M_\Delta, K)}) - \sum_{t=1}^{T} c_t(x_t^{K^*}, u_t^{K^*}) \\
&\leq \widetilde{\mathcal{O}}(\sqrt{T}) + T \cdot 4 G_c D W H \kappa_B^2 \kappa^6 (1-\gamma)^{H-1} \gamma^{-1} \\
&\leq \widetilde{\mathcal{O}}(\sqrt{T}).
\end{aligned}$$

The first inequality uses the optimality of $\arg\min_{\pi \in \Pi_{\text{DAC}}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi)$ and $\pi(M_\Delta, K)$ is a DAC policy with $M_\Delta = (M_\Delta^{[0]}, \ldots, M_\Delta^{[H-1]})$ defined by $M_\Delta^{[i]} = (K - K^\star)(A - BK^\star)^i$. The second inequality holds by Lemma 28, and the final inequality holds by setting $H = \mathcal{O}(\log T)$.

The above arguments hold for the known system setting. On the other hand, when the system dynamics are unknown, using the system identification yields an additional estimation overhead of order $\widetilde{\mathcal{O}}(T^{2/3})$ no matter which comparator set is chosen. Therefore, the overall regret remains $\widetilde{\mathcal{O}}(T^{2/3})$ for unknown systems. Hence, we complete the proof. ■

C.4 Supporting Lemmas

In this part, we provide several supporting lemmas used frequently in the analysis of online non-stochastic control. Most of them are due to the pioneering work of Agarwal et al. (2019), and we adapt them to our notations and provide the proofs to achieve self-containedness. Specifically,

- Lemma 24 establishes the norm relations between the ℓ_1, op norm and Frobenius norm used in the $\mathcal{M}$-space.
- Lemma 25 checks the boundedness of several variables of interest.
- Lemma 26 shows several properties of the truncated functions $\{f_t\}$ and the feasible set $\mathcal{M}$.
- Lemma 27 provides an upper bound for the norm of transfer matrix.
- Lemma 28 connects the DAC class and the strongly linear controller class.
- Lemmas 29 – 33 are useful for analysis in unknown systems.

Lemma 24 (Norm Relations)**.** *For any $M = (M^{[1]}, \dots, M^{[H]}) \in \mathcal{M} \subseteq (\mathbb{R}^{d_u \times d_x})^H$, its ℓ_1, op norm and Frobenius norm are defined by*

$$\|M\|_{\ell_1,\mathrm{op}} := \sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{op}}, \textit{ and } \|M\|_{\mathrm{F}} := \sqrt{\sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{F}}^2}.$$

We then have the following inequalities on their relations:

$$\|M\|_{\ell_1,\mathrm{op}} \leq \sqrt{H}\|M\|_{\mathrm{F}}, \textit{ and } \|M\|_{\mathrm{F}} \leq \sqrt{d}\|M\|_{\ell_1,\mathrm{op}}, \tag{62}$$

where $d = \min\{d_u, d_x\}$.

Proof [Proof of Lemma 24] Recall the matrix norm relations, we know that for any matrix $X \in \mathbb{R}^{m \times n}$,

$$\|X\|_{\mathrm{op}} \leq \|X\|_{\mathrm{F}} \leq \sqrt{d}\|X\|_{\mathrm{op}}.$$

Therefore, by definition and Cauchy-Schwarz inequality, we obtain

$$\|M\|_{\ell_1,\mathrm{op}} = \sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{op}} \leq \sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{F}} \leq \sqrt{H}\|M\|_{\mathrm{F}}.$$

On the other hand, we have

$$\|M\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{F}}^2} \leq \sum_{i=1}^{H} \|M^{[i]}\|_{\mathrm{F}} \leq \sum_{i=1}^{H} \sqrt{d}\|M^{[i]}\|_{\mathrm{op}} = \sqrt{d}\|M\|_{\ell_1,\mathrm{op}}.$$

We thus complete the proof. ■

Lemma 25. *Suppose K and $K^\star$ are two (κ,γ)-strongly stable linear controllers (cf. Definition 3). Define*

$$D := \frac{W(\kappa^3 + H\kappa_B\kappa^3\tau)}{\gamma(1-\kappa^2(1-\gamma)^{H+1})} + \frac{W\tau}{\gamma}.$$

Suppose there exists a $\tau > 0$ such that for every $i \in \{0,\dots,H-1\}$ and every $t \in [T]$, $\|M_t^{[i]}\|_{\mathrm{F}} \le \tau(1-\gamma)^i$. Then, we have

- $\|x_t^K(M_{0:t-1})\| \le D$, $\|y_t^K(M_{t-H-1:t-1})\| \le D$, *and* $\|x_t^{K^\star}\| \le D$.
- $\|u_t^K(M_{0:t})\| \le D$, *and* $\|v_t^K(M_{t-H-1:t})\| \le D$.
- $\|x_t^K(M_{0:t-1}) - y_t^K(M_{t-1-H:t-1})\| \le \kappa^2(1-\gamma)^{H+1}D$.
- $\|u_t^K(M_{0:t}) - v_t^K(M_{t-1-H:t})\| \le \kappa^3(1-\gamma)^{H+1}D$.

In above, the definitions of state $x_t^K(M_{0:t-1})$ and corresponding DAC control $u_t^K(M_{0:t})$ can be found in Proposition 4, and the definitions of truncated state $x_t^K(M_{0:t-1})$ and corresponding DAC control $v_t^K(M_{0:t})$ can be found in Definition 2. The definitions of state $x_t^{K^\star}$ can be found (and will be used) in Lemma 28.

Proof [of Lemma 25] We first study the state.

$$\begin{aligned}
\|x_t^K(M_{0:t-1})\| &= \left\|\widetilde{A}_K^{H+1}x_{t-H-1}^K(M_{0:t-H-2}) + \sum_{i=0}^{2H}\Psi_{t-1,i}^{K,H}(M_{t-H-1:t-1})w_{t-1-i}\right\| \\
&\le \kappa^2(1-\gamma)^{H+1}\|x_{t-H-1}^K(M_{0:t-H-2})\| + W\sum_{i=0}^{2H}\|\Psi_{t-1,i}^{K,H}(M_{t-H-1:t-1})\| \\
&\le \kappa^2(1-\gamma)^{H+1}\|x_{t-H-1}^K(M_{0:t-H-2})\| + W\sum_{i=0}^{2H}\left(\kappa^2(1-\gamma)^i + H\kappa_B\kappa^2\tau(1-\gamma)^{i-1}\right) \\
&\le \kappa^2(1-\gamma)^{H+1}\|x_{t-H}^K(M_{0:t-H-1})\| + W(\kappa^2 + H\kappa_B\kappa^2\tau)/\gamma \\
&\le \frac{W(\kappa^2 + H\kappa_B\kappa^2\tau)}{\gamma(1-\kappa^2(1-\gamma)^{H+1})} \le D, \qquad (63)
\end{aligned}$$

where inequality (63) is a summation of geometric series and the ratio of this series is $\kappa^2(1-\gamma)^{H+1}$. Similarly,

$$\begin{aligned}
\|y_t^K(M_{t-1-H:t-1})\| &= \left\|\sum_{i=0}^{2H}\Psi_{t-1,i}^{K,H}(M_{t-1-H:t-1})w_{t-1-i}\right\| \\
&\le W\sum_{i=0}^{2H}\|\Psi_{t-1,i}^{K,H}(M_{t-1-H:t-1})\|
\end{aligned}$$

$$\leq W \sum_{i=0}^{2H} \left(\kappa^2(1-\gamma)^i + H\kappa_B\kappa^2\tau(1-\gamma)^{i-1}\right)$$
$$\leq W\left(\frac{\kappa^2 + H\kappa_B\kappa^2\tau}{\gamma}\right) \leq D.$$

Besides,

$$\|x_t^{K^\star}\| = \left\|\sum_{i=0}^{t-1} \widetilde{A}_{K^\star}^i w_{t-1-i}\right\| \leq W \sum_{i=0}^{t-1} \kappa^2(1-\gamma)^i \leq \frac{W\kappa^2}{\gamma} \leq D.$$

So the difference can be evaluated as follows:

$$\|x_t^K(M_{0:t-1}) - y_t^K(M_{t-H-1:t-1})\| = \|\widetilde{A}_K^{H+1} x_{t-H-1}^K(M_{0:t-H-1})\| \leq \kappa^2(1-\gamma)^{H+1}D.$$

We now consider the action (or control signal).

$$\|u_t^K(M_{0:t})\| = \left\|-Kx_t^K(M_{0:t-1}) + \sum_{i=1}^{H} M_t^{[i-1]} w_{t-i}\right\|$$
$$\leq \kappa\|x_t^K(M_{0:t-1})\| + \sum_{i=1}^{H} W\tau(1-\gamma)^{i-1}$$
$$\leq \frac{W(\kappa^3 + H\kappa_B\kappa^3\tau)}{\gamma(1-\kappa^2(1-\gamma)^{H+1})} + \frac{W\tau}{\gamma} \leq D.$$

Similarly,

$$\|v_t^K(M_{t-H-1:t})\| \leq \kappa\|y_t^K(M_{t-H-1:t-1})\| + \sum_{i=1}^{H} W\tau(1-\gamma)^{i-1} \leq D.$$

The difference of the actions is

$$\|u_t^K(M_{0:t-1}) - v_t^K(M_{t-H-1:t-1})\| = \|-K(x_t^K(M_{0:t-1}) - y_t^K(M_{t-H-1:t-1}))\| \leq \kappa^3(1-\gamma)^{H+1}D.$$

■

To reduce the online non-stochastic control to OCO with memory, in Definition 2 we define the truncated loss $f_t : \mathcal{M}^{H+2} \mapsto \mathbb{R}$ as

$$f_t(M_{t-1-H:t}) = c_t(y_t^K(M_{t-1-H:t-1}), v_t^K(M_{t-1-H:t})),$$

where $y_{t+1}^K(M_{t-H:t}) = \sum_{i=0}^{2H} \Psi_{t,i}^{K,H}(M_{t-H:t})w_{t-i}$ and $v_{t+1}^K(M_{t-H:t+1}) = -Ky_{t+1}(M_{t-H:t}) + \sum_{i=1}^{H} M_{t+1}^{[i-1]} w_{t+1-i}$. In the following lemma, we show several properties of the truncated functions $\{f_t\}$ and the feasible set $\mathcal{M}$ such that we can further apply the results of OCO with memory.

Lemma 26. *The truncated loss $f_t : \mathcal{M}^{H+2} \mapsto \mathbb{R}$ and the feasible set $\mathcal{M}$ satisfy the following properties. For notational convenience, we first let D be defined the same as (41), and we restate it below*

$$D := \frac{W\kappa^3(1+H\kappa_B\tau)}{\gamma(1-\kappa^2(1-\gamma)^{H+1})} + \frac{W\tau}{\gamma}.$$

(i) The function is L_f-coordinate-wise Lipschitz with respect to the Euclidean (i.e., Frobenius) norm, namely,

$$|f_t(M_{t-H-1},\ldots,M_{t-k},\ldots,M_t)|-|f_t(M_{t-H-1},\ldots,\widetilde{M}_{t-k},\ldots,M_t)| \le L_f\|M_{t-k}-\widetilde{M}_{t-k}\|_{\mathrm{F}}.$$

Besides,

$$L_f \le 3\sqrt{H}G_cDW\kappa_B\kappa^3.$$

(ii) The gradient norm of surrogate loss $\widetilde{f}_t : \mathcal{M} \mapsto \mathbb{R}$ is bounded by G_f, namely, $\|\nabla_M \widetilde{f}_t(M)\|_{\mathrm{F}} \le G_f$ holds for any $M \in \mathcal{M}$ and any $t \in [T]$. Besides,

$$G_f \le 3Hd^2G_cW\kappa_B\kappa^3\gamma^{-1}.$$

(iii) The diameter of the feasible set is at most D_f, namely, $\|M - M'\|_{\mathrm{F}} \le D_f$ holds for any $M, M' \in \mathcal{M}$. Besides,

$$D_f \le 2\sqrt{d}\kappa_B\kappa^3\gamma^{-1}.$$

Proof [of Lemma 26] We first prove the claim (i), i.e., the L_f-coordinate-wise Lipschitz continuity. For simplicity, we will make use of the following definitions in the following arguments.

$$\begin{aligned}
M_{t-H-1:t} &:= \{M_{t-H-1}\ldots M_{t-k}\ldots M_t\}\\
M_{t-H-1:t-1} &:= \{M_{t-H-1}\ldots M_{t-k}\ldots M_{t-1}\}\\
\widetilde{M}_{t-H-1:t} &:= \{M_{t-H-1}\ldots \widetilde{M}_{t-k}\ldots M_t\}\\
\widetilde{M}_{t-H-1:t-1} &:= \{M_{t-H-1}\ldots \widetilde{M}_{t-k}\ldots M_{t-1}\}
\end{aligned}$$

By representing f_t using c_t, we have

$$\begin{aligned}
&f_t(M_{t-H-1:t}) - f_t(\widetilde{M}_{t-H-1:t})\\
={}& c_t\left(y_t^K(M_{t-H-1:t-1}), v_t^K(M_{t-H-1:t})\right) - c_t\left(y_t^K(\widetilde{M}_{t-H-1:t-1}), v_t^K(\widetilde{M}_{t-H-1:t})\right)\\
\le{}& G_cD\|y_t^K - \widetilde{y}_t^K\| + G_cD\|v_t^K - \widetilde{v}_t^K\|,
\end{aligned} \tag{64}$$

where for convenience we use the notations $y_t^K := y_t^K(\widetilde{M}_{t-H-1:t-1}), \widetilde{y}_t^K := y_t^K(\widetilde{M}_{t-H-1:t-1})$ and $v_t^K := v_t^K(M_{t-H-1:t}, \widetilde{v}_t^K := \widetilde{v}_t^K(M_{t-H-1:t}$. Besides, the last inequality holds because the norm of $\|y_t^K\|$, $\|\widetilde{y}_t^K\|$, $\|v_t^K\|$, $\|\widetilde{v}_t^K\|$ are all bounded by D, as shown in Lemma 25.

Then we try to bound $\|y_t^K - \widetilde{y}_t^K\|$ and $\|v_t^K - \widetilde{v}_t^K\|$.

$$\begin{aligned}
\|y_t^K - \widetilde{y}_t^K\| &= \left\|\sum_{i=0}^{2H}\left(\Psi_{t-1,i}^{K,H}(M_{t-H-1:t-1}) - \Psi_{t-1,i}^{K,H}(\widetilde{M}_{t-H-1:t-1})\right)w_{t-1-i}\right\|\\
&= \left\|\widetilde{A}_K^k B\sum_{i=0}^{2H}\left(M_{t-k}^{[i-k-1]} - \widetilde{M}_{t-k}^{[i-k-1]}\right)\mathbf{1}_{\{i-k\in[H]\}}w_{t-1-i}\right\|\\
&\le \kappa_B\kappa^2(1-\gamma)^kW\sum_{i=1}^{H}\|M_{t-k}^{[i-1]} - \widetilde{M}_{t-k}^{[i-1]}\|\\
&\le \kappa_B\kappa^2W\|M_{t-k} - \widetilde{M}_{t-k}\|,
\end{aligned} \tag{65}$$

and we have

$$\begin{aligned}\|v_t^K - \widetilde{v}_t^K\| &= \left\| -K(y_t^K - \widetilde{y}_t^K) + \mathbf{1}_{\{k=0\}} \sum_{i=1}^{H} \left(M_{t-k}^{[i-1]} - \widetilde{M}_{t-k}^{[i-1]} \right) \right\| \\ &\leq (\kappa_B \kappa^3 W + 1)\|M_{t-k} - \widetilde{M}_{t-k}\| \\ &\leq 2\kappa_B \kappa^3 W \|M_{t-k} - \widetilde{M}_{t-k}\|.\end{aligned} \tag{66}$$

Combining (64), (65), and (66), we obtain

$$\begin{aligned}f_t(M_{t-H-1:t}) - f_t(\widetilde{M}_{t-H-1:t}) &\leq G_c D \|y_t^K - \widetilde{y}_t^K\| + G_c D \|v_t^K - \widetilde{v}_t^K\| \\ &\leq G_c D \kappa_B \kappa^2 W \|M_{t-k} - \widetilde{M}_{t-k}\| + G_c D 2\kappa_B \kappa^3 W \|M_{t-k} - \widetilde{M}_{t-k}\| \\ &\leq 3 G_c D \kappa_B \kappa^3 W \|M_{t-k} - \widetilde{M}_{t-k}\|.\end{aligned}$$

So we have $L_f \leq 3G_c D W \kappa_B \kappa^3$.

Next, we prove the claim (ii), i.e., the boundedness of the gradient norm. Indeed, we will try to bound $\nabla_{M_{p,q}^{[r]}} \widetilde{f}_t(M)$ for every $p \in [d_u], q \in [d_x]$ and $r \in \{0, \ldots, H-1\}$,

$$\left| \nabla_{M_{p,q}^{[r]}} \widetilde{f}_t(M) \right| \leq G_c \left\| \frac{\partial y_t^K(M)}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} + G_c \left\| \frac{\partial v_t^K(M)}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}}. \tag{67}$$

So we will bound the two terms of the right hand side respectively.

$$\begin{aligned}\left\| \frac{\partial y_t^K(M)}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} &\leq \left\| \sum_{i=0}^{2H} \sum_{j=0}^{H} \left[\frac{\partial \widetilde{A}_K^j B M^{[i-j-1]}}{\partial M_{p,q}^{[r]}} \right] w_{t-1-i} \mathbf{1}_{\{i-j \in [H]\}} \right\|_{\mathrm{F}} \\ &\leq \sum_{i=r+1}^{r+H+1} \left\| \frac{\partial \widetilde{A}_K^{i-r-1} B M^{[r]}}{\partial M_{p,q}^{[r]}} w_{t-1-i} \right\|_{\mathrm{F}} \\ &\leq W \kappa_B \kappa^2 \left\| \frac{\partial M^{[r]}}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} \sum_{i=r+1}^{r+H+1} (1-\gamma)^{i-r-1} \\ &\leq \frac{W \kappa_B \kappa^2}{\gamma} \left\| \frac{\partial M^{[r]}}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} \\ &\leq \frac{W \kappa_B \kappa^2}{\gamma}\end{aligned} \tag{68}$$

$$\begin{aligned}\left\| \frac{\partial v_t^K(M)}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} &\leq \kappa \left\| \frac{\partial y_t^K(M)}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} + \sum_{i=1}^{H} \left\| \frac{\partial M^{[i-1]}}{\partial M_{p,q}^{[r]}} w_{t-i} \right\|_{\mathrm{F}} \\ &\leq \frac{W \kappa_B \kappa^3}{\gamma} + W \left\| \frac{\partial M^{[r]}}{\partial M_{p,q}^{[r]}} \right\|_{\mathrm{F}} \\ &\leq W \left(\frac{\kappa_B \kappa^3}{\gamma} + 1 \right)\end{aligned} \tag{69}$$

Combining (67), (68), and (69), we obtain

$$\left|\nabla_{M_{p,q}^{[r]}} \widetilde{f}_t(M)\right| \leq G_c \frac{W\kappa_B\kappa^2}{\gamma} + G_c W \left(\frac{\kappa_B\kappa^3}{\gamma} + 1\right) \leq 3G_c W\kappa_B\kappa^3\gamma^{-1}.$$

Thus, $\|\nabla_M \widetilde{f}_t(M)\|_{\mathrm{F}}$ at most $3Hd^2G_cW\kappa_B\kappa^3\gamma^{-1}$.

Finally, we prove the claim (iii), i.e., the upper bound of diameter of the feasible set.

Actually, the construction of feasible set $\mathcal{M}$ ensures that $\forall i, 0 \leq i \leq H-1$, $\|M\|_{\mathrm{op}}^{[i]} \leq \kappa_B\kappa^3(1-\gamma)^i$. Therefore, we have

$$\begin{aligned}
\max_{M_1,M_2\in\mathcal{M}} \|M_1 - M_2\|_{\mathrm{F}} &\overset{(62)}{\leq} \sqrt{d} \max_{M_1,M_2\in\mathcal{M}} \|M_1 - M_2\|_{\ell_1,\mathrm{op}} \\
&\leq \sqrt{d} \max_{M_1,M_2\in\mathcal{M}} (\|M_1\|_{\ell_1,\mathrm{op}} + \|M_2\|_{\ell_1,\mathrm{op}}) \\
&= \sqrt{d} \max_{M_1,M_2\in\mathcal{M}} \left(\sum_{i=0}^{H-1} \|M_1^{[i]}\|_{\mathrm{op}} + \|M_2^{[i]}\|_{\mathrm{op}}\right) \\
&\leq \sqrt{d} \max_{M_1,M_2\in\mathcal{M}} \left(2\sum_{i=0}^{H-1} \kappa_B\kappa^3(1-\gamma)^i\right) \\
&= 2\sqrt{d}\kappa_B\kappa^3 \sum_{i=0}^{H-1} (1-\gamma)^i \\
&\leq 2\sqrt{d}\kappa_B\kappa^3\gamma^{-1}.
\end{aligned}$$

Hence, we finish the proof of all three claims in the statement. ■

The following lemma provides an upper bound for the norm of transfer matrix.

Lemma 27. *Suppose K is (κ, γ)-strongly stable as defined in Definition 3. Suppose there exists a $\tau > 0$ such that for every $i \in \{0, \dots, H-1\}$ and every $t \in [T]$, $\|M_t^{[i]}\|_{\mathrm{F}} \leq \tau(1-\gamma)^i$. Then, we have*

$$\|\Psi_{t,i}^{K,h}\| \leq \kappa^2(1-\gamma)^i \mathbf{1}_{\{i\leq h\}} + H\kappa_B\kappa^2\tau(1-\gamma)^{i-1}. \tag{70}$$

Proof [Proof of Lemma 27] We first expand $\Psi_{t,i}^{K,h}$ by its definition (cf. Proposition 4 for its formal definition):

$$\begin{aligned}
\|\Psi_{t,i}^{K,h}\| &= \left\|\widetilde{A}_K^i \mathbf{1}_{\{i\leq h\}} + \sum_{j=0}^{h} \widetilde{A}_K^j B M_{t-j}^{[i-j-1]} \mathbf{1}_{\{1\leq i-j\leq H\}}\right\| \\
&\leq \|\widetilde{A}_K^i\| \mathbf{1}_{\{i\leq h\}} + \sum_{j=1}^{H} \|\widetilde{A}_K^j B M_{t-j}^{[i-j-1]}\| \qquad (71) \\
&\leq \kappa^2(1-\gamma)^i + \sum_{j=1}^{H} \kappa^2(1-\gamma)^j \kappa_B \tau (1-\gamma)^{i-j-1}
\end{aligned}$$

$$
\begin{aligned}
&\le \kappa^2(1-\gamma)^i + \kappa^2\kappa_B\tau\sum_{j=1}^{H}(1-\gamma)^{i-1} \\
&= \kappa^2(1-\gamma)^i + H\kappa^2\kappa_B\tau(1-\gamma)^{i-1},
\end{aligned}
$$

where inequality (71) has to be emphasized here that no matter what the index i is, once i is fixed, to satisfy the condition $1 \le i - j \le H$, there is at most H different values which j can take. And that is why we can take j in range $[H]$ as an upper bound. ■

In the following lemma, we show that minimizing the static policy regret over the DAC class is sufficient to deliver a policy regret competing with the strongly linear controller class (Agarwal et al., 2019, Lemma 5.2).

Lemma 28. *With $K, K^\star$ chosen as the (κ,γ)-strongly stable linear controllers as defined in Definition 3 and under Assumption 5, there exists a DAC policy $\pi(M_\Delta, K)$ with $M_\Delta = (M_\Delta^{[0]}, \dots, M_\Delta^{[H-1]})$ defined by*

$$
M_\Delta^{[i]} = (K - K^\star)(A - BK^\star)^i \tag{72}
$$

such that

$$
\sum_{t=1}^{T} c_t(x_t^K(M_\Delta), u_t^K(M_\Delta)) - \sum_{t=1}^{T} c_t(x_t^{K^\star}, u_t^{K^\star}) \le T \cdot 4G_c DWH\kappa_B^2\kappa^6(1-\gamma)^{H-1}\gamma^{-1}, \tag{73}
$$

where $x_t^{K^\star}$ is the state attained by executing a linear controller $K^\star$ which chooses the action $u_t^{K^\star} = -K^\star x_t^{K^\star}$.

Proof [of Lemma 28] The coordinate-wise Lipschitzness of the cost functions implies that

$$
c_t\left(x_t^K(M_\Delta), u_t^K(M_\Delta)\right) - c_t\left(x_t^{K^\star}, u_t^{K^\star}\right) \le G_c D\left\|x_t^K(M_\Delta) - x_t^{K^\star}\right\| + G_c D\left\|u_t^K(M_\Delta) - u_t^{K^\star}\right\|.
$$

By the linear dynamical equation (9), we have

$$
x_{t+1}^{K^\star} = \sum_{i=0}^{t}(A - BK^\star)^i w_{t-i} = \sum_{i=0}^{t}\widetilde{A}_{K^\star}^i w_{t-i} \tag{74}
$$

By the property of the DAC policy (Proposition 4), we have

$$
x_{t+1}^K(M_\Delta) = \widetilde{A}_K^{h+1}x_{t-h}^K(M_\Delta) + \sum_{i=0}^{H+h}\Psi_{t,i}^{K,h}(M_\Delta)w_{t-i}.
$$

Setting $h = t$ and combining the assumption that the starting state $x_0 = \mathbf{0}$, we achieve the following equation,

$$
x_{t+1}^K(M_\Delta) = \sum_{i=0}^{H}\Psi_{t,i}^{K,t}(M_\Delta)w_{t-i} + \sum_{i=H+1}^{t}\Psi_{t,i}^{K,t}(M_\Delta)w_{t-i}.
$$

Now we turn to calculate the transfer matrix $\Psi_{t,i}^{K,h}(M_\Delta)$ explicitly. Actually, for any $i \in \{0, \ldots, H\}$, $h \geq H$, i.e., $0 \leq i \leq H \leq h$, by definition we have

$$\begin{aligned}
\Psi_{t,i}^{K,h}(M_\Delta) &= \widetilde{A}_K^i \mathbf{1}_{\{i \leq h\}} + \sum_{j=0}^{h} \widetilde{A}_K^j B M_\Delta^{[i-j-1]} \mathbf{1}_{\{i-j \in [H]\}} \\
&= \widetilde{A}_K^i + \sum_{k=1}^{i} \widetilde{A}_K^{i-k} B M_\Delta^{[k-1]} \quad (75) \\
&= \widetilde{A}_K^i + \sum_{k=1}^{i} \widetilde{A}_K^{i-k} B (K - K^\star) \widetilde{A}_{K^\star}^{k-1} \quad (76) \\
&= \widetilde{A}_K^i + \sum_{k=1}^{i} \widetilde{A}_K^{i-k} (\widetilde{A}_{K^\star} - \widetilde{A}_K) \widetilde{A}_{K^\star}^{k-1} \\
&= \widetilde{A}_K^i + \sum_{k=1}^{i} \widetilde{A}_K^{i-k} \widetilde{A}_{K^\star}^{k} - \widetilde{A}_K^{i-k+1} \widetilde{A}_{K^\star}^{k-1} \\
&= \widetilde{A}_K^i + \widetilde{A}_{K^\star}^i - \widetilde{A}_K^i \\
&= \widetilde{A}_{K^\star}^i,
\end{aligned}$$

where (75) holds by introducing a new index $k = i - j$ and (76) can be obtained by plugging the construction of $M_\Delta^{[i]}$ (72). So we achieve the conclusion that

$$x_{t+1}^K(M_\Delta) = \sum_{i=0}^{H} \widetilde{A}_{K^\star}^i w_{t-i} + \sum_{i=H+1}^{t} \Psi_{t,i}^{K,t}(M_\Delta) w_{t-i}. \quad (77)$$

Combining (74) and (77) yields

$$\begin{aligned}
\left\| x_{t+1}^{K^\star} - x_{t+1}^K(M_\Delta) \right\| &= \left\| \sum_{i=H+1}^{t} \left(\Psi_{t,i}^{K,t}(M_\Delta) - \widetilde{A}_{K^\star}^i \right) w_{t-i} \right\| \\
&\leq W \left(\sum_{i=H+1}^{t} \| \Psi_{t,i}^{K,t}(M_\Delta) \| + \sum_{i=H+1}^{t} \| \widetilde{A}_{K^\star}^i \| \right) \\
&\leq W \left(\sum_{i=H+1}^{t} \left(2\kappa^2 (1-\gamma)^i + H \kappa_B^2 \kappa^5 (1-\gamma)^{i-1} \right) \right) \\
&\leq W \left(2\kappa^2 (1-\gamma)^{H+1} \gamma^{-1} + H \kappa_B^2 \kappa^5 (1-\gamma)^H \gamma^{-1} \right) \\
&\leq \kappa^2 W (1-\gamma)^H \gamma^{-1} \left(2(1-\gamma) + H \kappa_B^2 \kappa^3 \right) \\
&\leq H \kappa_B^2 \kappa^5 W (1-\gamma)^H \gamma^{-1} (2(1-\gamma) + 1) \\
&\leq 2 W H \kappa_B^2 \kappa^5 (1-\gamma)^H \gamma^{-1},
\end{aligned}$$

where the second inequality makes use of Lemma 25. Next, we investigate the difference between the control signals,

$$\| u_{t+1}^{K^\star} - u_{t+1}^K(M_\Delta) \| = \left\| -K^\star x_{t+1}^{K^\star} - \left(-K x_{t+1}^K(M_\Delta) + \sum_{i=1}^{H} M_\Delta^{[i-1]} w_{t+1-i} \right) \right\|$$

$$
\begin{aligned}
&= \left\| -K^\star x_{t+1}^{K^\star} + K x_{t+1}^{K}(M_\Delta) - \sum_{i=1}^{H} (K - K^\star) \widetilde{A}_{K^\star}^{i-1} w_{t+1-i} \right\| \\
&= \left\| -K^\star \left(x_{t+1}^{K^\star} - \sum_{i=0}^{H-1} \widetilde{A}_{K^\star}^{i} w_{t-i} \right) + K \left(x_{t+1}^{K}(M_\Delta) - \sum_{i=0}^{H-1} \widetilde{A}_{K^\star}^{i} w_{t-i} \right) \right\| \\
&= \left\| -K^\star \sum_{i=H}^{t} \widetilde{A}_{K^\star}^{i} w_{t-i} + K \sum_{i=H}^{t} \Psi_{t,i}^{K,h}(M_\Delta) w_{t-i} \right\| \\
&\le 2WH\kappa_B^2\kappa^6(1-\gamma)^{H-1}\gamma^{-1}.
\end{aligned}
$$

Using above inequalities and Lipschitz assumption as well as the boundedness result (Lemma 25), we complete the proof. ■

The remaining part of this section lists useful supporting lemmas for studying non-stochastic control in unknown systems.

Lemma 29 (Preservation of Stability)**.** *Under Assumption 6, if K is (κ, γ)-strongly stable for a linear dynamical system $S = (A, B, \{\mathbf{w}\})$, i.e., $A - BK = QLQ^{-1}$, and $\|A - \widehat{A}\|_{\mathrm{F}}, \|A - \widehat{A}\|_{\mathrm{F}} \le \varepsilon_{A,B}$, then the same linear controller K is $\left(\kappa, \gamma - 2\kappa^3\varepsilon_{A,B}\right)$-strongly stable for the estimated system $\widehat{S} = (\widehat{A}, \widehat{B}, \{\widehat{w}\})$, i.e., $\widehat{A} - \widehat{B}K = Q\widehat{L}Q^{-1}$, where $\|\widehat{L}\| \le 1 - \gamma + 2\kappa^3\varepsilon_{A,B}$.*

Proof [of Lemma 29] First, we try to express the strong stability of K with respect to $(\widehat{A}, \widehat{B})$ as

$$
\begin{aligned}
\widehat{A} - \widehat{B}K &= A - BK + (\widehat{A} - A) - (\widehat{B} - B)K \\
&= QLQ^{-1} + (\widehat{A} - A) - (\widehat{B} - B)K \\
&= Q\left(L + Q^{-1}\left((\widehat{A} - A) - (\widehat{B} - B)K\right) Q\right) Q^{-1} \\
&:= \widehat{Q}\widehat{L}\widehat{Q}^{-1},
\end{aligned}
$$

where the last equality is by defining $\widehat{L} = L + Q^{-1}((\widehat{A} - A) - (\widehat{B} - B)K)Q$. Further, the operator norm of $\widehat{L}$ can be bounded as

$$
\begin{aligned}
\|\widehat{L}\|_{\mathrm{op}} &= \|L + Q^{-1}\left((\widehat{A} - A) - (\widehat{B} - B)K\right) Q\|_{\mathrm{op}} \\
&\le \|L\|_{\mathrm{op}} + \|Q^{-1}\|_{\mathrm{op}} \left(\|\widehat{A} - A\|_{\mathrm{op}} + \|K\|_{\mathrm{op}}\|\widehat{B} - B\|_{\mathrm{op}}\right) \|Q\|_{\mathrm{op}} \\
&\le (1 - \gamma) + \kappa \cdot (\varepsilon_{A,B} + \kappa \cdot \varepsilon_{A,B}) \cdot \kappa \\
&\le 1 - \gamma + 2\kappa^3\varepsilon_{A,B}.
\end{aligned}
$$

By definition of strong stability, it holds that K is $\left(\kappa, \gamma - 2\kappa^3\varepsilon_{A,B}\right)$-strongly stable for the estimated system $\widehat{S} = (\widehat{A}, \widehat{B}, \{\widehat{w}\})$. ■

Lemma 30 (Bounds of disturbances in the fictitious system (Hazan et al., 2020, Lemma 18))**.** *Under Assumptions 4, 6, if it holds that $\varepsilon_{A,B} \le 10^{-3}\kappa^{-10}\gamma^2$, then for any $t \ge T_0 + 1$,*

$$
\|x_t\|_2 \le 20\sqrt{d_u}\kappa^{11}\gamma^{-3}W, \quad \|w_t - \widehat{w}_t\|_2 \le 42\sqrt{d_u}\kappa^{12}\gamma^{-3}W\varepsilon_{A,B}, \quad \|\widehat{w}_{t-1}\|_2 \le 2\sqrt{d_u}\kappa^3\gamma^{-1}W.
$$

Lemma 31. *Under Assumption 8, it holds that* $\sigma_{\min}(C_k) \geq 1/\sqrt{\kappa_c}$*, where* C_k *is defined in* (47).

Proof [of Lemma 31] Under Assumption 8, it holds that $\|(C_k C_k^{\mathrm{T}})^{-1}\|_{\mathrm{op}} \leq \kappa_c$, i.e.,

$$\sigma_{\max}((C_k C_k^{\mathrm{T}})^{-1}) \leq \kappa_c.$$

It is apparent that $\left((C_k C_k^{\mathrm{T}})^{-1}\right)^{\mathrm{T}} = \left((C_k C_k^{\mathrm{T}})^{\mathrm{T}}\right)^{-1} = (C_k C_k^{\mathrm{T}})^{-1}$, i.e., $(C_k C_k^{\mathrm{T}})^{-1}$ is a symmetric matrix. Then we have

$$\begin{aligned} \sigma_{\max}((C_k C_k^{\mathrm{T}})^{-1}) &= \lambda_{\max}\left((C_k C_k^{\mathrm{T}})^{-1}\left((C_k C_k^{\mathrm{T}})^{-1}\right)^{\mathrm{T}}\right) = \lambda_{\max}\left((C_k C_k^{\mathrm{T}})^{-1}(C_k C_k^{\mathrm{T}})^{-1}\right) \\ &= \lambda_{\max}^2\left((C_k C_k^{\mathrm{T}})^{-1}\right) \leq \kappa_c. \end{aligned}$$

Finally we have $\sigma_{\min}(C_k) = \lambda_{\min}(C_k C_k^{\mathrm{T}}) \geq 1/\sqrt{\kappa_c}$, which finished the proof. ■

Lemma 32 ((Hazan et al., 2020, Lemma 17))**.** *For any matrix pair* $L, \widehat{L}$*, such that* $\|L\|_{\mathrm{op}}, \|\widehat{L}\|_{\mathrm{op}} \leq 1-\gamma, \gamma \in (0,1)$*, we have* $\sum_{t=0}^{\infty}\|L^t - \widehat{L}^t\|_{\mathrm{op}} \leq 3\gamma^{-2}\|L-\widehat{L}\|_{\mathrm{op}}$.

Lemma 33 (Perturbation Analysis (Hazan et al., 2020, Lemma 22))**.** *Let* $x^\star$ *be the solution to linear system* $Ax = b$*, and* $\widehat{x}$ *be the solution to* $(A+\Delta A)x = b+\Delta b$*, then if it holds that* $\|\Delta A\| \leq \sigma_{\min}(A)$*, it is true that*

$$\|x^\star - \widehat{x}\| \leq \frac{\|\Delta b\| + \|\Delta A\|\|x^\star\|}{\sigma_{\min}(A) - \|\Delta A\|_{\mathrm{op}}}.$$