Non-stationary Online Learning with Memory and Non-stochastic Control

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Abstract
We study the problem of Online Convex Optimization (OCO) with memory, which allows loss functions to depend on past decisions and thus captures temporal effects of learning problems. In this paper, we introduce dynamic policy regret as the performance measure to design algorithms robust to non-stationary environments, which competes algorithms’ decisions with a sequence of changing comparators. We propose a novel algorithm for OCO with memory that provably enjoys an optimal dynamic policy regret. The key technical challenge is how to control the switching cost, the cumulative movements of player’s decisions, which is neatly addressed by a novel decomposition of dynamic policy regret and an appropriate meta-expert structure. Furthermore, we generalize the results to the problem of online non-stochastic control, i.e., controlling a linear dynamical system with adversarial disturbance and convex loss functions. We derive a novel gradient-based controller with dynamic policy regret guarantees, which is the first controller competitive to a sequence of changing policies.

1. Introduction
Online Convex Optimization (OCO) (Shalev-Shwartz, 2012; Hazan, 2016) is a versatile model of learning in adversarial environments, which can be regarded as a sequential game between a player and an adversary (environments). At each round, the player makes a prediction from a convex set \( w_t \in W \subseteq \mathbb{R}^d \), the adversary simultaneously selects a convex loss \( f_t : W \rightarrow \mathbb{R} \), and the player incurs a loss \( f_t(w_t) \). The goal of the player is to minimize the cumulative loss.

The standard OCO framework only considers memoryless adversary, in the sense that the resulting loss is only determined by the player’s current prediction without involving past ones. In real-world applications, particularly those involve online decision making, it is often the case that past predictions/decisions would also contribute to current loss, which makes the standard OCO framework not viable. To remedy this issue, Online Convex Optimization with Memory (OCO with Memory) was proposed as a simplified and elegant model to capture the temporal effects of learning problems (Merhav et al., 2002; Anava et al., 2015). Specifically, at each round, the player makes a prediction \( w_t \in W \), the adversary chooses a loss function \( f_t : W^{m+1} \rightarrow \mathbb{R} \), and the player will then suffer a loss \( f_t(w_{t-m}, \ldots, w_t) \). Notably, now the loss function depends on both current and past
predictions. The parameter \( m \) is the memory length, and evidently the OCO with memory model reduces to the standard memoryless OCO when setting \( m = 0 \). The performance measure for OCO with memory is the \textit{policy regret} (Dekel et al., 2012), defined as

\[
\text{S-Regret}_T = \sum_{t=1}^{T} f_t(w_{t-m}, \ldots, w_t) - \min_{v \in \mathcal{W}} \sum_{t=1}^{T} \tilde{f}_t(v),
\]

where \( \tilde{f}_t : \mathcal{W} \mapsto \mathbb{R} \) is the unary loss defined by \( \tilde{f}_t(v) = f_t(v, \ldots, v) \). We start the index from 1 for convenience. Recent studies apply online learners with provably low policy regret to a variety of related problems (Chen et al., 2018; Agarwal et al., 2019; Daniely and Mansour, 2019; Chen et al., 2020). However, the policy regret (1) only measures the performance versus a \textit{fixed} comparator and is thus not suitable for learning in non-stationary environments.

\textbf{Our results.} This paper introduces \textit{dynamic policy regret} to measure the competitive performance of a learner against an arbitrary sequence of \textit{changing} comparators, defined as

\[
\text{D-Regret}_T(v_{1:T}) = \sum_{t=1}^{T} f_t(w_{t-m}, \ldots, w_t) - \sum_{t=1}^{T} \tilde{f}_t(v_t),
\]

where \( v_1, \ldots, v_T \in \mathcal{X} \) is the comparator sequence and is shorthanded by \( v_{1:T} \). An upper bound of \( \text{D-Regret}_T(v_{1:T}) \) will be a function of the comparator sequence \( v_{1:T} \). We note that the proposed performance measure is very general and subsumes the static policy regret (1) as a special case by setting the comparators as the best predictor in hindsight, i.e., \( v_{1:T} = v^* \in \arg \min_{v \in \mathcal{W}} \sum_{t=1}^{T} \tilde{f}_t(v) \). Thus, dynamic policy regret bound automatically implies static regret. Our main contributions are:

- We design a novel algorithm for OCO with memory that achieves an optimal \( \mathcal{O}(\sqrt{T(1 + P_T)}) \) \textit{dynamic policy regret} where \( P_T = \sum_{t=2}^{T} ||v_{t-1} - v_t||_2 \) denotes the \textit{unknown} path-length of the comparators.

- We apply the method to \textit{online non-stochastic control}, which yields the first controller that is competitive to a sequence of changing “disturbance-action” policies.

\textbf{Technical contribution.} The fundamental challenge of dynamic policy regret optimization is how to simultaneously compete with all comparator sequences with vastly different level of non-stationarity. Our approach builds upon the idea of meta-expert aggregation (Zhang et al., 2018a; Zhao et al., 2020b) to hedge the uncertainty, along with several new ingredients specifically designed for the OCO with memory setting. In particular, the static policy regret analysis (Anava et al., 2015) critically relies on controlling \textit{switching cost}—the cumulative movement of player’s predictions; which could scale linearly due to the meta-expert structure needed for the dynamic regret analysis. We elegantly address this issue by a novel regret decomposition and a switching-cost-regularized surrogate loss, which avoids explicitly handling switching cost all together and renders a unified design by online mirror descent for both meta- and expert-algorithms.

\textbf{Application to online control.} We further investigate the problem of \textit{online non-stochastic control}, i.e., controlling a linear dynamical system with adversarial (non-stochastic) disturbance and adversarial convex cost functions (Agarwal et al., 2019; Hazan et al., 2020). To design controllers robust to non-stationary environments, we introduce the \textit{dynamic policy regret} to compete the controller’s performance with any sequence of \textit{changing} compared controllers within the controller
class. By adopting the “disturbance-action” policy parameterization (Agarwal et al., 2019), the online non-stochastic control is reduced to the OCO with memory problem, and thus its dynamic policy regret can be optimized by a similar meta-expert structure as developed before. Our designed controller attains $\tilde{O}(\sqrt{T(1 + P_T)})$ dynamic policy regret, where $P_T$ measures the fluctuation of compared controllers and $\tilde{O}$-notation hides the logarithmic dependence on time horizon $T$. To the best of our knowledge, this is the first controller competitive to a sequence of changing “disturbance-action” policies. Our current results focus on the known system with fully observation, which serves as a foundational step for future developments. In particular, our results readily extend to the unknown system by system identification techniques (Hazan et al., 2020), to partially observed setting by using the idea of “nature’s y’s” (Simchowitz et al., 2020), etc.

This paper is organized as follows. In Section 3 problem setup and preliminaries are introduced. Section 2 discusses some related work. Section 4 and Section 5 present results for OCO with memory and online non-stochastic control, respectively. Section 6 concludes the paper. All the proofs are deferred to the appendices.

2. Related Work

In this section we present more discussions on the related work, including OCO with memory, online non-stochastic control, as well as dynamic regret minimization for online learning.

**Online Convex Optimization with Memory.** OCO with memory is initiated by Merhav et al. (2002), who prove an $O(T^{2/3})$ policy regret for convex and Lipschitz functions by a blocking technique. Later, for Lipschitz functions, Anava et al. (2015) propose a simple gradient-based algorithm that provably achieves $O(\sqrt{T})$ and $O(\log T)$ policy regret for convex and strongly convex functions, respectively. Recent study discloses that the policy regret of OCO with memory over exp-concave functions is at least $\Omega(T^{1/3})$ (Simchowitz, 2020, Theorem 2.3). Online learning with memory is also studied in the prediction with expert advice setting (Geulen et al., 2010; György and Neu, 2014; Altschuler and Talwar, 2018; Cesa-Bianchi et al., 2013; Altschuler and Talwar, 2018) and bandit settings (Dekel et al., 2012, 2014; Altschuler and Talwar, 2018; Arora et al., 2019). One of the key concepts of OCO with memory is the switching cost, cumulative movement of the decisions, which is also concerned in smoothed online learning (Chen et al., 2018; Goel et al., 2019) and competitive online learning (Daniely and Mansour, 2019).

**Online Non-stochastic Control.** Recently, there is a surge of interest to apply modern statistical and algorithmic techniques to the control problem. We focus on the online non-stochastic control setting proposed by Agarwal et al. (2019), where the regret is chosen as the performance measure and the disturbance is allowed to be adversarially chosen. Under conditions of convex and Lipschitz cost functions as well as adversarial disturbance, Agarwal et al. (2019) obtain an $O(\sqrt{T})$ policy regret for known linear dynamical system by introducing the DAC parameterization and reducing the problem to OCO with memory. Hazan et al. (2020) show an $O(T^{2/3})$ policy regret for unknown system via system identification. In addition, Foster and Simchowitz (2020) propose the online learning with advantages techniques and obtain logarithmic regret for known system with quadratic cost and adversarial disturbance, and the results are strengthened by Simchowitz (2020) to accommodate arbitrary, changing costs. All mentioned results are developed for fully observed system, and Simchowitz et al. (2020) present a clear picture for non-stochastic control with partially observed system. We are still witnessing a variety of recent advances, for example, non-stochastic
control with bandit feedback (Gradu et al., 2020a; Cassel and Koren, 2020), adaptive regret minimization (Gradu et al., 2020b), etc.

**Dynamic Regret.** Benchmarking regret in term of changing comparators can date back to early development of prediction with expert advice (Herbster and Warmuth, 1998, 2001). In the OCO setting, Zinkevich (2003) pioneers the dynamic regret against any comparator sequence and shows that OGD can attain an \( O(\sqrt{T(1 + P_T)}) \) dynamic regret. It is revealed by Zhang et al. (2018b) that the result is not tight, who establish an \( \Omega(\sqrt{T(1 + P_T)}) \) minimax lower bound for convex functions and close the gap by proposing an algorithm with optimal \( O(\sqrt{T(1 + P_T)}) \) rate. Recent improvement achieves problem-dependent dynamic regret by further exploiting the smoothness (Zhao et al., 2020b). Dynamic regret of bandit convex optimization is studies in (Zhao et al., 2020a). We finally emphasize that the dynamic regret measure studied in this paper is also called the universal dynamic regret, in that the guarantee holds universally against any comparator sequence in the domain. Another special form called the worst-case dynamic regret is also frequently investigated in the literature (Besbes et al., 2015; Jadbabaie et al., 2015; Mokhtari et al., 2016; Zhang et al., 2017; Baby and Wang, 2019; Zhao and Zhang, 2020; Zhang et al., 2020b), which specifies comparators as the optimizers of online functions. The worst-case dynamic regret is less general than the universal one, and the reader is referred to the work of Zhang et al. (2018a) for more discussions.

**More Discussions.** We finally discuss two papers (Chen et al., 2018; Gradu et al., 2020b). In addition to OCO with memory, switching cost is also investigated in smoothed OCO, particularly there are some recently efforts devoted to dynamic regret of smoothed OCO (Chen et al., 2018, Section 5). We note that the settings of two problems are different: smoothed OCO requires to observe the cost function \( f_t \) first and then choose the decision \( w_t \in W \); while OCO with memory decides \( w_t \) without the knowledge of \( f_t \). Additionally, the dynamic regret bound of smoothed OCO (Chen et al., 2018, Corollary 11) needs prior knowledge of path-length \( P_T \), and our techniques might be useful in removing this undesired requirement. Besides, online non-stochastic control in non-stationary environments is recently studied in (Gradu et al., 2020b), where the measure is chosen as (weakly) adaptive policy regret (Hazan and Seshadhri, 2009) — the regret compared to the best policy on any interval in time horizon. We note that even for the standard online convex optimization setting, dynamic regret and adaptive regret reflect different perspective of environments and are incomparable. More discussions can be found in (Zhang et al., 2020a).

### 3. Problem Setup and Preliminaries

In this section, we formalize the problem setup and introduce some preliminaries.

#### 3.1 Problem Setup

Online Convex Optimization (OCO) with memory is a variant of standard OCO framework to capture long-term effects of past decisions. The protocol is shown as follows.

1. **for** \( t = m + 1, \ldots, T \) **do**
2. the player chooses a decision \( w_t \in W \);
3. the adversary reveals the loss \( f_t : W^{m+1} \rightarrow \mathbb{R} \) that applies to last \( m + 1 \) decisions;
4. the player suffers a loss of \( f_t(w_{t-m}, \ldots, w_t) \);
5. **end for**
In above, \( m \) is the memory length, and \( f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R} \) is convex in memory, i.e., its unary function \( \tilde{f}_t(w) = f_t(w, \ldots, w) \) is convex in \( w \). Clearly, OCO with memory recovers the standard memoryless OCO when \( m = 0 \). The standard measure in the literature is policy regret (Dekel et al., 2012), as shown in (1). We introduce and study the dynamic policy regret, defined as

\[
\text{D-Regret}_{T}(v_1:T) = \sum_{t=1}^{T} f_t(w_{t-m:t}) - \sum_{t=1}^{T} \tilde{f}_t(v_t),
\]

where \( v_1, \ldots, v_T \in \mathcal{W} \) is the comparator sequence arbitrarily chosen by the environments. We note that the dynamic policy regret holds universally against any comparator sequence, while the algorithm should be agnostic to the choice of comparators. So algorithms that optimize the dynamic regret are more adaptive to non-stationary environments, whereas the gain is accompanied with challenge on how to tackle the uncertainty of the environmental non-stationarity.

We conclude this part by introducing several standard assumptions that might be used in the theoretical analysis.

\textbf{Assumption 1} (coordinate-wise Lipschitzness). The function \( f_t : \mathcal{W}^{m+1} \mapsto \mathbb{R} \) is \( L \)-coordinate-wise Lipschitz, i.e.,

\[
|f_t(x_0, \ldots, x_m) - f_t(y_0, \ldots, y_m)| \leq L \sum_{i=0}^{m} ||x_i - y_i||_2.
\]

\textbf{Assumption 2} (bounded gradient). The gradient norm of the unary loss is at most \( G \), i.e., for all \( w \in \mathcal{W} \) and \( t \in [T] \),

\[
||\nabla \tilde{f}_t(w)||_2 \leq G.
\]

\textbf{Assumption 3} (bounded domain). The domain \( \mathcal{W} \) is convex, closed, and satisfies \( ||w - w'||_2 \leq D \) for any \( w, w' \in \mathcal{W} \). For convenience of analysis, we further assume \( 0 \in \mathcal{W} \).

### 3.2 Static Policy Regret of OCO with Memory

Before presenting the dynamic policy regret minimization of OCO with memory, we review the result of static policy regret. Anava et al. (2015) propose a simple approach based on the gradient descent, whose crucial observation is that when online functions are coordinate-wise Lipschitz, the policy regret \( R_T = \sum_{t=1}^{T} f_t(w_{t-m:t}) - \min_{v \in \mathcal{W}} \sum_{t=1}^{T} \tilde{f}_t(v) \) can be upper bounded by the switching cost and the vanilla regret over the unary loss, namely,

\[
R_T \leq \lambda \sum_{t=2}^{T} ||w_t - w_{t-1}||_2 + \sum_{t=1}^{T} \tilde{f}_t(w_t) - \min_{v \in \mathcal{W}} \sum_{t=1}^{T} \tilde{f}_t(v),
\]

where \( \lambda = m^2 L \). The first term is the switching cost measuring the cumulative movement of decisions \( w_{1:T} \), and the remaining term is the standard regret of memoryless OCO. As a result, it is natural to perform Online Gradient Descent (OGD) (Zinkevich, 2003) over the unary loss \( \tilde{f}_t \),

\[
w_{t+1} = \Pi_{\mathcal{W}}[w_t - \eta \nabla \tilde{f}_t(w_t)],
\]

where \( \eta > 0 \) is the step size, and \( \Pi_{\mathcal{W}}[\cdot] \) denotes projection onto the nearest point in \( \mathcal{W} \). It is well-known that with an appropriate step size OGD enjoys an \( \mathcal{O}\sqrt{T} \) regret in memoryless OCO.
Furthermore, Anava et al. (2015) argue that the produced decisions move sufficiently slowly. Indeed, switching cost
\[ \sum_{t=2}^{T} \| w_t - w_{t-1} \|_2^2 \leq O(\eta T), \]
which will not affect the final regret order by choosing \( \eta = O(1/\sqrt{T}) \). Thus, they prove an \( O(\sqrt{T}) \) policy regret as shown below.

**Theorem 1** (Theorem 3.1 of Anava et al. (2015)). Under Assumptions 1–3, running OGD over unary loss achieves
\[ \sum_{t=1}^{T} f_t(w_t) - \min_{v \in W} \sum_{t=1}^{T} \tilde{f}_t(v) \leq (G^2 + m^2 LG)\eta T + \frac{2D^2}{\eta}. \]

By setting step size optimally as \( \eta = \eta^* = \sqrt{\frac{2D^2}{(G^2 + m^2 LG)T}} \), we attain an \( O(\sqrt{T}) \) static policy regret.

### 4. Online Convex Optimization with Memory

In this section we present the results for dynamic regret of OCO with memory. We begin with the gentle case when path-length is known, and then handle the general case when it is unknown. We elucidate the challenge and show how to resolve it by novel algorithmic ingredients, and finally present the dynamic policy regret analysis.

#### 4.1 A Gentle Start: known path-length \( P_T \)

In the following, we generalize Theorem 1 by showing that the algorithm (5) also enjoys the dynamic policy regret.

**Theorem 2.** Under Assumptions 1–3, running OGD over unary loss satisfies that for any \( v_1, \ldots, v_T \in W \), we have
\[ \text{D-Regret}_T(v_{1:T}) \leq (G^2 + m^2 LG)\eta T + \frac{1}{2\eta} (D^2 + 2DP_T), \]
where \( P_T = \sum_{t=2}^{T} \| v_t - v_{t-1} \|_2^2 \) is the path-length measuring the fluctuation of the comparator sequence.

Suppose the path-length \( P_T \) were known, we could obtain an \( O(\sqrt{T(1 + P_T)}) \) dynamic policy regret by setting step size \( \eta = \sqrt{\frac{D^2 + 2DP_T}{(G^2 + m^2 LG)T}} \), which matches the \( \Omega(\sqrt{T(1 + P_T)}) \) lower bound of memoryless OCO (Zhang et al., 2018a).

However, the above step size setting is not realistic in that it requires the knowledge of path-length \( P_T \) a priori. In fact, the comparator sequence \( v_1, \ldots, v_T \) can be arbitrarily selected by the environments, and thus \( P_T = \sum_{t=2}^{T} \| v_{t-1} - v_t \|_2 \) reflects the environmental non-stationarity and is unknown to the player. The similar challenge also emerges in recent studies of memoryless OCO (Zhang et al., 2018a; Zhao et al., 2020b), inspired by which we employ the by-now-standard meta-expert framework to hedge the non-stationarity. In Section 4.2, we will elucidate the challenge of applying the framework to OCO with memory setting, and in Section 4.3 we demonstrate how to resolve the issue by designing several novel and necessary technical ingredients.

#### 4.2 Challenge: switching cost of meta-expert structure

In the development of dynamic regret of memoryless OCO, the meta-expert framework is proposed to deal with the unknown path-length emerging in the optimal step size tuning (Zhang et al., 2018a;
Zhao et al., 2020b). Below we briefly review the framework and elucidate the challenge of its application in OCO with memory.

**Meta-expert framework.** The framework is essentially an online ensemble method (Zhou, 2012), which consists of three components: the pool of candidate step sizes, the expert-algorithm, and the meta-algorithm. We need first design an appropriate pool of candidate step sizes \( \mathcal{H} = \{ \eta_1, \ldots, \eta_N \} \) and ensure that there exists a step size \( \eta_k \) that approximates the optimal step size \( \eta^* \) well. Then, multiple experts \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) are maintained where each performs OGD with a step size \( \eta_i \in \mathcal{H} \) and generates the decision sequence \( \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_T \), as

\[
\mathbf{w}_{t+1,i} = \Pi_{\mathcal{W}}[\mathbf{w}_{t,i} - \eta_t \nabla g_t(\mathbf{w}_{t,i})],
\]

where \( g_t : \mathcal{W} \rightarrow \mathbb{R} \) is a certain (surrogate) loss function to optimize. Finally, a meta-algorithm, supposed to be able to track the best expert-algorithm, is used to combine all the intermediate results of experts to produce the final decisions \( \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_T \), where \( \mathbf{w}_t = \sum_{i=1}^N p_t,i \mathbf{w}_{t,i} \).

Similar to static policy regret analysis (4), dynamic policy regret is also upper bounded by switching cost and dynamic regret of memoryless OCO over the unary loss \( \tilde{f}_t \),

\[
\text{D-Regret}_T \leq \lambda \sum_{t=2}^T \| \mathbf{w}_t - \mathbf{w}_{t-1} \|_2 + \sum_{t=1}^T \tilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \tilde{f}_t(\mathbf{v}_t).
\]

The results of memoryless OCO (Zhang et al., 2018a) ensure \( \sum_{t=1}^T \tilde{f}_t(\mathbf{w}_t) - \sum_{t=1}^T \tilde{f}_t(\mathbf{v}_t) \leq \mathcal{O}(\sqrt{T(1 + P_T)}) \), providing proper choices of the step size pool and meta-expert algorithms. Thus, it suffices to further control switching cost.

**Switching cost.** The switching cost is the pivot of the analysis for OCO with memory even for static policy regret optimization (Anava et al., 2015; Foster and Simchowitz, 2020). We elucidate that it is particularly challenging to control the switching cost in dynamic regret analysis, due to the meta-expert structure. Indeed, suppose we adopt the standard methods of dynamic regret optimization for memoryless OCO (Zhang et al., 2018a), then the switching cost \( S_T = \sum_{t=2}^T \| \mathbf{w}_t - \mathbf{w}_{t-1} \|_2 \) can be bounded by

\[
S_T \leq D \sum_{t=2}^T \| \mathbf{p}_t - \mathbf{p}_{t-1} \|_1 + \sum_{t=2}^T \sum_{i=1}^N p_{t,i} \| \mathbf{w}_{t,i} - \mathbf{w}_{t-1,i} \|_2.
\]

The first term is the switching cost of the meta-algorithm, which is at most \( \mathcal{O}(\sqrt{T}) \). However, the second term could be very large and even grow linearly with iterations. In fact, the switching cost of expert-algorithm \( \mathcal{E}_i \) (OGD with step size \( \eta_i \)) is \( \mathcal{O}(\eta_i T) \). On the other hand, to ensure a coverage of the optimal step size, the pool of candidate step sizes is usually set as \( \mathcal{H} = \{ \eta_i = \mathcal{O}(2^i \cdot T^{-\frac{1}{2}}), i \in [N] \} \) such that \( \eta_i = \mathcal{O}(T^{-\frac{1}{2}}) \) and \( \eta_N = \mathcal{O}(1) \). Therefore, experts with larger step sizes would incur unacceptable switching cost, for instance, the switching cost of expert \( \mathcal{E}_N \) could grow linearly, of order \( \mathcal{O}(T) \). As a result, the second term, the weighted combination of experts’ switching cost, could be enlarged by the experts whose step sizes are too large and therefore become difficult to control.

We further empirically illustrate the issue and the result is presented in Figure 1. The simulation records the switching cost of final decisions attained by the meta-expert structure, and we supply
To algorithmically enforce low switching cost, we present a new decomposition of dynamic policy regret in Section 4.3, and then develop appropriate expert- and meta-learners in Section 4.4 to deliver an optimal dynamic policy regret. The green line plots the switching cost attained by our method, which is drastically smaller than that of the standard method.

4.3 Algorithmically Enforcing Low Switching Cost: a new decomposition of dynamic policy regret

To avoid directly controlling the switching cost of final predictions, we propose a novel switching-cost-regularized meta-expert decomposition, which inspires us to design a new algorithm that algorithmically enforcing low switching cost by regularization, hence allowing us to address the challenge arisen in the last subsection.

Denote by $g_t(w) = \langle \nabla f_t(w_t), w \rangle$ the linearized function of $f_t$ over the decision $w_t$, then clearly we have

$$D\text{-Regret}_T \leq \lambda \sum_{t=2}^{T} ||w_t - w_{t-1}||_2 + \sum_{t=1}^{T} g_t(w_t) - \sum_{t=1}^{T} g_t(v_t) = \sum_{t=1}^{T} M_{t,i} + \sum_{t=1}^{T} E_{t,i},$$

where $\sum_{t=1}^{T} M_{t,i}$ and $\sum_{t=1}^{T} E_{t,i}$ are the meta-regret and expert-regret, with the instantaneous quantity defined as,

$$M_{t,i} = (\lambda ||w_t - w_{t-1}||_2 + g_t(w_t)) - (\lambda ||w_{t-1,i} - w_{t-1,i}||_2 + g_t(w_{t-1,i})),
\quad E_{t,i} = \lambda ||w_{t,i} - w_{t-1,i}||_2 + g_t(w_{t,i}) - g_t(v_t).$$

The decomposition holds for any expert index $i \in [N]$. Notice that the key characteristic here is to incorporate a switching-cost regularizer into the regret decomposition. Then, clearly the expert-
regret is now the dynamic regret with switching cost of each expert $E_t$, namely,

$$
\sum_{t=1}^{T} E_{t,i} = \lambda \sum_{t=2}^{T} \|w_{t,i} - w_{t-1,i}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i}) - \sum_{t=1}^{T} g_t(v_t). \tag{6}
$$

We further show that the meta-regret can be upper bounded by static regret with switching cost of the meta-algorithm.

**Proposition 3.** The meta-regret is at most

$$
\sum_{t=1}^{T} M_{t,i} \leq \lambda \sum_{t=2}^{T} \|p_{t} - p_{t-1}\|_1 + \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i}, \tag{7}
$$

where the loss vector $\ell_t \in \mathbb{R}^N$ is defined by

$$
\ell_{t,i} := \lambda \|w_{t,i} - w_{t-1,i}\|_2 + g_t(w_{t,i}), \tag{8}
$$

and the coefficients are $\lambda = m^2 L$ and $\lambda' = m^2 DL$.

Notably, the key algorithmic novelty of the meta-algorithm is to optimize the switching-cost-regularized loss (8). The intuition is that the meta-algorithm would impose more penalty on expert-algorithms with larger switching cost, to enforce low switching cost of the sequence final decisions.

Owing to the new decomposition, we avoid explicitly handling switching cost all together (i.e., $\sum_{t=2}^{T} \|w_{t} - w_{t-1}\|_2$). Instead, we only need to tackle switching cost of individual expert-algorithm (i.e., $\sum_{t=2}^{T} \|w_{t,i} - w_{t-1,i}\|_2$) and that of meta-algorithm (i.e., $\sum_{t=2}^{T} \|p_{t} - p_{t-1}\|_1$), as shown in (6) and (7), which turns out to be much easier to control. In the next part, we will present a unified algorithmic design for expert- and meta-learners with tolerance of switching cost.

### 4.4 A Unified Design by Online Mirror Descent

We present a unified view for algorithmic design of meta- and expert-algorithms by Online Mirror Descent (OMD). OMD starts from any $w_1 \in \mathcal{W}$, and at iteration $t$, the algorithm performs the following update:

$$
w_{t+1} = \arg \min_{w \in \mathcal{W}} \eta \langle \nabla f_t(w_t), w \rangle + \mathcal{D}_\psi(w, w_t), \tag{9}
$$

where $\eta > 0$ is the step size. The regularizer $\psi : \mathcal{W} \mapsto \mathbb{R}$ is a differentiable convex function defined on $\mathcal{W}$ and is assumed (without loss of generality) to be 1-strongly convex with respect to some norm $\| \cdot \|$ over $\mathcal{W}$. The induced Bregman divergence $\mathcal{D}_\psi$ is defined by $\mathcal{D}_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$. The following generic result gives an upper bound of the dynamic regret with switching cost of OMD algorithm.

**Theorem 4.** Online Mirror Descent (9) satisfies that

$$
\lambda \sum_{t=2}^{T} \|w_{t} - w_{t-1}\| + \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(v_t) \leq \frac{1}{\eta} (R^2 + \gamma P_T) + \eta (\lambda G + G^2) T, \tag{10}
$$

provided that $\mathcal{D}_\psi(x, z) - \mathcal{D}_\psi(y, z) \leq \|x - y\|$ holds for any $x, y, z \in \mathcal{W}$. In above, $R^2 = \sup_{x, y \in \mathcal{W}} \mathcal{D}_\psi(x, y)$, and $G = \sup_{w \in \mathcal{W}, \ell \in [T]} \|\nabla f_t(w)\|_*$. Note that the above result holds for any comparator sequence $v_1, \ldots, v_T \in \mathcal{W}$.
Remark 1. The dynamic regret of Theorem 4 holds against any comparator sequence in the domain, in particular, we can set comparators as the best fixed decision in hindsight and thus obtain static regret with switching cost, \[ R^2 \sum_{t=2}^{T} \| w_t - w_{t-1} \| + \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(w^*) \leq \frac{R^2}{\eta} + \eta (\lambda G + G^2) T, \] that holds for any \( w^* \in \mathcal{W} \). A technical caveat is that when deriving the static regret, the Bregman divergence is not required to satisfy the Lipschitz condition.

Theorem 4 exhibits general dynamic regret analysis for OMD algorithm. By flexibly choosing the regularizer \( \psi \) and comparator sequence \( v_1, \ldots, v_T \), we can obtain the following two implications, which corresponds to expert-regret (dynamic regret with switching cost of OGD) and meta-regret (static regret with switching cost of Hedge).

Implication 1 (expert-algorithm). To attain an appropriate expert-algorithm, we set

- the online function \( f_t \) as the linearized loss \( g_t(w) = \langle \nabla \tilde{f}_t(w), w \rangle \) and the feasible set as \( \mathcal{W} \);
- the regularizer as \( \psi(w) = \frac{1}{2} \| w \|_2^2 \), which is 1-strongly convex with respect to \( \| \cdot \|_2 \) norm;
- the step size as \( \eta_i > 0 \) for expert \( E_i \).

Then, OMD recovers the expert-algorithm OGD, where expert \( E_i \) performs the gradient descent with step size \( \eta_i \):

\[ w_{t+1,i} = \Pi_{\mathcal{W}}[w_{t,i} - \eta_i \nabla g_t(w_{t,i})] = \Pi_{\mathcal{W}}[w_{t,i} - \eta_i \nabla \tilde{f}_t(w_t)]. \]

The second equality holds by definition of the linearized loss, and we can observe that at iteration \( t \) all the experts perform OGD with the same gradient \( \nabla \tilde{f}_t(w_t) \), which means our algorithm only needs to query the gradient of the surrogate loss \( \tilde{f}_t \) only once at each round. Moreover, it can be verified that under such settings \( \gamma = D, R_{max}^2 = D^2 / 2 \) and \( \| \nabla f_t(w) \|_2 \leq G \). As a result, Theorem 4 implies that the expert-regret satisfies

\[ \sum_{t=1}^{T} E_{t,i} \leq \frac{1}{2\eta_i} (D^2 + 2DP_T) + \eta_i (\lambda G + G^2) T, \] (11)

which holds for any comparator sequence \( v_1, \ldots, v_T \in \mathcal{W} \).

Implication 2 (meta-algorithm). To achieve an appropriate meta-algorithm, we set

- the online function as \( f_t(p) = \langle p, \ell_t \rangle \) and the feasible set as the simplex \( \mathcal{W} = \Delta_N \);
- the regularizer as negative entropy \( \psi(p) = \langle p, \ln p \rangle \), which is 1-strongly convex with respect to \( \| \cdot \|_1 \) norm;
- the step size (or we call it learning rate) as \( \varepsilon > 0 \).

Then, OMD recovers Hedge with a learning rate \( \varepsilon \).

\[ p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell_{t,i})}{\sum_{i=1}^{N} p_{t,i} \exp(-\varepsilon \ell_{t,i})}. \] (12)

For technical considerations, we adopt a non-uniform initialization by setting \( p_t \in \Delta_N \) with \( p_{t,i} = \frac{1}{i(i+1) \cdot N+1} \cdot \frac{N+1}{N} \). Besides, it can be verified that \( R^2 = \ln N \) and \( \| \nabla f_t(p) \|_\infty = \| \ell_t \|_\infty \leq \lambda G + DG \). By
setting a fixed comparator \( v_{1:T} = e_i \), the \( i \)-th standard basis of \( \mathbb{R}^N \)-space, Theorem 4 (with a slight modification) implies that for any \( i \in [N] \),
\[
\sum_{t=1}^{T} M_{t,i} \leq \frac{\ln(1/p_{1,i})}{\varepsilon} + \varepsilon(2\lambda + G)(\lambda + G)D^2T = \mathcal{O}\left(\lambda \sqrt{T} \ln(1/p_{1,i})\right),
\]
where we choose the learning rate \( \varepsilon \propto \sqrt{1/(\lambda^2 T)} \). Note that the dependence of learning rate on \( T \) can be removed by either a time-varying tuning or doubling trick.

### 4.5 Overall Algorithm and Theoretical Guarantee

We propose the Switching-Cost-Regularized meta-Expert Aggregation for OCO with Memory (SCREAM) approach, which is based on the online mirror descent algorithm and admits a structure of meta-expert aggregation.

We first specify the step size pool in the following:
\[
\mathcal{H} = \left\{ \eta_i \mid \eta_i = 2^{i-1}, \sqrt{\frac{D^2}{(\lambda G + G^2)T}}, i \in [N] \right\},
\]
where \( N = \lceil \frac{1}{2} \log_2(1 + T) \rceil + 1 = \mathcal{O}(\log T) \) is the number of experts. The overall algorithm consists of meta-algorithm (Algorithm 1) and expert-algorithm (Algorithm 2). Our method enjoys the following dynamic policy regret bound.

**Theorem 5.** Under Assumptions 1–3, by setting learning rate as \( \varepsilon = \sqrt{2/(2(\lambda + G)(\lambda + G)D^2T)} \) and the step size pool \( \mathcal{H} \) as (14), our proposed algorithm enjoys
\[
\sum_{t=1}^{T} f_t(w_{t-m,t}) - \sum_{t=1}^{T} \tilde{f}_t(v_t) \leq D\sqrt{2(2\lambda + G)(\lambda + G)T} \left(1 + \ln(\lceil \log_2(1 + 2P_T/D) \rceil + 2)\right) + 2\sqrt{2(G^2 + \lambda G)(D^2 + 2DP_T)T}
\]
\[
= \mathcal{O}\left(\sqrt{T}(1 + P_T)\right),
\]
where \( \lambda = m^2L \) and \( P_T = \sum_{t=2}^{T} \|v_{t-1} - v_t\|_2 \). The result holds for any comparator sequence \( v_1, \ldots, v_T \in \mathcal{W} \).

**Remark 2.** First, since the dynamic policy regret holds for any comparator sequence, by simply setting comparators as the fixed best decision in hindsight (and now \( P_T = 0 \)), our dynamic policy regret implies the \( \mathcal{O}(\sqrt{T}) \) static regret in Theorem 1. Second, the attained dynamic policy regret is minimax optimal in terms of the dependence on time horizon \( T \) and the path-length \( P_T \), because an \( \Omega(\sqrt{T}(1 + P_T)) \) lower bound has been established for the dynamic regret of memoryless OCO (Zhang et al., 2018a), which is a special case of OCO with memory when setting \( m = 0 \). Finally, we note that it remains unclear about the optimal memory dependence and will left as future work for investigation.

### 5. Online Non-stochastic Control

In this section we present the algorithm for optimizing the dynamic policy regret of online non-stochastic control.
Algorithm 1 SCREAM: Meta-algorithm

**Input:** step size pool $\mathcal{H} = \{\eta_1, \ldots, \eta_N\}$; learning rate $\varepsilon$

1. Initialization: any feasible $w_1, w_2, \ldots, w_m \in \mathcal{W}$
2. Initialization: non-uniform weight $p_{m+1} \in \Delta_N$,
   \[
   p_{m+1,i} = \frac{1}{i(i+1)} \cdot \frac{N+1}{N}, \text{ for } i \in [N]
   \]
3. for $t = m+1$ to $T$ do
4.   Receive $w_{t,i}$ from expert $E_i$
5.   Compute the surrogate loss of expert $E_i$:
   \[
   \ell_{t,i} = \lambda \|w_{t,i} - w_{t-1,i}\|_2 + g_t(w_{t,i})
   \]
6.   Submit the decision $w_t = \sum_{i=1}^N p_{t,i} w_{t,i}$
7.   Update weight $p_{t+1,i}$ by
   \[
   p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell_{t,i})}{\sum_{i=1}^N p_{t,i} \exp(-\varepsilon \ell_{t,i})}
   \]
8. end for

Algorithm 2 SCREAM: Expert-algorithm

**Input:** step size $\eta_i \in \mathcal{H}$

1. Initialization: pick any feasible decision $w_{m,i} \in \mathcal{W}$
2. for $t = m+1$ to $T$ do
3.   Perform $w_{t+1,i} = \Pi_{\mathcal{W}}[w_{t,i} - \eta_i \nabla \tilde{f}_t(w_t)]$
4. end for

5.1 Problem Statement and Performance Measure

**Problem Setting.** We study the online control problem: at iteration $t$, the controller provides the control $u_t$ upon the observed dynamical state $x_t$ and suffers a cost of $c_t(x_t, u_t)$, where $c_t : \mathbb{R}^{d_x} \times \mathbb{R}^{d_u} \mapsto \mathbb{R}$ is a convex function. We focus on the linear dynamical system (LDS), which is governed by the following dynamics equation:
\[
x_{t+1} = Ax_t + Bu_t + w_t,
\]
where $A$, $B$ are system matrices and $w_t$ is the disturbance (noise) not known to the controller in advance. Note that following the notational convention of previous works, throughout the section we will use unbold fonts to denote vectors (including control signal, state, disturbance, etc).

We focus on the online non-stochastic control setting (Agarwal et al., 2019; Hazan et al., 2020), namely, no statistical assumption is imposed on the disturbance distribution. The adversarial nature of the disturbance hinders an a priori computation of the optimal policy as in the settings of classical control theory (Kalman, 1960), and we will leverage recent advance in online control and the results of OCO with memory presented in the last section to address the issue.

**Policy Regret.** The standard measure for online non-stochastic control is the game-theoretic policy regret (Agarwal et al., 2019; Hazan et al., 2020), defined as the difference between cumulative
loss of the designed controller $\mathcal{A}$ and that of the compared controller $\pi \in \Pi$,

$$\text{Regret}_T = J_T(A) - \min_{\pi \in \Pi} J_T(\pi) = \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi). \quad (16)$$

Note that the compared controller $\pi$ could be chosen with complete foreknowledge of the disturbance and loss functions within the compared policy class $\Pi$.

In this paper, we generalize the standard measure (16) to the dynamic policy regret, to make controllers more robust to non-stationary environments. Specifically, instead of competing with a fixed controller, we benchmark the algorithm with a sequence of time-varying controllers $\pi_1, \ldots, \pi_T$ from a certain controller class $\Pi$, and thereby define the dynamic policy regret for online non-stochastic control,

$$\text{D-Regret}_T = J_T(A) - J_T(\pi_1, \ldots, \pi_T) = \sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}), \quad (17)$$

where $\pi_1, \ldots, \pi_T \in \Pi$ is the sequence of compared controllers. Note that the measure subsumes static policy regret (16) when choosing compared controllers as a fixed one, namely, $\pi_1 = \ldots = \pi_T = \pi^\star \in \arg \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi)$. In this work, we compete with benchmark class of the disturbance-action controllers (DAC) (cf. Definition 1), which encompasses many controllers of interest.

5.2 Reduction to OCO with Memory

Following the pioneering work (Agarwal et al., 2019), we will work on the Disturbance-Action Controller (DAC) class, which parametrizes the executed action as a linear function of the past disturbances. By doing so, we can reduce the online non-stochastic control to OCO with memory so that the results of Section 4 can be leveraged to design robust controllers with provably dynamic policy regret guarantees.

**Definition 1** (Disturbance-Action Controller, DAC). A disturbance-action controller $\pi(K, M)$ with memory $H$ is specified by a fixed matrix $K$ (required to be strongly stable) and parameters $M = (M^{[1]}, \ldots, M^{[H]})$. At each iteration $t$, controller $\pi(K, M)$ chooses the action $u_t$ as a linear map of past disturbances with an offset linear controller, formally,

$$u_t = -K x_t + \sum_{i=1}^{H} M^{[i]} w_{t-i}.$$ 

For convenience, we define disturbance $w_i = 0$ for $i < 0$.

Note that the DAC controller can be implemented because the disturbance can be perfectly recovered by $w_{t-1} = x_t - A_t x_{t-1} - B_t u_{t-1}$ as system dynamics $A_t$ and $B_t$ are known.

As shown by Agarwal et al. (2019), the dynamical state obtained by executing any DAC controller can be represented by a linear function of the parameters of the policy.
Proposition 6. Suppose the initial state is $x_0 = 0$ and one chooses the DAC controller $\pi(K, M_t)$ at iteration $t$, the reaching state and the corresponding DAC control are

$$x^K_t(M_0:t-1) = \sum_{i=0}^{H+t-1} \Psi^K_{t-1,i}(M_0:t-1)w_{t-1-i},$$

$$u^K_t(M_0:t) = -Kx^K_t(M_0:t-1) + \sum_{i=1}^{H} M^{[i-1]}_t w_{t-i},$$

where $A_K = A - BK$ and

$$\Psi^{K,h}_{t,i}(M_{t-h:t}) = \tilde{A}^i_K \mathbf{1}_{i \leq h} + \sum_{j=0}^{h} \tilde{A}^j_K BM_{i-j}^{[i-j-1]} \mathbf{1}_{i-j \leq H}.$$

Evidently, both state $x_t$ and control signal $u_t$ are linear functions of DAC parameters $M_0, \ldots, M_t$, so the cost $c_t(x^K_t(M_0:t-1), u^K_t(M_0:t))$ as a function of $M_0:t$ is convex. The problem is reminiscent of online convex optimization with memory (Anava et al., 2015). However, there is one big caveat in applying the technique—the current memory length is not fixed but growing with time, which is not feasible in OCO with memory setting. A truncated method is proposed by Agarwal et al. (2019) to address the issue, which truncates the state with a fixed memory length $H$ and thereby defines the truncated loss.

Definition 2 (Truncated Loss). The truncated loss $f_t : \mathcal{M}^{H+2} \mapsto \mathbb{R}$ is defined as

$$f_t(M_{t-1-H:t}) = c_t(y^K_t(M_{t-1-H:t-1}), v^K_t(M_{t-1-H:t})), $$

where truncated state $y_t$ and DAC control $v_t$ are

$$y^{K}_{t+1}(M_{t-H:t+1}) = \sum_{i=0}^{2H} \Psi^{K,H}_{t,i}(M_{t-H:t})w_{t-i},$$

$$v^{K}_{t+1}(M_{t-H:t+1}) = -Ky^{K}_{t+1}(M_{t-H:t}) + \sum_{i=1}^{H} M^{[i-1]}_{t+1} w_{t+1-i}. $$

The truncated loss $f_t$ is then fed to the OCO with memory framework for regret minimization with a memory length of $H + 2$. On the other hand, the error introduced by the truncation (the gap between $f_t$ and $c_t$) can be precisely controlled. As a result, we make the reduction from online non-stochastic control to OCO with memory.

5.3 Dynamic Regret of Online Non-stochastic Control

Based on the above reduction, we can leverage the results in Section 4 to design an online controller to compete with a sequence of changing compared policies. Our main algorithm, SCREAM.CONTROL, combines the two ideas:

1. DAC parameterization for reduction: using DAC control $u_t = \pi(K, M_t)$ to parametrize the space and define the unary loss of the truncated loss, i.e., $\tilde{f}_t : \mathcal{M} \mapsto \mathbb{R}$ with $\tilde{f}_t(M) = f_t(M, \ldots, M)$, defined in Definition 2.
Algorithm 3 SCREAM.CONTROL: Meta-algorithm

**Input:** step size pool $\mathcal{H} = \{\eta_1, \ldots, \eta_N\}$; learning rate $\varepsilon$; memory length $H$; linear controller $K$; feasible set $\mathcal{M}$

1: Initialization: any feasible $M_1, M_2, \ldots, M_m \in \mathcal{M}$
2: Initialization: non-uniform weight $p_{m+1} \in \Delta_N$,
$$p_{m+1,i} = \frac{1}{i(i+1)} \cdot \frac{N+1}{N}, \text{ for } i \in [N]$$
3: for $t = H + 1$ to $T$ do
4:   Receive $M_{t,i}$ from expert $E_i$
5:   Compute the surrogate loss of expert $E_i$:
$$\ell_{t,i} = \lambda \|M_{t,i} - M_{t-1,i}\|_F + g_t(M_{t,i})$$
6:   Obtain the parameter $M_t = \sum_{i=1}^N p_{t,i} M_{t,i}$
7:   Update the weight $p_{t+1,i}$ by
$$p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell_{t,i})}{\sum_{i=1}^N p_{t,i} \exp(-\varepsilon \ell_{t,i})}$$
8:   Output DAC control $u_t = -K x_t + \sum_{i=1}^H M_{t-i,w_{t-i}}$
9:   Observe the new state $x_{t+1}$ and calculate the disturbance $w_t = x_{t+1} - A x_t - B u_t$
10: end for

Algorithm 4 SCREAM.CONTROL: Expert-algorithm

**Input:** step size $\eta_i \in \mathcal{H}$

1: Initialization: pick any feasible decision $M_{H,i} \in \mathcal{M}$
2: for $t = H + 1$ to $T$ do
3:   Perform $M_{t+1,i} = \Pi_M[M_{t,i} - \eta_i \nabla \tilde{f}_t(M_t)]$, where $\Pi_M[\cdot]$ denotes the Euclidean projection
4: end for

(2) Meta-expert aggregation for OCO with memory: performing SCREAM algorithm of Section 4 over the unary loss $\tilde{f}_t$, and combining intermediate parameters $M_{t,1}, \ldots, M_{t,N}$ from all experts $E_1, \ldots, E_N$ to produce the final parameter $M_t$ by the meta-algorithm.

Therefore, SCREAM.CONTROL algorithm also consists of meta-algorithm and expert-algorithm. We describe the meta-algorithm in Algorithm 3, and present the expert-algorithm in Algorithm 4.

Before showing the dynamic policy regret of our proposed algorithm, we introduce several common assumptions used in the literature of online non-stochastic control (Agarwal et al., 2019; Hazan et al., 2020; Gradu et al., 2020a).

Assumption 4. The system matrices are bounded, i.e., $\|A\|_{op} \leq \kappa_A$ and $\|B\|_{op} \leq \kappa_B$. Besides, the disturbance $w_t$ is bounded by $W$, i.e., $\|w_t\| \leq W$ holds for any $t \in [T]$. 
Assumption 5. The costs $c_t(x, u)$ are convex. Further, as long as it is guaranteed that $\|x\|, \|u\| \leq D$, it holds that
\[ |c_t(x, u)| \leq \beta D^2, \text{ and } \|\nabla_x c_t(x, u)\|, \|\nabla_u c_t(x, u)\| \leq G_c D. \]

Assumption 6. The DAC controller $\pi(K, M)$ satisfies:

1. the matrix $K$ is $(\kappa, \gamma)$-strongly stable, whose precise definition is provided in Definition 3 of Appendix A.2;
2. parameters satisfy $M \in \mathcal{M}$, with domain $\mathcal{M} = \{M = (M^{[1]}, \ldots, M^{[H]}) \mid \|M^{[i]}\|_{op} \leq \kappa B \kappa^3 (1 - \gamma)^i\}$.

Theorem 7. Under Assumptions 4–6, by setting the learning rate optimally and the step size pool $\mathcal{H}$ as
\[ \mathcal{H} = \left\{ \eta_i \mid \eta_i = 2^{i-1} \cdot \sqrt{\frac{D_f^2}{(\lambda G_f + G_f^2)T}}, i \in [N] \right\}, \] (18)
where $N = \lceil \frac{1}{2} \log_2 (1 + T) \rceil + 1 = O(\log T)$ is the number of experts, and $\lambda = (H + 2)^2 L_f$. The parameters $L_f, G_f, D_f$ are defined in Lemma 19 and only depend on the natural parameters of the linear dynamical system and the hyperparameter $H$. By choosing the truncated memory length $H = \Theta(\log T)$, our SCREAM.CONTROL algorithm enjoys
\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \leq \tilde{O}(\sqrt{T(1 + P_T)}), \]
where $\pi_1, \ldots, \pi_T \in \Pi$ is any comparator sequence from the compared DAC policy class $\Pi = \{\pi(K, M) \mid M \in \mathcal{M}\}$. The path-length $P_T$ is the cumulative variation of compared policies, defined as $P_T = \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F$. The $\tilde{O}(\cdot)$-notation hides poly-logarithmic factors in time horizon $T$.

Our dynamic policy regret can recover the $\tilde{O}(\sqrt{T})$ static policy regret of the pioneering paper (Agarwal et al., 2019).

Corollary 8. Under the same assumptions of Theorem 7, SCREAM.CONTROL enjoys the following static policy regret,
\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \leq \tilde{O}(\sqrt{T}), \]
where the comparator set $\Pi$ can be chosen as either the set of DAC policies or the set of strongly linear controllers.

6. Conclusion

In this paper, we investigate the dynamic policy regret of online convex optimization with memory and online non-stochastic control. For OCO with memory, we propose the SCREAM algorithm and prove an optimal $O(\sqrt{T(1 + P_T)})$ dynamic policy regret, where $P_T$ is the path-length of the comparator sequence that reflects the environmental non-stationarity. Our approach admits a structure
of meta-expert aggregation to deal with the unknown environments, and introduces a novel meta-expert decomposition via switching-cost regularized surrogate loss so that the switching cost can be successfully controlled. The method is further used to design robust controllers for online non-stochastic control, where the underlying disturbance could be chosen adversarially. We adopt the DAC parameterization and then develop the SCREAM.CONTROL algorithm that provably achieves an $O(\sqrt{T(1 + P_T)})$ dynamic policy regret, where $P_T$ is the path-length of compared controllers. Minimizing dynamic policy regret facilitates our controller with more robustness, in that it can compete with any sequence time-varying controllers instead of a fixed one in the controller class.

In the future, we will explore the possibility of extension to bandit feedback, where the only feedback to the controller is the loss value (Gradu et al., 2020a; Cassel and Koren, 2020).

Acknowledgment

The work was partially done while Peng Zhao remotely visited UC Santa Barbara. The authors thank Ming Yin and Dheeraj Baby for helpful discussions.

References


A. Preliminaries

A.1 Dynamic Regret of Memoryless OCO

In this part we present the dynamic regret analysis of the online gradient descent (OGD) algorithm for memoryless online convex optimization (Zinkevich, 2003; Zhang et al., 2018a).

We first specify the problem settings and notations of memoryless online convex optimization. Specifically, the player iteratively selects a decision \( w \in W \) from a convex set \( W \subseteq \mathbb{R}^d \) and then suffers a loss of \( f_t(w) \), in which the loss function \( f_t : W \rightarrow \mathbb{R} \) is assumed to be convex and chosen adversarially by the environments. The performance measure we are concerned with is the dynamic regret, defined as

\[
\text{D-Regret}_T(v_1, \ldots, v_T) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(v_t),
\]

where \( v_1, \ldots, v_T \in W \) is the comparator sequence arbitrarily chosen in the domain by the adversaries. The critical advantage of the above dynamic regret measure is that it supports to compete with a sequence of time-varying comparators, instead of a fixed one as specified in the standard (static) regret.

In the development of dynamic regret minimization of memoryless OCO, one of the most crucial building blocks is the well-known Online Gradient Descent (OGD) algorithm (Zinkevich, 2003), which starts from any \( w_1 \in W \) and performs the following update,

\[
w_{t+1} = \Pi_W[w_t - \eta \nabla f_t(w_t)].
\] (19)

Here, \( \eta > 0 \) is step size and \( \Pi_W[\cdot] \) denotes the Euclidean projection onto the nearest point in \( W \).

The standard textbooks of online convex optimization (Shalev-Shwartz, 2012; Hazan, 2016) show that OGD can achieves an optimal \( O(\sqrt{T}) \) static regret for convex functions, providing with appropriate step size settings. Furthermore, such a simple algorithm actually also enjoys the following dynamic regret guarantee (Zinkevich, 2003, Theorem 2), and we supply the proof for self-containedness.

**Theorem 9.** Let \( W \subseteq \mathbb{R}^d \) be a bounded convex and compact set in Euclidean space, and we denote by \( D \) an upper bound of the diameter of the domain, i.e., \( \| w - w' \|_2 \leq D \) holds for any \( w, w' \in W \). Suppose the gradient norm of \( f_t \) over \( W \) is bounded by \( G \), i.e., \( \| \nabla f_t(w) \|_2 \leq G \) holds for any \( w \in W \) and \( t \in [T] \). Then, OGD (19) enjoys the following dynamic regret,

\[
\text{D-Regret}_T(v_1, \ldots, v_T) \leq \frac{\eta}{2} G^2 T + \frac{1}{2\eta}(D^2 + 2DP_T),
\]

which holds for any comparator sequence \( v_1, \ldots, v_T \in W \), and \( P_T = \sum_{t=2}^{T} \| v_{t-1} - v_t \|_2 \) is the path-length that measures the cumulative movements of the comparator sequence.

**Proof** [of Theorem 9] Since the online functions are convex, we have

\[
\text{D-Regret}_T(v_1, \ldots, v_T) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(v_t) \leq \sum_{t=1}^{T} \langle \nabla f_t(w_t), w_t - v_t \rangle.
\]
Thus, it suffices to bound the sum of $\langle \nabla f_t(w_t), w_t - v_t \rangle$ over iterations. Note that from the update rule in (43),

\[
\|w_{t+1} - v_t\|^2 \leq \|w_t - \eta \nabla f_t(w_t) - v_t\|^2
\]

The inequality holds due to Pythagorean theorem (Hazan, 2016, Theorem 2.1). After rearranging, we obtain

\[
\langle \nabla f_t(w_t), w_t - v_t \rangle \leq \frac{\eta}{2} \|\nabla f_t(w_t)\|^2 + \frac{1}{2\eta} (\|w_t - v_t\|^2 - \|w_{t+1} - v_t\|^2).\]

Summing the above inequality from $t = 1$ to $T$ yields,

\[
\text{D-Regret}_T(v_1, \ldots, v_T) \leq \frac{\eta}{2} \sum_{t=1}^T \|\nabla f_t(w_t)\|^2 + \frac{1}{2\eta} \left(\sum_{t=1}^T (\|w_t - v_t\|^2 - \|w_{t+1} - v_t\|^2)\right).\]

We further provide an upper bound for the second term in the right hand side. Indeed,

\[
\sum_{t=1}^T (\|w_t - v_t\|^2 - \|w_{t+1} - v_t\|^2) \leq \sum_{t=1}^T \|w_t - v_t\|^2 - \sum_{t=2}^T \|w_t - v_{t-1}\|^2
\]

\[
\leq \|w_1 - v_1\|^2 + \sum_{t=2}^T (\|w_t - v_t\|^2 - \|w_t - v_{t-1}\|^2)
\]

\[
= \|w_1 - v_1\|^2 + \sum_{t=2}^T (v_{t-1} - v_t, 2w_t - v_{t-1} - v_t)
\]

\[
\leq D^2 + 2D \sum_{t=2}^T \|v_{t-1} - v_t\|_2.
\]

Combining all above inequalities, we have

\[
\text{D-Regret}_T(v_1, \ldots, v_T) \leq \frac{\eta}{2} \sum_{t=1}^T \|\nabla f_t(w_t)\|^2 + \frac{1}{2\eta} \left(D^2 + 2D \sum_{t=2}^T \|v_{t-1} - v_t\|_2\right)
\]

\[
\leq \frac{\eta}{2} G^2 T + \frac{1}{2\eta} (D^2 + 2DP_T).
\]

Hence, we complete the proof.

\[\blacksquare\]

**A.2 Additional Notions**

We introduce the formal definition of strongly stable linear controllers (Cohen et al., 2018; Agarwal et al., 2019). Indeed, the stable condition can guarantee the convergence, but nothing can be ensured about the rate of convergence. While working on the class of strongly stable controllers, we can establish the non-asymptotic convergence rate.
Definition 3. A linear controller $K$ is $(\kappa, \gamma)$-strongly stable if there exist matrices $L, H$ satisfying $A - BK = HLH^{-1}$, such that the following two conditions are satisfied:

1. The spectral norm of $L$ satisfies $\|L\| \leq 1 - \gamma$.
2. The controller and transforming matrices are bounded, i.e., $\|K\| \leq \kappa$ and $\|H\|, \|H^{-1}\| \leq \kappa$.

A.3 Technical Lemmas

The following lemma plays an important role in analyzing algorithms based on the mirror descent.

Lemma 10 (Lemma 3.2 of Chen and Teboulle (1993)). Let $X$ be a convex set in a Banach space $B$. Let $f : X \mapsto \mathbb{R}$ be a closed proper convex function on $X$. Given a convex regularizer $\psi : X \mapsto \mathbb{R}$, we denote its induced Bregman divergence by $D_{\psi}(\cdot, \cdot)$. Then, any update of the form

$$x_k = \arg \min_{x \in X} \left\{ f(x) + D_{\psi}(x, x_{k-1}) \right\}$$

satisfies the following inequality

$$f(x_k) - f(u) \leq D_{\psi}(u, x_{k-1}) - D_{\psi}(u, x_k) - D_{\psi}(x_k, x_{k-1})$$

for any $u \in X$.

Lemma 11. If the regularizer $\psi : X \mapsto \mathbb{R}$ is $\lambda$-strongly convex with respect to a norm $\|\cdot\|$, then we have the following lower bound for the induced Bregman divergence: $D_{\psi}(x, y) \geq \frac{1}{2}\|x - y\|$.

B. Omitted Details for Section 4 (OCO with Memory)

In this section, we present omitted details for Section 4 OCO with memory, including proofs of Theorem 2, Proposition 3, Theorem 4, and Theorem 5. Moreover, we supply more details of the empirical validation.

B.1 Proof of Theorem 2

Proof. The coordinate-Lipschitz continuity of $f_t$ (Assumption 1) implies that

$$|f_t(w_{t-m}, \ldots, w_t) - \tilde{f}_t(w_t)| \leq L \cdot \sum_{i=1}^{m} \|w_t - w_{t-i}\|_2 \leq mL \sum_{i=1}^{m} \|w_{t-i+1} - w_{t-i}\|_2$$

Therefore, we have

$$\sum_{t=m}^{T} f_t(w_{t-m}, \ldots, w_t) - \sum_{t=m}^{T} \tilde{f}_t(w_t) \leq m^2 L \sum_{t=m}^{T} \|w_t - w_{t-1}\|_2,$$

and the dynamic policy regret can be thus upper bounded by

$$\sum_{t=1}^{T} f_t(w_{t-m}, \ldots, w_t) - \sum_{t=1}^{T} \tilde{f}_t(v_t)$$
where we define $\lambda := m^2 L$ for notational convenience. As a result, to derive the dynamic policy regret, it is sufficient to examine the switching cost of the algorithm, i.e., $\sum_{t=2}^{T} \|w_{t-1} - w_t\|_2$, as well as the dynamic regret over the surrogate loss, i.e., $\sum_{t=1}^{T} \tilde{f}_t(w_t) - \tilde{f}_t(v_t)$.

First, by the non-expansive property of the projection operator, we can derive an upper bound for the switching cost:

$$\sum_{t=1}^{T} \|w_t - w_{t-1}\|_2 = \sum_{t=1}^{T} \|\Pi_W[w_{t-1} - \eta g_{t-1}] - w_{t-1}\|_2 \leq \sum_{t=1}^{T} \|\tilde{w}_t - w_{t-1}\|_2 = \eta \sum_{t=1}^{T} \|g_{t-1}\|_2 \leq \eta GT. \quad (21)$$

Next, since the sequence of surrogate loss $\{\tilde{f}_t\}_{t=1}^{T}$ is convex and memoryless, from the standard dynamic regret analysis (Zinkevich, 2003; Zhang et al., 2018a), as shown in Theorem 9, we have

$$\sum_{t=1}^{T} \tilde{f}_t(w_t) - \sum_{t=1}^{T} \tilde{f}_t(v_t) \leq \frac{\eta}{2} G^2 T + \frac{1}{2\eta} (D^2 + 2DP_T), \quad (22)$$

where $P_T = \sum_{t=2}^{T} \|v_t - v_{t-1}\|_2$ is the path-length measuring the fluctuation of the comparator sequence $v_1, v_2, \ldots, v_T$. Combining above two inequalities (21) and (22) yields

$$\sum_{t=m}^{T} f_t(w_{t-m}, \ldots, w_t) - \sum_{t=m}^{T} \tilde{f}_t(v_t) \leq \frac{\eta}{2} (G^2 + 2\lambda G) T + \frac{1}{2\eta} (D^2 + 2DP_T),$$

with $\lambda = m^2 L$. We thus complete the proof. \hfill \blacksquare

### B.2 Proof of Proposition 3

**Proof** We first analyze the switching cost of the final prediction sequence.

$$\|w_t - w_{t-1}\|_2 = \left\| \sum_{i=1}^{N} p_{t,i} w_{t,i} - \sum_{i=1}^{N} p_{t-1,i} w_{t-1,i} \right\|_2 \leq \left\| \sum_{i=1}^{N} p_{t,i} w_{t,i} - \sum_{i=1}^{N} p_{t,i} w_{t-1,i} \right\|_2 + \left\| \sum_{i=1}^{N} p_{t,i} w_{t-1,i} - \sum_{i=1}^{N} p_{t-1,i} w_{t-1,i} \right\|_2 \leq \sum_{i=1}^{N} p_{t,i} \|w_{t,i} - w_{t-1,i}\|_2 + D \sum_{i=1}^{N} |p_{t,i} - p_{t-1,i}|$$
\begin{equation}
\sum_{i=1}^{N} p_{t,i} \| w_{t,i} - w_{t-1,i} \|_2 + D \| p_t - p_{t-1} \|_1 \tag{23}
\end{equation}

We introduce the surrogate loss vector \( \ell_t \in \mathbb{R}^N \), defined as \( \ell_{t,i} := \lambda \| w_{t,i} - w_{t-1,i} \|_2 + g_t(w_{t,i}) \). By Assumptions 2 and 3, we can assure that the loss \( |\ell_{t,i}| \leq \lambda D + GD \) in that

\[
|\ell_{t,i}| \leq \lambda \| w_{t,i} - w_{t-1,i} \|_2 + \| \nabla f_t(w_t) \|_2 \| w_t \|_2 \\
\leq \lambda \| w_{t,i} - w_{t-1,i} \|_2 + \| \nabla \tilde{f}_t(w_t) \|_2 \| w_t \|_2 \\
\leq \lambda D + GD.
\]

Therefore, we have the following upper bound for the instantaneous meta-regret with respect to the expert \( i \).

\[
M_{t,i} = (\lambda \| w_t - w_{t-1} \|_2 + g_t(w_t)) - (\lambda \| w_t - w_{t-1} \|_2 + g_t(w_t)) \\
\leq \lambda D \| p_t - p_{t-1} \|_1 + \lambda \sum_{i=1}^{N} p_{t,i} \| w_{t,i} - w_{t-1,i} \|_2 + \sum_{i=1}^{N} p_{t,i} g_t(w_{t,i}) - (\lambda \| w_{t,i} - w_{t-1,i} \|_2 + g_t(w_{t,i})) \\
= \lambda D \| p_t - p_{t-1} \|_1 + \langle p_t, \ell_t \rangle - \ell_{t,i},
\]

where the coefficient is \( \lambda' = \lambda D = m^2 DL \). Summing the instantaneous meta-regret over all the iterations yields,

\[
\sum_{t=1}^{T} M_{t,i} \leq \lambda' \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1 + \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i},
\]

which completes the proof.

\[\blacksquare\]

**B.3 Proof of Theorem 4**

Before presenting the proof of Theorem 4, we first analyze the switching cost of the online mirror descent, as demonstrated in the following stability lemma.

**Lemma 12.** For Online Mirror Descent (9), the instantaneous switching cost is at most

\[
\| w_t - w_{t+1} \| \leq \eta \| \nabla f_t(w_t) \|.
\]

**Proof** [of Lemma 12] From the update procedure of OMD (9) and Lemma 10, we know that

\[
\langle w_{t+1} - w_t, \eta \nabla f_t(w_t) \rangle \leq D_\psi(w_t, w_{t+1}) - D_\psi(w_t, w_{t+1}) - D_\psi(w_{t+1}, w_t),
\]

which implies

\[
D_\psi(w_t, w_{t+1}) + D_\psi(w_{t+1}, w_t) \leq \langle w_t - w_{t+1}, \eta \nabla f_t(w_t) \rangle.
\]
Since the regularizer \( \psi \) is chosen as a 1-strongly convex function with respect to the norm \( \| \cdot \| \), by Lemma 11 we have

\[
\mathcal{D}_\psi(w_t, w_{t+1}) + \mathcal{D}_\psi(w_{t+1}, w_t) \geq \|w_t - w_{t+1}\|^2.
\]

Combining above two inequalities and further applying the H"older’s inequality, we obtain that

\[
\|w_t - w_{t+1}\|^2 \leq \langle w_t - w_{t+1}, \eta \nabla f_t(w_t) \rangle \leq \|w_t - w_{t+1}\| \|\eta \nabla f_t(w_t)\|_*.
\]

Therefore, we conclude that \( \|w_t - w_{t+1}\| \leq \eta \|\eta \nabla f_t(w_t)\|_* \) and finish the proof.

Based on the above stability lemma, we can now prove Theorem 4 regarding dynamic regret with switching cost for OMD.

**Proof** [of Theorem 4] Notice that the dynamic regret can be decomposed as follows:

\[
\sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(v_t) \leq \sum_{t=1}^{T} \langle \nabla f_t(w_t), w_t - v_t \rangle
\]

\[
= \sum_{t=1}^{T} \langle \nabla f_t(w_t), w_t - w_{t+1} \rangle + \sum_{t=1}^{T} \langle \nabla f_t(w_t), w_{t+1} - v_t \rangle.
\]

From Lemma 12 and H"older’s inequality, we have

\[
\text{term (a)} \leq \sum_{t=1}^{T} \|\nabla f_t(w_t)\|_* \|w_t - w_{t+1}\| \leq \eta \sum_{t=1}^{T} \|\nabla f_t(w_t)\|_*^2. \tag{25}
\]

Next, we investigate the term (b):

\[
\text{term (b)} \leq \frac{1}{\eta} \sum_{t=1}^{T} (\mathcal{D}_\psi(v_t, w_t) - \mathcal{D}_\psi(v_t, w_{t+1}) - \mathcal{D}_\psi(w_{t+1}, w_t))
\]

\[
\leq \frac{1}{\eta} \sum_{t=2}^{T} (\mathcal{D}_\psi(v_t, w_t) - \mathcal{D}_\psi(v_{t-1}, w_t)) + \mathcal{D}_\psi(v_1, w_1)
\]

\[
\leq \frac{\gamma}{\eta} \sum_{t=2}^{T} \|v_t - v_{t-1}\| + \frac{1}{\eta} R^2, \tag{26}
\]

where the first inequality holds due to Lemma 10, and the second inequality makes uses of the non-negativity of the Bregman divergence. The last inequality holds due to the assumption of Lipschitz property that \( \mathcal{D}_\psi(x, z) - \mathcal{D}_\psi(y, z) \leq \gamma \|x - y\| \) holds for any \( x, y, z \in W \).

Furthermore, the switching cost can be bounded by Lemma 12,

\[
\sum_{t=2}^{T} \|w_t - w_{t-1}\| \leq \eta \sum_{t=2}^{T} \|\nabla f_{t-1}(w_{t-1})\|_* \tag{27}
\]
Combining (25), (26), and (27), we can attain that
\[
\begin{align*}
\lambda \sum_{t=2}^{T} \|w_t - w_{t-1}\| + \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(v_t) \\
\leq \frac{1}{\eta} (R^2 + \gamma P_T) + \eta \sum_{t=1}^{T} (\lambda \|
abla f_t(w_t)\|_* + \|
abla f_{t-1}(w_{t-1})\|_2^2) \\
\leq \frac{1}{\eta} (R^2 + \gamma P_T) + \eta (\lambda G + G^2) T,
\end{align*}
\]
which finishes the proof.

In the following, we further present a corollary regarding the static regret with switching cost for the meta-algorithm, which is essentially a specialization of OMD algorithm by setting the negative-entropy regularizer.

**Corollary 13.** Setting the negative-entropy regularizer \(\psi(p) = \sum_{i=1}^{N} p_i \log p_i\) and learning rate \(\varepsilon > 0\) for OMD. Suppose \(\|\ell_t\|_\infty \leq G\) holds for any \(t \in [T]\) and the algorithm starts from the initial weight \(p_1 \in \Delta_N\), then we have
\[
\begin{align*}
\lambda \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1 + \sum_{t=1}^{T} (p_t, \ell_t) - \sum_{t=1}^{T} \ell_{t,i} &\leq \frac{\ln(1/p_{1,i})}{\varepsilon} + \varepsilon (\lambda G + G^2) T. \\
\end{align*}
\]

**Proof** [of Corollary 13] From the proof of Theorem 4, we can easily obtain that
\[
\begin{align*}
\lambda \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1 + \sum_{t=1}^{T} (p_t, \ell_t) - \sum_{t=1}^{T} \ell_{t,i} &\leq \frac{D_\psi(e_i, p_1)}{\varepsilon} + \varepsilon (\lambda G + G^2) T.
\end{align*}
\]
When choosing the negative-entropy regularizer, the induced Bregman divergence becomes Kullback-Leibler divergence, i.e., \(D_\psi(q, p) = KL(q, p) = \sum_{i=1}^{N} q_i \ln \frac{q_i}{p_i}\). Therefore, \(D_\psi(e_i, p_1) = \ln(1/p_{1,i})\), which implies the desired result.

**B.4 Proof of Theorem 5**

**Proof** Dynamic policy regret can be related to switching cost and dynamic regret of surrogate loss:
\[
\begin{align*}
\sum_{t=1}^{T} f_t(w_{t-m:t}) - \sum_{t=1}^{T} \tilde{f}_t(v_t) &\leq \lambda \sum_{t=1}^{T} \|w_t - w_{t-1}\|_2 + \sum_{t=1}^{T} \tilde{f}_t(w_t) - \sum_{t=1}^{T} \tilde{f}_t(v_t) \\
&\leq \lambda \sum_{t=1}^{T} \|w_t - w_{t-1}\|_2 + \sum_{t=1}^{T} g_t(w_t) - \sum_{t=1}^{T} g_t(v_t),
\end{align*}
\]
where the last step uses the convexity of \(\tilde{f}_t\) and the definition of linearized loss \(g_t(w) = \langle \nabla \tilde{f}_t(w_t), w \rangle\). We will formally prove that our proposed algorithm essentially optimizes the right hand side.
We next decompose the overall dynamic regret into two parts: meta-regret and expert-regret, and then evaluate each term individually. Indeed, the following decomposition holds for any expert index $i \in [N]$,

$$
\text{D-Regret}_T \leq \lambda \sum_{t=1}^{T} \|w_t - w_{t-1}\|_2 + \sum_{t=1}^{T} g_t(w_t) - \sum_{t=1}^{T} g_t(v_t)
= \left( \lambda \sum_{t=2}^{T} \|w_t - w_{t-1}\|_2 + \sum_{t=1}^{T} g_t(w_t) \right) - \left( \lambda \sum_{t=2}^{T} \|w_{t,i} - w_{t-1,i}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i}) \right)
+ \left( \lambda \sum_{i=2}^{T} \|w_{t,i} - w_{t-1,i}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i}) - \sum_{t=1}^{T} g_t(v_t) \right).
$$

For convenience, in the following we use the notation $M_{t,i}$ to denote the instantaneous meta-regret with respect to the expert $i$,

$$
M_{t,i} = \left( \lambda \|w_t - w_{t-1}\|_2 + g_t(w_t) \right) - \left( \lambda \|w_{t,i} - w_{t-1,i}\|_2 + g_t(w_{t,i}) \right).
$$

Similar, we use the notation $E_{t,i}$ to denote the instantaneous expert-regret of the expert $i$,

$$
E_{t,i} = \lambda \|w_{t,i} - w_{t-1,i}\|_2 + g_t(w_{t,i}) - g_t(v_t).
$$

**Bounding meta-regret.** We first examine the meta-regret. In fact, Proposition 3 implies the following upper bound of meta-regret,

$$
\text{meta-regret} = \sum_{t=1}^{T} M_{t,i} \leq \lambda' \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1 + \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i},
$$

where the coefficient is $\lambda' = m^2 DL$.

Notice that our meta-algorithm actually performs the Hedge algorithm over the surrogate loss vector $\{\ell_t\}_{t=1}^{T}$, and meanwhile as shown in Implication 2 of Section 4.4, Hedge can be actually regarded as a specialization of OMD algorithm, by setting

- the online function $f_t$ as $f_t(p) = \langle p, \ell_t \rangle$ and the feasible set as $\mathcal{W} = \Delta_N$;
- the regularizer as negative entropy $\psi(p) = \langle p, \ln p \rangle$, which is 1-strongly convex with respect to $\|\cdot\|_1$ norm;
- the step size (or we call it learning rate) as $\varepsilon > 0$.

Under such settings, OMD recovers Hedge with a learning rate $\varepsilon$:

$$
p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell_{t,i})}{\sum_{i=1}^{N} p_{t,i} \exp(-\varepsilon \ell_{t,i})},
$$
It can be verified that $R_{\text{meta}}^2 = \ln N$ and $G_{\text{meta}} = \sup_{p \in \Delta_N} \|\nabla f_t(p)\|_\infty = \max_{t \in [T]} \|\ell_t\|_\infty \leq (\lambda + G)D$. Moreover, we will adopt a non-uniform weight initialization, specifically, the initial weight $p_1 \in \Delta_N$ is set as $p_{1,i} = \frac{1}{i(i+1)} \cdot \frac{N+1}{N}$, for $i \in [N]$.

By setting a fixed comparator $v_1 = v_2 = \ldots = v_T = e_i$, the $i$-th standard basis of $\mathbb{R}^N$-space, Theorem 4 implies that for any $i \in [N]$, we have

$$\lambda' \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1 + \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i} \leq \varepsilon (\lambda' G_{\text{meta}} + G_{\text{meta}}^2)T + \frac{D_\psi(e_i, p_1)}{\varepsilon}$$

$$= \varepsilon (2\lambda + G)(\lambda + G)D^2T + \frac{\ln(1/p_{1,i})}{\varepsilon}$$

$$\leq \varepsilon (2\lambda + G)(\lambda + G)D^2T + \frac{2\ln(i+1)}{\varepsilon}.$$ 

By choosing the learning rate as $\varepsilon = \varepsilon^\ast = \sqrt{\frac{2}{(2\lambda+G)(\lambda+G)DT}}$, we can obtain that

$$\text{meta-regret} = \sum_{t=1}^{T} M_{t,i} \leq D\sqrt{2(2\lambda + G)(\lambda + G)T(1 + \ln(i + 1))}.$$ (31)

Note that the dependence of learning rate tuning on $T$ can be removed by either a time-varying tuning or doubling trick.

**Bounding expert-regret.** Our next objective is to bound the expert-regret.

$$\text{expert-regret} = \sum_{t=1}^{T} E_{t,i} = \lambda \sum_{t=2}^{T} \|w_{t,i} - w_{t-1,i}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i}) - \sum_{t=1}^{T} g_t(v_t).$$

Notice that our expert-algorithm actually performs the OGD algorithm over the linearized loss function $g_t : \mathcal{W} \mapsto \mathbb{R}$, and meanwhile as shown in Implication 1 of Section 4.4, OGD is a special case of OMD algorithm, by setting

- the online function $f_t$ as $g_t$ defined as the linearized surrogate loss and the feasible set as $\mathcal{W}$;
- the regularizer as $\psi(w) = \frac{1}{2} \|w\|_2^2$, which is 1-strongly convex w.r.t $\|\cdot\|_2$ norm;
- the step size as $\eta_i > 0$ for expert $i$.

Under such settings, OMD recovers OGD with a constant step size $\eta_i$ of expert $E_i$:

$$w_{t+1,i} = \Pi_{\mathcal{W}}[w_{t,i} - \eta_i \nabla g_t(w_{t,i})] = \Pi_{\mathcal{W}}[w_{t,i} - \eta_i \nabla f_t(w_t)].$$

As a result, denote by $P_T = \sum_{t=2}^{T} \|v_{t-1} - v_t\|_2$ the path-length, Theorem 4 implies that the expert-regret satisfies that

$$\text{expert-regret} = \sum_{t=1}^{T} E_{t,i} \leq \eta_i (G^2 + \lambda G)T + \frac{1}{2\eta_i} (D^2 + 2DP_T),$$ (32)

which holds for any comparator sequence $v_1, \ldots, v_T \in \mathcal{W}$ as well as any expert $i \in [N]$. 

29
Bounding overall dynamic regret. Due to the boundedness of the path-length, we know that the optimal step size \( \eta^* \) provably lies in the range of \([\eta_1, \eta_N]\). Furthermore, by the construction of the pool of candidate step sizes, we can confirm that there exists an index \( i^* \in [N] \) ensuring \( \eta_{i^*} \leq \eta_{i^*+1} = 2\eta_{i^*} \). Therefore, we know that \( i^* \leq \lceil \frac{1}{2} \log_2 \left( 1 + \frac{2PT}{D} \right) \rceil + 1 \).

Notice that the dynamic regret decomposition (29) holds for any expert index \( i \in [N] \). Thus, in particular, we can choose the index \( i^* \) and achieve the following result by using the upper bounds of meta-regret (31) and expert-regret (32).

\[
D\text{-Regret}_T = \sum_{t=1}^{T} f_t(w_{t-m:t}) - \sum_{t=1}^{T} \tilde{f}_t(v_t) \leq \left( \lambda \sum_{t=2}^{T} \|w_t - w_{t-1}\|_2 + \sum_{t=1}^{T} g_t(w_t) \right) - \left( \lambda \sum_{t=2}^{T} \|w_{t,i^*} - w_{t-1,i^*}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i^*}) \right) 
+ \left( \lambda \sum_{t=2}^{T} \|w_{t,i^*} - w_{t-1,i^*}\|_2 + \sum_{t=1}^{T} g_t(w_{t,i^*}) - \sum_{t=1}^{T} g_t(v_t) \right)
\leq D \sqrt{2(2\lambda + G)(\lambda + G)T(1 + \ln(i^* + 1)) + \eta_{i^*}(G^2 + \lambda G)T + \frac{1}{2\eta_{i^*}}(D^2 + 2DP_T)}
\leq D \sqrt{2(2\lambda + G)(\lambda + G)T(1 + \ln(i^* + 1)) + \eta_{i^*}(G^2 + \lambda G)T + \frac{1}{\eta_{i^*}}(D^2 + 2DP_T)}
\leq D \sqrt{2(2\lambda + G)(\lambda + G)T \left( 1 + \ln \left( \left\lceil \log_2 \left( 1 + \frac{2PT}{D} \right) \right\rceil + 2 \right) \right)}
\leq O(\sqrt{T(1 + \log \log P_T)})
+ 2\sqrt{2 \left( \frac{G^2 + \lambda G}{D^2 + 2DP_T} \right) T}
\leq O(\sqrt{T(1 + P_T)})
\leq O(\sqrt{T(1 + P_T)}).
\]

We hence complete the proof of Theorem 5.

B.5 More Details of Empirical Validation of Section 4.2

In this part, we supply Section 4.2 with more details of empirical validation to show the challenge of switching cost due to the meta-expert structure. In the experiments, the online functions are set as

\[
f_t(w) = \begin{cases} 
  w & \text{w.p. 0.5,} \\
  -w & \text{w.p. 0.5.}
\end{cases}
\]

The dimension is set as \( d = 10 \), and time horizon \( T = 10^5 \). There are \( N = \lceil 0.5 + \log_2(T) \rceil + 1 = 10 \) experts, denoted by \( E_1, \ldots, E_N \). The expert \( E_i \) performs OGD over the linearized loss function
\[ g_t(w) = \langle \nabla f_t(w_t), w \rangle \] with a step size \( \eta_i = T^{-1/2} \cdot 2^{i-1} \),

\[ w_{t+1,i} = \Pi_W[w_t,i - \eta_i \nabla g_t(w_{t,i})] = \Pi_W[w_t,i - \eta_i \nabla f_t(w_t)] \]

where the domain \( W \) is set as a ball with diameter \( D = 1 \), i.e., \( W = \{w \mid \|w\|_2 \leq 1\} \). The initial decision is set as \( w_{1,i} = w_1 = 0 \) for all \( i \in [N] \).

Next, we will specify the meta-algorithm used in the standard method and our method. Essentially, both methods can be regarded as performing the Hedge algorithm but over different surrogate loss functions. Specifically, at each round, the surrogate loss vectors \( \ell_t \) and \( \ell^\lambda_t \) are set as:

standard method: \( \ell_{t,i} = \langle \nabla f_t(w_t), w_{t,i} \rangle \),

our method: \( \ell^\lambda_{t,i} = \langle \nabla f_t(w_t), w_{t,i} \rangle + \lambda \|w_{t,i} - w_{t-1,i}\|_2 \).

Notice that the key characteristic of our method is the regularizer in terms of the switching cost of each expert, which is of great importance for algorithmically enforcing a low switching cost of final decisions. In the experiments, we set the regularizer coefficient \( \lambda = 150 \), which is roughly set according to \( m^2 = (\log T)^2 \approx 132.5 \).

Then, the meta-algorithm starts from a uniform distribution \( p_1 = 1/N \cdot 1 \in \Delta_N \) and performs the following update:

standard method: \( p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell_{t,i})}{\sum_{i=1}^N p_{t,i} \exp(-\varepsilon \ell_{t,i})} \),

our method: \( p_{t+1,i} = \frac{p_{t,i} \exp(-\varepsilon \ell^\lambda_{t,i})}{\sum_{i=1}^N p_{t,i} \exp(-\varepsilon \ell^\lambda_{t,i})} \).

The learning rate of the meta-algorithm is set as \( \varepsilon = \sqrt{(\ln N)/T} \). Then, the final decision is made as the weighted combination of the intermediate decisions returned by the experts, i.e., \( w_{t+1} = \sum_{i=1}^N p_{t+1,i} w_{t+1,i} \). We record the switching cost \( S_T = \sum_{t=2}^T \|w_t - w_{t-1}\|_2 \). The experiments are conducted for 20 trials, and we plot the mean and standard variance of the switching cost with respect to iterations in Figure 1.

**C. Omitted Details for Section 5 (Online Non-stochastic Control)**

In this section, we present omitted details for Section 5 online non-stochastic control, including proofs of Proposition 6, Theorem 7, and Corollary 8.

**C.1 Proof of Proposition 6 (DAC Parametrization)**

We will prove the following statement that gives the state recurrence for any \( h \leq t \), which is essentially a strengthened result of Proposition 6.

**Proposition 14.** Suppose one chooses the DAC controller \( \pi(M_t, K) \) at iteration \( t \), the reaching state is

\[ x_{t+1} = \hat{A}_K^{h+1} x_{t-h} + \sum_{i=0}^{H+h} \Psi_{t,i}^{K,h}(M_{t-h,i}) w_{t-i}, \]  

(34)
where \( \bar{A}_K = A - BK \), and \( \Psi_{t,i}^{K,h}(M_{t-h:t}) \) is the transfer matrix defined as

\[
\Psi_{t,i}^{K,h}(M_{t-h:t}) = \bar{A}_K^i 1_{i \leq h} + \sum_{j=0}^{h} \bar{A}_K^j BM_{t-j}^{i-j-1} 1_{1 \leq i-j \leq H}.
\]  

(35)

The evolving equation holds for any \( h \in \{0, \ldots, t\} \).

**Proof** [of Proposition 14] First, by substituting the DAC policy into the dynamics equation, we have

\[
x_{t+1} = Ax_t + Bu_t + w_t = (A - BK)x_t + \sum_{i=1}^{H} BM_{t-i}^{i-1} w_t - i + w_t
\]

\[
= \bar{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \bar{A}_K^{j} \left( \sum_{i=1}^{H} BM_{t-j}^{i-1} w_{t-i} + w_{t-j} \right)
\]

\[
= \bar{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \sum_{i=1}^{H} \bar{A}_K^{j} BM_{t-j}^{i-1} w_{t-i} + \sum_{j=0}^{h} \bar{A}_K^{j} w_{t-j}.
\]

Exchanging the summation index yields,

\[
\sum_{j=0}^{h} \sum_{i=1}^{H} \bar{A}_K^{j} BM_{t-j}^{i-1} w_{t-j-i} = \sum_{i=1}^{H} \sum_{k=i}^{i+h} \bar{A}_K^{k-i} BM_{t-k+i}^{i-1} w_{t-k} \tag{36}
\]

\[
= \sum_{i=1}^{H} \sum_{k=i}^{H+h} \bar{A}_K^{k-i} BM_{t-k+i}^{i-1} w_{t-k} 1_{1 \leq i \leq H} \tag{37}
\]

\[
= \sum_{k=1}^{H+h} \sum_{i=0}^{k} \bar{A}_K^{k-i} BM_{t-i+k}^{i-1} w_{t-k} 1_{1 \leq i \leq (k-h) \leq H} \tag{38}
\]

\[
= \sum_{m=0}^{H} \sum_{k=1}^{H+h} \bar{A}_K^{k-m} BM_{t-k}^{k-m-1} w_{t-k} 1_{1 \leq k-m \leq H} \tag{39}
\]

\[
= \sum_{i=0}^{H} \sum_{j=0}^{h} \bar{A}_K^{j} BM_{t-j}^{i-j-1} w_{t-i} 1_{1 \leq i-j \leq H} \tag{40}
\]

where (36) holds by defining a third variable \( k = j + i \), and (37) is obtained by exchanging the summation index \( i \) and \( k \) and the new range of \( i \) is from inequality \( i \leq k \leq i + h \). Moreover, the equation (38) is obtained by another change of variable \( l = i - k + h \), (39) is obtained by replacing \( l \) by \( h - m \), and the equation (40) is true by setting \( i = k, j = m \). Therefore, we can obtain that

\[
x_{t+1} = \bar{A}_K^{h+1} x_{t-h} + \sum_{j=0}^{h} \sum_{i=1}^{H} \bar{A}_K^{j} BM_{t-j}^{i-1} w_{t-i} + \sum_{j=0}^{h} \bar{A}_K^{j} w_{t-j}
\]

\[
= \bar{A}_K^{h+1} x_{t-h} + \sum_{i=0}^{H} \sum_{j=0}^{h} \bar{A}_K^{j} BM_{t-j}^{i-j-1} w_{t-i} 1_{1 \leq i-j \leq H} + \sum_{i=0}^{h} \bar{A}_K^{j} w_{t-i}
\]

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\[
= \tilde{A}^{h+1}_K x_{t-h} + \sum_{i=0}^{H+h} \left( \tilde{A}^i_K 1_{i \leq h} + \sum_{j=0}^{h} \tilde{A}^j_K B M^{[i-j-1]}_{t-j} 1_{1 \leq i-j \leq H} \right) w_{t-i}
\]
and hence complete the proof.

C.2 Proof of Theorem 7

To prove the dynamic policy regret of online non-stochastic control (Theorem 7), we will first present theoretical analysis of the reduction to OCO with memory in Section C.2.1, then give the dynamic regret analysis over the \( M \)-space in Section C.2.2, and finally present the overall proof of Theorem 7 in Section C.2.3.

C.2.1 Reduction to OCO with Memory & Approximation Theorems

In Section 5.2 of the main paper, we have presented how to reduce from online non-stochastic control to OCO with memory, by employing the DAC parameterization and introducing the truncated loss functions. In this part, we introduce the following approximation theorem that discloses that the truncation loss \( f_t \) approximates the original cost function \( c_t \) well.

Theorem 15 (Theorem 5.3 of Agarwal et al. (2019)). Suppose the disturbance are bounded by \( W \). For any \((\kappa, \gamma)\)-strongly stable linear controller \( K \), and any \( \tau > 0 \) such that the sequence of \( M_1, \ldots, M_T \) satisfies \( \| M_t \|_{op} \leq \tau (1 - \gamma)^t \), the approximation error between original loss and truncated loss is at most

\[
\left| \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} f_t(M_{t-1:H:t}) \right| \leq 2TG_cD^2\kappa^3(1 - \gamma)^{H+1}, \tag{41}
\]

where

\[
D := \frac{WK^3(1 + H\kappa\tau)}{\gamma(1 - \kappa^2(1 - \gamma)^{H+1})} + \frac{W\tau}{\gamma}. \tag{42}
\]

Proof [of Theorem 15] By Lipschitzness and definition of the truncated loss, we get that

\[
\begin{align*}
c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - f_t(M_{t-1:H-1:t}) &= c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - c_t(y_t^K(M_{t-H-1:t-1}), u_t^K(M_{t-H-1:t})) \\
&\leq G_cD (\| x_t^K(M_{0:t-1}) - y_t^K(M_{t-H-1:t-1}) \| + \| u_t^K(M_{0:t}) - u_t^K(M_{t-H-1:t}) \|) \\
&\leq G_cD(\kappa^2(1 - \gamma)^{H+1} + \kappa^3(1 - \gamma)^{H+1}D) \\
&\leq 2G_cD^2\kappa^3(1 - \gamma)^{H+1},
\end{align*}
\]

where the last two inequalities use the Lipschitzness and the boundedness presented in Lemma 18. We complete the proof by summing over the iterations from \( t = 1, \ldots, T \).
C.2.2 Dynamic Regret Analysis over $\mathcal{M}$-Space

In previous sections, we have analyzed the dynamic regret of OGD over the $\mathbb{R}^d$-space. However, after reducing online non-stochastic control to OCO with memory, we need to apply their results to the $\mathcal{M}$-space and thus require to generalize the arguments of previous sections from Euclidean norm for $\mathbb{R}^d$-space to Frobenius norm for $\mathcal{M}$-space. For completeness, we present the proof here.

At first place, we analyze the dynamic regret of the online gradient descent (OGD) algorithm over the $\mathbb{R}^d$-space. OGD begins with any $M_1 \in \mathcal{M}$ and performs the following update procedure,

$$M_{t+1} = \Pi_\mathcal{M}[M_t - \eta \nabla_M \tilde{f}_t(M_t)]$$

(43)

where $\eta > 0$ is the step size and $\Pi_\mathcal{M}[:]$ denotes the projection onto the nearest point in the feasible set $\mathcal{M}$. We have the following dynamic regret regarding its dynamic regret.

**Theorem 16.** Suppose the function $\tilde{f} : \mathcal{M} \mapsto \mathbb{R}$ is convex; the gradient norm $\|\nabla_M \tilde{f}_t(M)\|_F \leq G_f$ holds for any $M \in \mathcal{M}$ and $t \in [T]$; and the Euclidean diameter of $\mathcal{M}$ is at most $D_f$, i.e.,

$$\sup_{M, M' \in \mathcal{M}} \|M - M'\|_F \leq D_f.$$

Then, OGD with a step size $\eta > 0$ as shown in (43) satisfies that

$$\lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F + \sum_{t=1}^{T} \tilde{f}_t(M_t) - \sum_{t=1}^{T} \tilde{f}_t(M_t^*) \leq \frac{\eta}{2} (G^2_f + 2\lambda G_f) T + \frac{1}{2\eta} (D_f^2 + 2D_f P_T),$$

(44)

which holds for any comparator sequence $M_1, \ldots, M_T \in \mathcal{M}$. Besides, the path-length $P_T = \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_F$ measures the non-stationarity of the comparator sequence.

**Proof** [of Theorem 16] Denote the gradient by $G_t = \nabla_M \tilde{f}_t(M_t)$. The convexity of online surrogate loss functions implies that

$$\sum_{t=1}^{T} \tilde{f}_t(M_t) - \sum_{t=1}^{T} \tilde{f}_t(M_t^*) \leq \sum_{t=1}^{T} \langle G_t, M_t - M_t^* \rangle.$$ 

Thus, it suffices to bound the sum of $\langle G_t, M_t - M_t^* \rangle$. From the OGD update rule and the non-expensive property, we have

$$\|M_{t+1} - M_t^*\|_F^2 = \|\Pi_\mathcal{M} [M_t - \eta G_t] - M_t^*\|_F^2 \leq \|M_t - \eta G_t - M_t^*\|_F^2$$

$$= \eta^2 \|G_t\|_F^2 - 2\eta \langle G_t, M_t - M_t^* \rangle + \|M_t - M_t^*\|_F^2,$$

After rearranging, we obtain

$$\langle G_t, M_t - M_t^* \rangle \leq \frac{\eta}{2} \|G_t\|_F^2 + \frac{1}{2\eta} (\|M_t - M_t^*\|_F^2 - \|M_{t+1} - M_t^*\|_F^2).$$

Next, we turn to analyze the second term in the right hand side. Indeed,

$$\sum_{t=1}^{T} (\|M_t - M_t^*\|_F^2 - \|M_{t+1} - M_t^*\|_F^2) \leq \sum_{t=1}^{T} \|M_t - M_t^*\|_F^2 - \sum_{t=2}^{T} \|M_t - M_{t-1}^*\|_F^2$$

$$\leq \|M_1 - M_1^*\|_F^2 + \sum_{t=2}^{T} (\|M_t - M_t^*\|_F^2 - \|M_{t-1} - M_{t-1}^*\|_F^2).$$

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= \|M_1 - M_1^*\|^2_F + \sum_{t=2}^{T} \langle M_{t-1}^* - M_t^*, 2M_t - M_{t-1}^* - M_t^* \rangle \\
\leq D_f^2 + 2D_f \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_F.

Hence, combining all above inequalities, we have

\[ \sum_{t=1}^{T} \tilde{f}_t(M_t) - \sum_{t=1}^{T} \tilde{f}_t(M_t^*) \leq \frac{\eta}{2} \sum_{t=1}^{T} \|G_t\|^2_F + \frac{1}{2\eta} \left(D_f^2 + 2D_f \sum_{t=2}^{T} \|M_{t-1}^* - M_t^*\|_F \right) \]
\[ \leq \frac{\eta}{2} G_f^2 T + \frac{1}{2\eta} (D_f^2 + 2D_f P_T). \]

On the other hand, the switching cost can be bounded by

\[ \sum_{t=2}^{T} \|M_t - M_{t-1}\|_F = \|\Pi_A [M_{t-1} - \eta G_{t-1}] - M_{t-1}\|^2_F \leq \|M_{t-1} - \eta G_{t-1} - M_{t-1}\|_F \leq \eta G_f T, \]

which together with the previous dynamic regret bound yields the desired result. \(\square\)

C.2.3 PROOF OF THEOREM 7

**Proof** We begin with the following dynamic policy regret decomposition,

\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \sum_{t=1}^{T} c_t(x_t^{\pi_t}, u_t^{\pi_t}) \]
\[ = \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}^{\pi_t}), u_t^K(M_{0:t}^{\pi_t})) \]
\[ = \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}), u_t^K(M_{0:t})) - \sum_{t=1}^{T} f_t(M_{t-1-H:t}) \]
\[ := A_T \]
\[ + \sum_{t=1}^{T} f_t(M_{t-1-H:t}) - \sum_{t=1}^{T} f_t(M_{t-1-H:t}^*) + \sum_{t=1}^{T} f_t(M_{t-1-H:t}^*) - \sum_{t=1}^{T} c_t(x_t^K(M_{0:t-1}^*), u_t^K(M_{0:t}^*)) \]
\[ := B_T + C_T \]

(45)

Notice that both \(A_T\) and \(C_T\) essentially represent the approximation error introduced by the truncated loss, so we can apply Theorem 15 and obtain that

\[ A_T + C_T \leq 4T G_f D_f^2 \kappa^3 (1 - \gamma)^{H+1}. \]
We now focus on the quantity $B_T$, which is the dynamic policy regret over the truncated loss functions $\{f_t\}_{t=1,\ldots, T}$. Indeed,

$$B_T = \sum_{t=1}^{T} f_t(M_{t-1-H:t}) - \sum_{t=1}^{T} f_t(M^*_{t-1-H:t})$$

$$\leq \sum_{t=1}^{T} \tilde{f}_t(M_t) - \sum_{t=1}^{T} \tilde{f}_t(M^*_t) + \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F + \lambda \sum_{t=2}^{T} \|M^*_{t-1} - M^*_t\|_F$$

$$\leq \sum_{t=1}^{T} \langle \nabla_M \tilde{f}_t(M_t), M_t - M^*_t \rangle + \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F + \lambda \sum_{t=2}^{T} \|M^*_{t-1} - M^*_t\|_F$$

$$= \sum_{t=1}^{T} g_t(M_t) - \sum_{t=1}^{T} g_t(M^*_t) + \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F + \lambda \sum_{t=2}^{T} \|M^*_{t-1} - M^*_t\|_F, \quad (47)$$

where $\lambda = (H + 2)^2 L_f$ and $g_t(M) = \langle \nabla_M \tilde{f}_t(M_t), M \rangle$ is the surrogate linearized loss. As a consequence, we are reduced to proving an dynamic regret over the sequence of functions $\{g_t\}_{t=1,\ldots, T}$ with switching cost, namely, the first three terms in the right hand side. We thus make use of the techniques developed in Section 4.4 (dynamic policy regret minimization for OCO with memory) to decompose the terms into meta-regret and expert-regret:

$$\sum_{t=1}^{T} g_t(M_t) - \sum_{t=1}^{T} g_t(M^*_t) + \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F$$

$$= \left( \lambda \sum_{t=2}^{T} \|M_{t-1} - M_t\|_F + \sum_{t=1}^{T} g_t(M_t) \right) - \left( \lambda \sum_{t=2}^{T} \|M_{t-1,i} - M_{t,i}\|_F + \sum_{t=1}^{T} g_t(M_{t,i}) \right)$$

$$+ \left( \lambda \sum_{t=2}^{T} \|M_{t-1,i} - M_{t,i}\|_F + \sum_{t=1}^{T} g_t(M_{t,i}) - \sum_{t=1}^{T} g_t(M^*_t) \right)$$

We further introduce the following notations to indicate the instantaneous meta-regret and expert-regret:

$$\text{MR}_{t,i} := (\lambda \|M_{t-1,i} - M_{t,i}\|_F + g_t(M_{t,i})) - (\lambda \|M_{t-1,i} - M_{t,i}\|_F + g_t(M_{t,i})),$$

and

$$\text{ER}_{t,i} := \lambda \|M_{t-1,i} - M_{t,i}\|_F + g_t(M_{t,i}) - g_t(M^*_t).$$

Clearly, meta-regret $= \sum_{t=1}^{T} \text{MR}_{t,i}$ and expert-regret $= \sum_{t=1}^{T} \text{ER}_{t,i}$. We remark that the regret decomposition holds for any expert index $i \in [N]$.

Theorem 16 ensures the expert-regret satisfies that

$$\text{expert-regret} = \sum_{t=1}^{T} \text{ER}_{t,i} \leq \frac{\eta_i}{2} (G_f^2 + 2\lambda G_f) T + \frac{1}{2\eta_i} (D_f^2 + 2D_f P_T),$$

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where \( P_T = \sum_{t=2}^T \| M_{t-1}^* - M_t^* \|_F \) is the path-length of the comparator sequence. On the other hand, similar to Proposition 3, we can show that the meta-regret satisfies that

\[
\text{meta-regret} = \sum_{t=1}^T \text{MR}_{t,i} \leq \lambda \sum_{t=2}^T \| p_{t-1} - p_t \|_1 + \sum_{t=1}^T (p_t, \ell_t) - \sum_{t=1}^T \ell_{t,i},
\]

where the surrogate loss vector \( \ell_t \in \Delta_N \) of the meta-algorithm is defined as

\[
\ell_{t,i} = \lambda \| M_{t-1,i} - M_{t,i} \|_F + g_t(M_{t,i}), \text{ for } i \in [N].
\]

Then, we can make use the static regret with switching cost of online mirror descent for the prediction with expert advice setting (c.f. Corollary 13 in Section B.3) and obtain that

\[
\tilde{\epsilon} = \lambda \| M_{t-1,i} - M_{t,i} \|_F + g_t(M_{t,i}), \text{ for } i \in [N].
\]

The specific values of \( \tilde{\epsilon} \) such that the optimal step size provably satisfies \( \epsilon \leq \eta \) for the dynamic policy regret is at most \( \tilde{\epsilon} \leq \eta \leq 2\eta^* \). As a result, we have

\[
\sum_{t=1}^T g_t(M_t) - \sum_{t=1}^T g_t(M_t^*) + \lambda \sum_{t=2}^T \| M_{t-1} - M_t \|_F
\]

\[
= \sum_{t=1}^T \text{MR}_{t,i} + \sum_{t=1}^T \text{ER}_{t,i^*}
\]

\[
\leq \frac{\eta_i^*}{2} (G_f^2 + 2\lambda G_f) T + \frac{1}{2\eta_i^*} (D_f^2 + 2D_f P_T) + D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f) T (1 + \ln(1 + i))}
\]

\[
\leq \frac{\eta_i^*}{2} (G_f^2 + 2\lambda G_f) T + \frac{1}{\eta_i^*} (D_f^2 + 2D_f P_T) + D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f) T (1 + \ln(1 + i))}
\]

\[
\leq \frac{3}{2} \sqrt{(G_f^2 + 2\lambda G_f)(D_f^2 + 2D_f P_T) T + D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f) T (1 + \ln(\lceil \log_2(1 + 2P_T/D) \rceil + 2))}}.
\]

Combining this result with the regret decomposition (45) and the upper bounds (46), (47), we have

\[
\sum_{t=1}^T c_t(x_t, u_t) - \sum_{t=1}^T c_t(x_t^*, u_t^*)
\]

\[
\leq 4T G_c D^2 c_3 (1 - \gamma)^{H+1} + \frac{3}{2} \sqrt{(G_f^2 + 2\lambda G_f)(D_f^2 + 2D_f P_T) T}
\]

\[
+ D_f \sqrt{2(2\lambda + G_f)(\lambda + G_f) T (1 + \ln(\lceil \log_2(1 + 2P_T/D) \rceil + 2))} + \lambda P_T.
\]

The specific values of \( D, L_f, G_f, D_f \) can be found in Lemma \ref{lemma:specific_values}. By setting \( H = O(\log T) \), we can ensure the final dynamic policy regret is at most \( O(\sqrt{T(1 + P_T)}) \) and hence complete the proof.
C.3 Proof of Corollary 8

In this part, we present the proof of Corollary 8, i.e., the static policy regret of the controller. Notice that Corollary 8 states that SCREAM\_CONTROL enjoys the following static policy regret,

\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \leq \tilde{O}(\sqrt{T}), \]  

(48)

where the comparator set \( \Pi \) can be chosen as either the set of DAC policies or the set of strongly linear controllers. Let us denote the two comparator sets as \( \Pi_{DAC} \) and \( \Pi_{SLC} \), respectively. In the following, we will prove the statement for two comparator sets separately.

**Proof** When the comparator set \( \Pi \) is chosen as the set of DAC policies, i.e., \( \pi \in \Pi_{DAC} = \{ \pi(K, M) \mid M \in \mathcal{M} \} \), the result of (48) can be easily obtained from Theorem 7 by setting \( \pi_1 = \ldots = \pi_T = \pi^* \in \arg\min_{\pi \in \Pi} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \). Under such a case, the path-length \( P_T = \sum_{t=2}^{T} \| M_{t-1} - M_t \|_F = 0 \), and thus

\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{DAC}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \leq \tilde{O}(\sqrt{T}). \]

On the other hand, when choosing the comparator set \( \Pi_{SL} \), i.e, \( \pi = K \in \Pi_{SL} = \{ K \mid K \text{ is } (\kappa, \gamma)\text{-strongly stable} \} \), we will need some efforts to prove the statement.

We show that the statement can be obtained by further incorporating Lemma 21, which demonstrates that minimizing static policy regret over the DAC class is sufficient to deliver a policy regret competing with the strongly linear controller class (Agarwal et al., 2019, Lemma 5.2). In fact, denote by \( \pi^* = K^* = \arg\min_{K \in \Pi_{SL}} \sum_{t=1}^{T} c_t(x_t^K, u_t^K) \), and we have

\[ \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{DAC}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \]

\[ = \sum_{t=1}^{T} c_t(x_t, u_t) - \min_{\pi \in \Pi_{DAC}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) + \min_{\pi \in \Pi_{DAC}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) - \sum_{t=1}^{T} c_t(x_t^K, u_t^K) \]

\[ \leq \tilde{O}(\sqrt{T}) + \sum_{t=1}^{T} c_t(x_t^{\pi(M_{\Delta}, K)}, u_t^{\pi(M_{\Delta}, K)}) - \sum_{t=1}^{T} c_t(x_t^{K^*}, u_t^{K^*}) \]

\[ \leq \tilde{O}(\sqrt{T}) + T \cdot 4G_cDWH\kappa^2_B\kappa^6(1 - \gamma)^{H-1}\gamma^{-1} \]

\[ \leq \tilde{O}(\sqrt{T}). \]

The first inequality uses the optimality of \( \arg\min_{\pi \in \Pi_{DAC}} \sum_{t=1}^{T} c_t(x_t^\pi, u_t^\pi) \) and \( \pi(M_{\Delta}, K) \) is a DAC policy with \( M_{\Delta} = (M_{\Delta}^{[0]}, \ldots, M_{\Delta}^{[H-1]}) \) defined by \( M_{\Delta}^{[i]} = (K - K^*)(A - BK^*)^i \). The second inequality holds by Lemma 21, and the final inequality holds by setting \( H = O(\log T) \).  

C.4 Supporting Lemmas

In this part, we provide several supporting lemmas used frequently in the analysis of online non-stochastic control (Agarwal et al., 2019). Specifically,
Lemma 17 establishes the norm relations between the $\ell_1, \text{op}$ norm and Frobenius norm used in the $\mathcal{M}$-space.

Lemma 18 checks the boundedness of several variables of interest.

Lemma 19 shows several properties of the truncated functions $\{f_t\}$ and the feasible set $\mathcal{M}$.

Lemma 20 provides an upper bound for the norm of transfer matrix.

Lemma 21 connects the DAC class and the strongly linear controller class.

**Lemma 17 (Norm Relations).** For any $M = (M[1], \ldots, M[H]) \in \mathcal{M} \subseteq (\mathbb{R}^{d_u \times d_x})^H$, its $\ell_1, \text{op}$ norm and Frobenius norm are defined by

$$
\|M\|_{\ell_1, \text{op}} := \sum_{i=1}^{H} \|M[i]\|_{\text{op}}, \text{ and } \|M\|_F := \sqrt{\sum_{i=1}^{H} \|M[i]\|^2_F}.
$$

We then have the following inequalities on their relations:

$$
\|M\|_{\ell_1, \text{op}} \leq \sqrt{H} \|M\|_F, \text{ and } \|M\|_F \leq \sqrt{d} \|M\|_{\ell_1, \text{op}},
$$

where $d = \min\{d_u, d_x\}$.

**Proof** [of Lemma 17] Recall the matrix norm relations, we know that for any matrix $X \in \mathbb{R}^{m \times n}$,

$$
\|X\|_{\text{op}} \leq \|X\|_F \leq \sqrt{d} \|X\|_{\text{op}}.
$$

Therefore, by definition and Cauchy-Schwarz inequality, we obtain

$$
\|M\|_{\ell_1, \text{op}} = \sum_{i=1}^{H} \|M[i]\|_{\text{op}} \leq \sum_{i=1}^{H} \|M[i]\|_F \leq \sqrt{H} \|M\|_F.
$$

On the other hand, we have

$$
\|M\|_F = \sqrt{\sum_{i=1}^{H} \|M[i]\|^2_F} \leq \sum_{i=1}^{H} \|M[i]\|_F \leq \sum_{i=1}^{H} \sqrt{d} \|M[i]\|_{\text{op}} = \sqrt{d} \|M\|_{\ell_1, \text{op}}.
$$

We thus complete the proof.

**Lemma 18.** Suppose $K$ and $K^*$ are two $(\kappa, \gamma)$-strongly stable linear controllers (cf. Definition 3). Define

$$D := \frac{W(\kappa^3 + H\kappa B\kappa^3 \tau)}{\gamma(1 - \kappa^2(1 - \gamma)^H + 1)} + \frac{W\tau}{\gamma}.
$$

Suppose there exists a $\tau > 0$ such that for every $i \in \{0, \ldots, H - 1\}$ and every $t \in [T]$, $\|M[i]\|_F \leq \tau(1 - \gamma)^i$. Then, we have

- $\|x^K_t(M_0:t-1)\| \leq D$, $\|y^K_t(M_t-H-1:t-1)\| \leq D$, and $\|x^K_t\| \leq D$. 

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• \( \|u^K_t(M_{0:t})\| \leq D, \) and \( \|v^K_t(M_{t-H-1:t})\| \leq D. \)
• \( \|x^K_t(M_{0:t-1}) - y^K_t(M_{t-1-H:t-1})\| \leq \kappa^2(1 - \gamma)^{H+1} D. \)
• \( \|u^K_t(M_{0:t}) - v^K_t(M_{t-1-H:t})\| \leq \kappa^3(1 - \gamma)^{H+1} D. \)

In above, the definitions of state \( x^K_t(M_{0:t-1}) \) and corresponding DAC control \( u^K_t(M_{0:t}) \) can be found in Proposition 6, and the definitions of truncated state \( x^K_t(M_{0:t-1}) \) and corresponding DAC control \( v^K_t(M_{0:t}) \) can be found in Definition 2. The definitions of state \( x^K_t \) can be found (and will be used) in Lemma 21.

**Proof** [of Lemma 18] We first study the state.

\[
\|x^K_t(M_{0:t-1})\| = \left\| A^K_{t-1}x^K_{t-1}w_{t-1} + \sum_{i=0}^{2H} \Psi^K_{t-1,i}(M_{t-1:t-1})w_{t-1-i} \right\|
\]

\[
\leq \kappa^2(1 - \gamma)^{H+1}\|x^K_{t-1}w_{t-1}\| + W\sum_{i=0}^{2H}\|\Psi^K_{t-1,i}(M_{t-1:t-1})\|
\]

\[
\leq \kappa^2(1 - \gamma)^{H+1}\|x^K_{t-1}w_{t-1}\| + W\sum_{i=0}^{2H}(\kappa^2(1 - \gamma)^i + H\kappa^2\tau(1 - \gamma)^{i-1})
\]

\[
\leq \kappa^2(1 - \gamma)^{H+1}\|x^K_{t-1}w_{t-1}\| + W(\kappa^2 + H\kappa^2\tau) / \gamma
\]

\[
\leq \frac{W(\kappa^2 + H\kappa^2\tau)}{\gamma(1 - \kappa^2(1 - \gamma)^{H+1})} \leq D,
\]

(50)

where inequality (50) is a summation of geometric series and the ratio of this series is \( \kappa^2(1 - \gamma)^{H+1}. \)

Similarly,

\[
\|y^K_t(M_{t-1:t-1})\| = \left\| \sum_{i=0}^{2H} \Psi^K_{t-1,i}(M_{t-1:t-1})w_{t-1-i} \right\|
\]

\[
\leq W\sum_{i=0}^{2H}\|\Psi^K_{t-1,i}(M_{t-1:t-1})\|
\]

\[
\leq W\sum_{i=0}^{2H}(\kappa^2(1 - \gamma)^i + H\kappa^2\tau(1 - \gamma)^{i-1})
\]

\[
\leq W \left( \frac{\kappa^2 + H\kappa^2\tau}{\gamma} \right) \leq D.
\]

Besides,

\[
\|x^K_{t}\| = \left\| \sum_{i=0}^{t-1} A^K_{t-i}w_{t-1-i} \right\| \leq W\sum_{i=0}^{t-1}\kappa^2(1 - \gamma)^i \leq \frac{W\kappa^2}{\gamma} \leq D.
\]

So the difference can be evaluated as follows:

\[
\|x^K_t(M_{0:t-1}) - y^K_t(M_{t-1:t-1})\| = \left\| A^K_{t-1}x^K_{t-1}w_{t-1} \right\| \leq \kappa^2(1 - \gamma)^{H+1} D.
\]
Lemma 19. The truncated loss \( f_t^K \) defined in Definition 2 we define the truncated loss \( f_t^K \) as

\[
f_t^K = g_t^K(M_{t-1}+1), v_t^K(M_{t-1}+1),
\]

where \( g_t^K(M_{t-1}+1) = \sum_{i=0}^{2H} \psi_t^K(M_{t-1}+1) \) and \( v_t^K(M_{t-1}+1) = -K y_t+1(M_{t-1}+1) + \sum_{i=1}^{H} M_{t-1}^{i-1} u_{i-1} \). In the following lemma, we show several properties of the truncated functions \( \{ f_t \} \) and the feasible set \( \mathcal{M} \) such that we can further apply the results of OCO with memory.

Lemma 19. The truncated loss \( f_t : \mathcal{M} \rightarrow \mathbb{R} \) and the feasible set \( \mathcal{M} \) satisfy the following properties. For notational convenience, we first let \( D \) be defined the same as (42), and we restate it below

\[
D := \frac{W_{\kappa}^3(1 + H_{\kappa}^3)}{(1 - \kappa^2(1 - \gamma)^{H+1})} + \frac{W_{\tau}}{\gamma}.
\]

(i) The function is \( L_f \)-coordinate-wise Lipschitz with respect to the Euclidean (i.e., Frobenius) norm, namely,

\[
|f_t(M_{t-1}, \ldots, M_{t-k}, \ldots, M_t)| - |f_t(M_{t-1}, \ldots, \widetilde{M}_{t-k}, \ldots, M_t)| \leq L_f \|M_{t-k} - \widetilde{M}_{t-k}\| F.
\]

Besides,

\[
L_f \leq 3\sqrt{HG_c D W_{\kappa} \kappa}.
\]

(ii) The gradient norm of surrogate loss \( \tilde{f}_t : \mathcal{M} \rightarrow \mathbb{R} \) is bounded by \( G_f \), namely, \( \| \nabla_M \tilde{f}_t(M) \|_F \leq G_f \) holds for any \( M \in \mathcal{M} \) and any \( t \in [T] \). Besides,

\[
G_f \leq 3H d^2 G_c W_{\kappa} \kappa^3 \gamma^{-1}.
\]
(iii) The diameter of the feasible set is at most $D_f$, namely, $\|M - M'\|_F \leq D_f$ holds for any $M, M' \in \mathcal{M}$. Besides,

$$D_f \leq 2\sqrt{d}K_3\gamma^{-1}.$$ 

Proof [of Lemma 19] We first prove the claim (i), i.e., the $L_f$-coordinate-wise Lipschitz continuity. For simplicity, we will make use of the following definitions in the following arguments.

$$M_{t-H-1:t} := \{M_{t-H-1} \ldots M_{t-k} \ldots M_t\}$$
$$M_{t-H-1:t-1} := \{M_{t-H-1} \ldots M_{t-k} \ldots M_{t-1}\}$$
$$\tilde{M}_{t-H-1:t} := \{M_{t-H-1} \ldots \tilde{M}_{t-k} \ldots M_t\}$$
$$\tilde{M}_{t-H-1:t-1} := \{M_{t-H-1} \ldots \tilde{M}_{t-k} \ldots M_{t-1}\}$$

By representing $f_t$ using $c_t$, we have

$$f_t(M_{t-H-1:t}) - f_t(\tilde{M}_{t-H-1:t}) = c_t(y^K_t(M_{t-H-1:t-1}), v^K_t(M_{t-H-1:t})) - c_t(y^K_t(\tilde{M}_{t-H-1:t-1}), v^K_t(\tilde{M}_{t-H-1:t}))$$

$$\leq G_c D\|y^K_t - \tilde{y}^K_t\| + G_c D\|v^K_t - \tilde{v}^K_t\|,$$

where for convenience we use the notations $y^K_t := y^K_t(\tilde{M}_{t-H-1:t-1}), \tilde{y}^K_t := y^K_t(\tilde{M}_{t-H-1:t-1})$ and $v^K_t := v^K_t(M_{t-H-1:t}), \tilde{v}^K_t := \tilde{v}^K_t(M_{t-H-1:t}).$ Besides, the last inequality holds because the norm of $\|y^K_t\|, \|\tilde{y}^K_t\|, \|v^K_t\|, \|\tilde{v}^K_t\|$ are all bounded by $D$, as shown in Lemma 18.

Then we try to bound $\|y^K_t - \tilde{y}^K_t\|$ and $\|v^K_t - \tilde{v}^K_t\|$.

$$\|y^K_t - \tilde{y}^K_t\| = \left\| \sum_{i=0}^{2H} \left( \Psi_{i-1,i}^{K,H}(M_{t-H-1:t-1}) - \Psi_{i-1,i}^{K,H}(\tilde{M}_{t-H-1:t-1}) \right) w_{t-1-i} \right\|$$

$$= \left\| \hat{A}_K B \sum_{i=0}^{2H} \left( M_{t-k}^{i-k-1} - \tilde{M}_{t-k}^{i-k-1} \right) 1\{i-k \in [H]\} w_{t-1-i} \right\|$$

$$\leq \kappa_B K_2(1 - \gamma)^K W \sum_{i=1}^{H} \|M_{t-k}^{i-1} - \tilde{M}_{t-k}^{i-1}\|$$

$$\leq \kappa_B K_2 W \|M_{t-k} - \tilde{M}_{t-k}\|,$$

and we have

$$\|v^K_t - \tilde{v}^K_t\| = \left\| -K(y^K_t - \tilde{y}^K_t) + 1_{\{k=0\}} \sum_{i=1}^{H} \left( M_{t-k}^{i-1} - \tilde{M}_{t-k}^{i-1} \right) \right\|$$

$$\leq (\kappa_B K_3 W + 1) \|M_{t-k} - \tilde{M}_{t-k}\|$$

$$\leq 2\kappa_B K_3 W \|M_{t-k} - \tilde{M}_{t-k}\|,$$

Combining (51), (52), and (53), we obtain

$$f_t(M_{t-H-1:t}) - f_t(\tilde{M}_{t-H-1:t}) \leq G_c D\|y^K_t - \tilde{y}^K_t\| + G_c D\|v^K_t - \tilde{v}^K_t\|.$$
\[ \begin{align*}
\leq G_c D\kappa \kappa^2 W \| M_{t-k} - \bar{M}_{t-k} \| + G_c D^2 \kappa^3 \kappa W \| M_{t-k} - \bar{M}_{t-k} \| \\
\leq 3G_c D\kappa \kappa^3 W \| M_{t-k} - \bar{M}_{t-k} \|.
\end{align*} \]

So we have \( L_I \leq 3G_c D\kappa \kappa^3 \).

Next, we prove the claim (ii), i.e., the boundedness of the gradient norm. Indeed, we will try to bound \( \nabla_{M_{\rho,q}^{[\rho]}} f_t(M) \) for every \( p \in [d_u], q \in [d_x] \) and \( r \in \{0, \ldots, H-1\} \).

\[ \left| \nabla_{M_{\rho,q}^{[\rho]}} f_t(M) \right| \leq G_c \left| \frac{\partial y^{[\rho]}(M)}{\partial M_{\rho,q}^{[\rho]}} \right|_F + G_c \left| \frac{\partial v^{[\rho]}(M)}{\partial M_{\rho,q}^{[\rho]}} \right|_F. \tag{54} \]

So we will bound the two terms of the right hand side respectively.

\[ \left| \frac{\partial y^{[\rho]}(M)}{\partial M_{\rho,q}^{[\rho]}} \right|_F \leq \sum_{r=0}^{2H} \sum_{j=0}^{H} \left| \frac{\partial \hat{A}_1^{[\rho]} B M^{[\rho]}_{\rho,q}^{[\rho]}}{\partial M_{\rho,q}^{[\rho]}} \right| w_{i-1-1} \mathbb{1}_{\{i-j[H]\}}_F \]

\[ \leq \frac{W \kappa \kappa^2}{\gamma} \left| \frac{\partial M^{[\rho]}_{\rho,q}}{\partial M_{\rho,q}^{[\rho]}} \right|_F \]

\[ \leq \frac{W \kappa \kappa^3 \gamma}{\gamma}. \tag{55} \]

\[ \left| \frac{\partial v^{[\rho]}(M)}{\partial M_{\rho,q}^{[\rho]}} \right|_F \leq \kappa \left| \frac{\partial y^{[\rho]}(M)}{\partial M_{\rho,q}^{[\rho]}} \right|_F + \sum_{r=0}^{H} \left| \frac{\partial M^{[\rho]}_{\rho,q}}{\partial M_{\rho,q}^{[\rho]}} \right| w_{i-1} \left| \frac{\partial M^{[\rho]}_{\rho,q}}{\partial M_{\rho,q}^{[\rho]}} \right|_F \]

\[ \leq \frac{W \kappa \kappa^3}{\gamma} + W \left| \frac{\partial M^{[\rho]}_{\rho,q}}{\partial M_{\rho,q}^{[\rho]}} \right|_F \tag{56} \]

\[ \leq W \left( \frac{\kappa \kappa^3}{\gamma} + 1 \right). \]

Combining (54), (55), and (56), we obtain

\[ \left| \nabla_{M_{\rho,q}^{[\rho]}} f_t(M) \right| \leq G_c \left( \frac{W \kappa \kappa^3}{\gamma} + \frac{W \kappa \kappa^3}{\gamma} + 1 \right) \leq 3G_c W \kappa \kappa^3 \gamma^{-1}. \]

Thus, \( \| \nabla_M \tilde{f}_t(M) \|_F \) at most \( 3Hd^2G_c W \kappa \kappa^3 \gamma^{-1} \).

Finally, we prove the claim (iii), i.e., the upper bound of diameter of the feasible set.

Actually, the construction of feasible set \( \mathcal{M} \) ensures that \( \forall i, 0 \leq i \leq H-1, \| M^{[i]} \|_W \leq \kappa \kappa^3 (1-\gamma)^i \). Therefore, we have

\[ \max_{M_1, M_2 \in \mathcal{M}} \| M_1 - M_2 \|_F \overset{\text{(49)}}{=} \sqrt{d} \max_{M_1, M_2 \in \mathcal{M}} \| M_1 - M_2 \|_{\ell_1,\text{op}}. \]

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\[
\leq \sqrt{a} \max_{M_1, M_2 \in \mathcal{M}} \left( \|M_1\|_{\ell_1, \text{op}} + \|M_2\|_{\ell_1, \text{op}} \right)
\]
\[
= \sqrt{a} \max_{M_1, M_2 \in \mathcal{M}} \left( \sum_{i=0}^{H-1} \|M_i^{[i]}\|_{\text{op}} + \|M_2^{[i]}\|_{\text{op}} \right)
\]
\[
\leq \sqrt{d} \max_{M_1, M_2 \in \mathcal{M}} \left( 2 \sum_{i=0}^{H-1} \kappa_B \kappa^3 (1 - \gamma)^i \right)
\]
\[
= 2\sqrt{d} \kappa_B \kappa^3 \sum_{i=0}^{H-1} (1 - \gamma)^i
\]
\[
\leq 2\sqrt{d} \kappa_B \kappa^3 \gamma^{-1}.
\]

Hence, we finish the proof of all three claims in the statement. 

The following lemma provides an upper bound for the norm of transfer matrix.

**Lemma 20.** Suppose \( K \) is \((\kappa, \gamma)\)-strongly stable as defined in Definition 3. Suppose there exists a \( \tau > 0 \) such that for every \( i \in \{0, \ldots, H - 1\} \) and every \( t \in [T] \), \( \|M_t^{[i]}\|_{F} \leq \tau (1 - \gamma)^i \). Then, we have
\[
\|\Psi_{t,i}^{K,h}\| \leq \kappa^2 (1 - \gamma)^i \mathbf{1}_{\{i \leq h\}} + H \kappa_B \kappa^2 \tau (1 - \gamma)^{i-1}.
\]  

**Proof** [of Lemma 20] We first expand \( \Psi_{t,i}^{K,h} \) by its definition (cf. Proposition 6 for its formal definition):
\[
\|\Psi_{t,i}^{K,h}\| = \left\| \widetilde{A}_K^i 1_{\{i \leq h\}} + \sum_{j=0}^{h} \widetilde{A}_K^j BM_{t-j}^{[i-j]} 1_{\{1 \leq i-j \leq H\}} \right\|
\]
\[
\leq \|\widetilde{A}_K^i\| 1_{\{i \leq h\}} + \sum_{j=1}^{H} \|\widetilde{A}_K^j BM_{t-j}^{[i-j-1]}\| 
\]  

\[
\leq \kappa^2 (1 - \gamma)^i + \sum_{j=1}^{H} \kappa^2 (1 - \gamma)^j \kappa_B \tau (1 - \gamma)^{i-j-1}
\]
\[
\leq \kappa^2 (1 - \gamma)^i + \kappa^2 \kappa_B \tau \sum_{j=1}^{H} (1 - \gamma)^{i-1}
\]
\[
= \kappa^2 (1 - \gamma)^i + H \kappa^2 \kappa_B \tau (1 - \gamma)^{i-1},
\]

where inequality (58) has to be emphasized here that no matter what the index \( i \) is, once \( i \) is fixed, to satisfy the condition \( 1 \leq i - j \leq H \), there is at most \( H \) different values which \( j \) can take. And that is why we can take \( j \) in range \([H]\) as an upper bound. 

In the following lemma, we show that minimizing the static policy regret over the DAC class is sufficient to deliver a policy regret competing with the strongly linear controller class (Agarwal et al., 2019, Lemma 5.2).
Lemma 21. With $K, K^* \text{ chosen as the } (\kappa, \gamma)\text{-strongly stable linear controllers as defined in Definition 3 and under Assumption 5, there exists a DAC policy } \pi(M_\Delta, K) \text{ with } M_\Delta = (M_\Delta^{[0]}, \ldots, M_\Delta^{[H-1]}) \text{ defined by }

\begin{equation}
M_\Delta^{[i]} = (K - K^*)(A - BK^*)^i
\end{equation}

such that

\begin{equation}
\sum_{t=1}^{T} c_t(x_t^K(M_\Delta), u_t^K(M_\Delta)) - \sum_{t=1}^{T} c_t(x_t^{K^*}, u_t^{K^*}) \leq T \cdot 4G_cDWH^2\kappa^2\kappa(1 - \gamma)^{H-1}\gamma^{-1},
\end{equation}

where $x_t^{K^*}$ is the state attained by executing a linear controller $K^*$ which chooses the action $u_t^{K^*} = -K^*x_t^{K^*}$.

Proof [of Lemma 21] The coordinate-wise Lipschitzness of the cost functions implies that

\[ c_t(x_t^K(M_\Delta), u_t^K(M_\Delta)) - c_t(x_t^{K^*}, u_t^{K^*}) \leq G_cD \| x_t^K(M_\Delta) - x_t^{K^*} \| + G_cD \| u_t^K(M_\Delta) - u_t^{K^*} \|. \]

By the linear dynamical equation (15), we have

\begin{equation}
x_{t+1} = \sum_{i=0}^{t} (A - BK^*)^iw_{t-i} = \sum_{i=0}^{t} \tilde{A}_K^i w_{t-i}
\end{equation}

By the property of the DAC policy (Proposition 6), we have

\[ x_t^K(M_\Delta) = \tilde{A}_K^{h+1}x_{t-h}^{K}(M_\Delta) + \sum_{i=0}^{H+h} \Psi_{t,i}^{K,h}(M_\Delta)w_{t-i}. \]

Setting $h = t$ and combining the assumption that the starting state $x_0 = 0$, we achieve the following equation,

\[ x_t^K(M_\Delta) = \sum_{i=0}^{H} \Psi_{t,i}^{K,t}(M_\Delta)w_{t-i} + \sum_{i=H+1}^{t} \Psi_{t,i}^{K,t}(M_\Delta)w_{t-i}. \]

Now we turn to calculate the transfer matrix $\Psi_{t,i}^{K,h}(M_\Delta)$ explicitly. Actually, for any $i \in \{0, \ldots, H\}$, $h \geq H$, i.e., $0 \leq i \leq H \leq h$, by definition we have

\[ \Psi_{t,i}^{K,h}(M_\Delta) = \tilde{A}_K^i \mathbf{1}_{\{i \leq h\}} + \sum_{j=0}^{h} \tilde{A}_K^j BM_\Delta^{[i-j-1]} \mathbf{1}_{\{i-j \in [H]\}} \]

\[ = \tilde{A}_K^i + \sum_{k=1}^{i} \tilde{A}_K^{i-k} BM_\Delta^{[k-1]} \]

\[ = \tilde{A}_K^i + \sum_{k=1}^{i} \tilde{A}_K^{i-k} (K - K^*) \tilde{A}_K^{k-1} \]

\[ = \tilde{A}_K^i + \sum_{k=1}^{i} \tilde{A}_K^{i-k} (\tilde{A}_K^* - \tilde{A}_K) \tilde{A}_K^{k-1} \]

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Combining (61) and (64) yields

\[
\begin{align*}
M = & \ A_{i} + \sum_{k=1}^{t} A_{i-k} A_{i-k}^* - A_{i-k} A_{i-k}^* \\
= & \ A_{i} + A_{i}^* - A_{i} \\
= & \ A_{i}^*,
\end{align*}
\]

where (62) holds by introducing a new index \( k = i - j \) and (63) can be obtained by plugging the construction of \( M^\Delta \) (59). So we achieve the conclusion that

\[
x_{t+1}(M) = \sum_{i=0}^{H} A_{i} w_{t-i} + \sum_{i=H+1}^{t} \Psi_{t,i} (M) w_{t-i}. \tag{64}
\]

Combining (61) and (64) yields

\[
\begin{align*}
\left\| x_{t+1} - x_{t+1}(M) \right\| &= \left\| \sum_{i=H+1}^{t} \left( \Psi_{t,i} (M) - A_{i}^* \right) w_{t-i} \right\| \\
&\leq W \left( \sum_{i=H+1}^{t} \left\| \Psi_{t,i} (M) \right\| + \sum_{i=H+1}^{t} \left\| A_{i} \right\| \right) \\
&\leq W \left( \sum_{i=H+1}^{t} (2\kappa^2 (1 - \gamma)^i + H \kappa_B^2 \kappa^5 (1 - \gamma)^{i-1}) \right) \\
&\leq W \left( 2\kappa^2 (1 - \gamma)^H \gamma^{-1} + H \kappa_B^2 \kappa^5 (1 - \gamma)^H \gamma^{-1} \right) \\
&\leq \kappa^2 W (1 - \gamma)^H \gamma^{-1} (2(1 - \gamma) + H \kappa_B^2 \kappa^3) \\
&\leq H \kappa_B^2 \kappa^5 W (1 - \gamma)^H \gamma^{-1} (2(1 - \gamma) + 1) \\
&\leq 2WH \kappa_B^2 \kappa^5 (1 - \gamma)^H \gamma^{-1},
\end{align*}
\]

where the second inequality makes use of Lemma 18. Next, we investigate the difference between the control signals,

\[
\begin{align*}
\left\| u_{t+1}^* - u_{t+1}(M) \right\| &= \left\| -K^* x_{t+1}^* - \left( -K x_{t+1}(M) + H \sum_{i=1}^{H} M_{i}^{H-i} w_{t+1-i} \right) \right\| \\
&= \left\| -K^* x_{t+1}^* + K x_{t+1}(M) - H \sum_{i=1}^{H} (K - K^*) A_{i}^* w_{t+1-i} \right\| \\
&= \left\| -K^* \left( x_{t+1}^* - \sum_{i=0}^{H-1} A_{i} w_{t-i} \right) + K \left( x_{t+1}(M) - \sum_{i=0}^{H-1} A_{i}^* w_{t-i} \right) \right\| \\
&= \left\| -K^* \sum_{i=H}^{t} A_{i} w_{t-i} + K \sum_{i=H}^{t} \Psi_{t,i} (M) w_{t-i} \right\| \\
&\leq 2WH \kappa_B^2 \kappa^6 (1 - \gamma)^H \gamma^{-1}.
\end{align*}
\]

Using above inequalities and Lipschitz assumption as well as the boundedness result (Lemma 18), we complete the proof. 

\[\square\]