Running Time Analysis of the (1+1)-EA for OneMax and LeadingOnes under Bit-wise Noise

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ABSTRACT

Previous running time analyses of evolutionary algorithms (EAs) in noisy environments often studied the one-bit noise model, which flips a randomly chosen bit of a solution before evaluation. In this paper, we study a natural extension of one-bit noise, the bit-wise noise model, which independently flips each bit of a solution with some probability. We analyze the running time of the (1+1)-EA solving OneMax and LeadingOnes under bit-wise noise for the first time, and derive the ranges of the noise level for polynomial and super-polynomial running time bounds. The analysis on LeadingOnes under bit-wise noise can be easily transferred to one-bit noise, and improves the previously known results.

CCS CONCEPTS

•Theory of computation → Theory of randomized search heuristics;

KEYWORDS

Noisy optimization, evolutionary algorithms, running time analysis, computational complexity

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1 INTRODUCTION

Evolutionary algorithms (EAs) have been widely applied to solve real-world optimization tasks, where the exact objective (i.e., fitness) evaluation of candidate solutions is often impossible, while we can obtain only a noisy one [3, 13]. However, previous theoretical analyses of EAs mainly focused on noise-free optimization, where the fitness evaluation is exact.

For running time analysis, a leading theoretical aspect [2, 14], only a few pieces of work on noisy evolutionary optimization have been reported. Droste [6] first analyzed the (1+1)-EA on the OneMax problem in the presence of one-bit noise and showed the maximal noise probability $p = \log n/n$ allowing a polynomial running time, where n is the problem size. Gießen and Kötzing [11] recently studied the LeadingOnes problem, and proved that the expected running time is polynomial if $p \le 1/(6en^2)$ and exponential if $p = 1/2$.

For inefficient optimization of the (1+1)-EA under high noise levels, some implicit mechanisms of EAs were proved to be robust to noise. In [11], it was shown that the $(\mu$+1)-EA with a small population of size $\Theta(\log n)$ can solve OneMax in polynomial time even if the probability of one-bit noise reaches 1. The robustness of populations to noise was also proved in the setting of non-elitist EAs [4, 17]. However, Friedrich et al. [8] showed the limitation of populations by proving that the $(\mu$+1)-EA needs super-polynomial time for solving OneMax under additive Gaussian noise $\mathcal{N}(0, \sigma^2)$ with $\sigma^2 \ge n^3$. This difficulty can be overcome by the compact genetic algorithm (cGA) [8] and a simple Ant Colony Optimization (ACO) algorithm [9], both of which find the optimal solution in polynomial time with a high probability.

The ability of explicit noise handling strategies was also theoretically studied. Qian et al. [19] proved that the threshold selection strategy is robust to noise: the expected running time of the (1+1)-EA using threshold selection on OneMax under one-bit noise is always polynomial regardless of the noise level. For the (1+1)-EA solving OneMax and LeadingOnes under one-bit or additive Gaussian noise, the resampling strategy was shown able to reduce the running time from exponential to polynomial in high noise levels [18]. Akimoto et al. [1] also proved that resampling with a large sample size can make optimization under additive unbiased noise behave as optimization in a noise-free environment. The interplay between resampling and implicit noise-handling mechanisms has been statistically studied in [10].

The studies mentioned above mainly considered the one-bit noise model, which flips a random bit of a solution before

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Table 1: For the running time of the (1+1)-EA solving OneMax and LeadingOnes under prior noise models, the ranges of noise parameters for a polynomial upper bound and a super-polynomial lower bound are shown below.

(1+1)-EA	bit-wise noise $(p, \frac{1}{n})$	bit-wise noise $(1, q)$	one-bit noise
OneMax	$O(\log n/n), \omega(\log n/n)$	$O(\log n/n^2), \omega(\log n/n^2)$ [11]	$O(\log n/n), \omega(\log n/n)$ [6]
LeadingOnes	$O(\log n/n^2), \omega(\log n/n)$	$O(\log n/n^3), \omega(\log n/n^2)$	$\leq 1/(6en^2), = 1/2$ [11]; $O(\log n/n^2), \omega(\log n/n)$

evaluation with probability p. However, the noise model, which can change several bits of a solution simultaneously, may be more realistic and needs to be studied, as mentioned in the first noisy theoretical work [6].

In this paper, we study the bit-wise noise model, which is characterized by a pair (p, q) of parameters. It happens with probability p, and independently flips each bit of a solution with probability q before evaluation. We analyze the running time of the (1+1)-EA solving OneMax and LeadingOnes under bit-wise noise with two specific parameter settings $(p, \frac{1}{n})$ and $(1, q)$. The ranges of p and q for a polynomial upper bound and a super-polynomial lower bound are derived, as shown in the middle two columns of Table 1. For the (1+1)-EA on LeadingOnes, we also transfer the running time bounds from bit-wise noise $(p, \frac{1}{n})$ to one-bit noise by using the same proof procedure. As shown in the bottom right of Table 1, our results improve the previously known ones [11].

The rest of this paper is organized as follows. Section 2 introduces some preliminaries. The running time analysis on OneMax and LeadingOnes is presented in Sections 3 and 4, respectively. Section 5 concludes the paper.

2 PRELIMINARIES

In this section, we first introduce the optimization problems, evolutionary algorithms and noise models studied in this paper, respectively, and then present the analysis tools that we use throughout this paper.

2.1 OneMax and LeadingOnes

In this paper, we use two well-known pseudo-Boolean functions OneMax and LeadingOnes. The OneMax problem as presented in Definition 2.1 aims to maximize the number of 1-bits of a solution. The LeadingOnes problem as presented in Definition 2.2 aims to maximize the number of consecutive 1-bits counting from the left of a solution. Their optimal solution is $11\ldots1$ (briefly denoted as 1^n). It has been shown that the expected running time of the (1+1)-EA on OneMax and LeadingOnes is $\Theta(n \log n)$ and $\Theta(n^2)$, respectively [7].

Definition 2.1 (OneMax). The OneMax Problem of size n is to find an n bits binary string x^* such that

$$x^* = \arg\max_{x\in\{0,1\}^n} \left(f(x) = \sum\nolimits_{i=1}^{n} x_i \right).$$

Definition 2.2 (LeadingOnes). The LeadingOnes Problem of size n is to find an n bits binary string x^* such that

$$x^* = \arg\max_{x\in\{0,1\}^n} \left(f(x) = \sum\nolimits_{i=1}^{n} \prod\nolimits_{j=1}^{i} x_j \right).$$

2.2 Bit-wise Noise

There are mainly two kinds of noise models: prior and posterior [11, 13]. The prior noise comes from the variation on a solution, while the posterior noise comes from the variation on the fitness of a solution. Previous theoretical analyses often focused on a specific prior noise model, one-bit noise. As presented in Definition 2.3, it flips a random bit of a solution before evaluation with probability p. However, in many realistic applications, noise can change several bits of a solution simultaneously rather than only one bit. We thus consider the bit-wise noise model. As presented in Definition 2.4, it happens with probability p, and independently flips each bit of a solution with probability q before evaluation.

To the best of our knowledge, only bit-wise noise with $p = 1$ and $q \in [0, 1]$ has been recently studied. Gießen and Kötzing [11] proved that for the (1+1)-EA on OneMax, the expected running time is polynomial if $q = O(\log n/n^2)$ and super-polynomial if $q = \omega(\log n/n^2)$. In this paper, we will study two specific bit-wise noise models: $p \in [0, 1] \wedge q = \frac{1}{n}$ and $p = 1 \wedge q \in [0, 1]$, which are briefly denoted as bit-wise noise $(p, \frac{1}{n})$ and bit-wise noise $(1, q)$, respectively.

Definition 2.3 (One-bit Noise). Given a parameter $p \in [0, 1]$, let $f^n(x)$ and $f(x)$ denote the noisy and true fitness of a binary solution $x \in \{0, 1\}^n$, respectively, then

$$f^n(x) = \begin{cases} f(x) & \text{with probability } 1 - p, \\ f(x') & \text{with probability } p, \end{cases}$$

where x' is generated by flipping a uniformly randomly chosen bit of x.

Definition 2.4 (Bit-wise Noise). Given parameters $p, q \in [0, 1]$, let $f^n(x)$ and $f(x)$ denote the noisy and true fitness of a binary solution $x \in \{0, 1\}^n$, respectively, then

$$f^n(x) = \begin{cases} f(x) & \text{with probability } 1 - p, \\ f(x') & \text{with probability } p, \end{cases}$$

where x' is generated by independently flipping each bit of x with probability q.

2.3 (1+1)-EA

The (1+1)-EA as described in Algorithm 1 is studied in this paper. For noisy optimization, only a noisy fitness value $f^n(x)$ instead of the exact one $f(x)$ can be accessed, and thus step 4 of Algorithm 1 changes to be "if $f^n(x') \geq f^n(x)$". Note that the reevaluation strategy is used as in [5, 6, 11]. That is, besides evaluating $f^n(x')$, $f^n(x)$ will be reevaluated in each iteration of the (1+1)-EA. The running time is usually defined as the number of fitness evaluations needed to find an optimal solution w.r.t. the true fitness function f for the first time [1, 6, 11].

ALGORITHM 1 ((1+1)-EA). *Given a function f over $\{0,1\}^n$ to be maximized, it consists of the following steps:*

1. *$x :=$ uniformly randomly selected from $\{0,1\}^n$.*
2. *Repeat until the termination condition is met*
3. *$x' :=$flip each bit of x independently with prob. $1/n$.*
4. *if $f(x') \geq f(x)$*
5. *$x := x'$.*

2.4 Analysis Tools

The process of the (1+1)-EA solving OneMax or LeadingOnes can be directly modeled as a Markov chain $\{\xi_t\}_{t=0}^{+\infty}$. We only need to take the solution space $\{0,1\}^n$ as the chain's state space (i.e., $\xi_t \in \mathcal{X} = \{0,1\}^n$), and take the optimal solution 1^n as the chain's optimal state (i.e., $\mathcal{X}^* = \{1^n\}$). Given a Markov chain $\{\xi_t\}_{t=0}^{+\infty}$ and $\xi_{\hat{t}} = x$, we define its *first hitting time* (FHT) as $\tau = \min\{t \mid \xi_{\hat{t}+t} \in \mathcal{X}^*, t \geq 0\}$. The mathematical expectation of τ, $\mathbb{E}[\![\tau \mid \xi_{\hat{t}} = x]\!] = \sum_{i=0}^{+\infty} iP(\tau = i)$, is called the *expected first hitting time* (EFHT) starting from $\xi_{\hat{t}} = x$. If ξ_0 is drawn from a distribution π_0, $\mathbb{E}[\![\tau \mid \xi_0 \sim \pi_0]\!] = \sum_{x\in\mathcal{X}} \pi_0(x)\mathbb{E}[\![\tau \mid \xi_0 = x]\!]$ is called the EFHT of the Markov chain over the initial distribution π_0. Thus, the expected running time of the (1+1)-EA starting from $\xi_0 \sim \pi_0$ is equal to $1 + 2 \cdot \mathbb{E}[\![\tau \mid \xi_0 \sim \pi_0]\!]$, where the term 1 corresponds to evaluating the initial solution, and the factor 2 corresponds to evaluating the offspring solution x' and reevaluating the parent solution x in each iteration. Note that we consider the expected running time of the (1+1)-EA starting from a uniform initial distribution in this paper.

In the following, we give three drift theorems that will be used to derive the EFHT of Markov chains in the paper.

LEMMA 2.5 (ADDITIVE DRIFT [12]). *Given a Markov chain $\{\xi_t\}_{t=0}^{+\infty}$ and a distance function $V(x)$, if for any $t \geq 0$ and any ξ_t with $V(\xi_t) > 0$, there exists a real number $c > 0$ such that*

$$\mathbb{E}[\![V(\xi_t) - V(\xi_{t+1}) \mid \xi_t]\!] \geq c,$$

then the EFHT satisfies that $\mathbb{E}[\![\tau \mid \xi_0]\!] \leq V(\xi_0)/c$.

LEMMA 2.6 (SIMPLIFIED DRIFT [15, 16]). *Let X_t, $t \geq 0$, be real-valued random variables describing a stochastic process. Suppose there exists an interval $[a,b] \subseteq \mathbb{R}$, two constants $\delta, \epsilon > 0$ and, possibly depending on $l := b - a$, a function $r(l)$ satisfying $1 \leq r(l) = o(l/\log(l))$ such that for all $t \geq 0$ the following two conditions hold:*

1. $\mathbb{E}[\![X_t - X_{t+1} \mid a < X_t < b]\!] \leq -\epsilon$,
2. $P(|X_{t+1} - X_t| \geq j \mid X_t > a) \leq \dfrac{r(l)}{(1+\delta)^j}$ *for $j \in \mathbb{N}_0$.*

Then there is a constant $c > 0$ such that for $T := \min\{t \geq 0 : X_t \leq a \mid X_0 \geq b\}$ it holds $P(T \leq 2^{cl/r(l)}) = 2^{-\Omega(l/r(l))}$.

LEMMA 2.7 (SIMPLIFIED DRIFT WITH SELF-LOOPS [20]). *Let X_t, $t \geq 0$, be real-valued random variables describing a stochastic process. Suppose there exists an interval $[a,b] \subseteq \mathbb{R}$, two constants $\delta, \epsilon > 0$ and, possibly depending on $l := b - a$, a function $r(l)$ satisfying $1 \leq r(l) = o(l/\log(l))$ such that for all $t \geq 0$ the following two conditions hold:*

1. $\forall a < i < b : \mathbb{E}[\![X_t - X_{t+1} | X_t = i]\!] \leq -\epsilon \cdot P(X_{t+1} \neq i | X_t = i)$,
2. $\forall i > a, j \in \mathbb{N}_0$:

$$P(|X_{t+1} - X_t| \geq j \mid X_t = i) \leq \frac{r(l)}{(1+\delta)^j} \cdot P(X_{t+1} \neq i | X_t = i).$$

Then there is a constant $c > 0$ such that for $T := \min\{t \geq 0 : X_t \leq a \mid X_0 \geq b\}$ it holds $P(T \leq 2^{cl/r(l)}) = 2^{-\Omega(l/r(l))}$.

3 THE ONEMAX PROBLEM

In this section, we analyze the running time of the (1+1)-EA on OneMax under bit-wise noise. Note that for bit-wise noise $(1, q)$, it has been proved that the maximal value of q allowing a polynomial running time is $\log n/n^2$, as shown in Theorem 3.1.

THEOREM 3.1. *[11] For the (1+1)-EA on OneMax under bit-wise noise $(1, q)$, the expected running time is polynomial if $q = O(\log n/n^2)$ and super-polynomial if $q = \omega(\log n/n^2)$.*

For bit-wise noise $(p, \frac{1}{n})$, we prove in Theorems 3.4 and 3.5 that the maximum value of p allowing a polynomial running time is $\log n/n$. Instead of using the original drift theorems, we apply the upper and lower bounds for the (1+1)-EA on stochastic OneMax in [11]. Let x^k denote any solution with k number of 1-bits, and $f^n(x^k)$ denote its noisy objective value, which is a random variable. Lemma 3.2 intuitively means that if the probability of recognizing the true better solution by noisy evaluation is large, the running time can be polynomially upper bounded. On the contrary, Lemma 3.3 shows that if the probability of making a right comparison is small, the running time can be exponentially lower bounded. Both of them are proved by applying standard drift theorems, and can be used to simplify our analysis.

LEMMA 3.2. *[11] Suppose there is a positive constant $c \leq 1/15$ and some $2 < l \leq n/2$ such that*

$$\forall k < n : P(f^n(x^k) < f^n(x^{k+1})) \geq 1 - \frac{l}{n};$$

$$\forall k < n - l : P(f^n(x^k) < f^n(x^{k+1})) \geq 1 - c\frac{n-k}{n},$$

then the (1+1)-EA optimizes f in expectation in $O(n \log n) + n2^{O(l)}$ iterations.

LEMMA 3.3. *[11] Suppose there is some $l \leq n/4$ and a constant $c \geq 16$ such that*

$$\forall n - l \leq k < n : P(f^n(x^k) < f^n(x^{k+1})) \leq 1 - c\frac{n-k}{n},$$

then the (1+1)-EA optimizes f in $2^{\Omega(l)}$ iterations with a high probability.

THEOREM 3.4. *For the (1+1)-EA on OneMax under bit-wise noise $(p, \frac{1}{n})$, the expected running time is polynomial if $p = O(\log n/n)$.*

PROOF. We prove it by using Lemma 3.2. For any positive constant b, suppose that $p \leq b\log n/n$. We set the two parameters in Lemma 3.2 as $c = \min\{\frac{1}{15}, b\}$ and $l = \frac{2b\log n}{c} \in (2, \frac{n}{2}]$.

For any $k < n$, $f^n(x^k) \geq f^n(x^{k+1})$ implies that $f^n(x^k) \geq k+1$ or $f^n(x^{k+1}) \leq k$, either of which happens with probability at most p. By the union bound, we get

$$P(f^n(x^k) \geq f^n(x^{k+1})) \leq 2p \leq \frac{2b \log n}{n} = \frac{lc}{n} \leq \frac{l}{n}.$$

For any $k < n - l$, we easily get

$$P(f^n(x^k) \geq f^n(x^{k+1})) \leq \frac{lc}{n} < c\frac{n-k}{n}.$$

By Lemma 3.2, we know that the expected running time is $O(n \log n) + n2^{O(2b \log n/c)}$, i.e., polynomial. □

THEOREM 3.5. *For the (1+1)-EA on OneMax under bit-wise noise* $(p, \frac{1}{n})$*, the expected running time is super-polynomial if* $p = \omega(\log n/n) \cap 1 - \omega(\log n/n)$ *and exponential if* $p = 1 - O(\log n/n)$.

PROOF. We use Lemma 3.3 to prove it. Let $c = 16$. The case $p = \omega(\log n/n) \cap 1 - \omega(\log n/n)$ is first analyzed. For any positive constant b, let $l = b \log n$. For any $k \geq n - l$, we get

$$P(f^n(x^k) \geq f^n(x^{k+1})) \geq P(f^n(x^k) = k) \cdot P(f^n(x^{k+1}) \leq k).$$

To make $f^n(x^k) = k$, it is sufficient that the noise does not happen, i.e., $P(f^n(x^k) = k) \geq 1 - p$. To make $f^n(x^{k+1}) \leq k$, it is sufficient to flip one 1-bit and keep other bits unchanged by noise, i.e., $P(f^n(x^{k+1}) \leq k) \geq p \cdot \frac{k+1}{n}(1 - \frac{1}{n})^{n-1}$. Thus,

$$P(f^n(x^k) \geq f^n(x^{k+1})) \geq (1-p) \cdot p\frac{k+1}{en} = \omega(\log n/n).$$

Since $c\frac{n-k}{n} \leq c\frac{l}{n} = \frac{cb \log n}{n}$, the condition of Lemma 3.3 holds. Thus, the expected running time is $2^{\Omega(b \log n)}$ (where b is any constant), i.e., super-polynomial.

For the case $p = 1 - O(\log n/n)$, let $l = \sqrt{n}$. We use another lower bound $p(1-\frac{1}{n})^n$ for $P(f^n(x^k) = k)$, since it is sufficient that no bit flips by noise. Thus, we have

$$P(f^n(x^k) \geq f^n(x^{k+1})) \geq p\left(1 - \frac{1}{n}\right)^n \cdot p\frac{k+1}{en} = \Omega(1).$$

Since $c\frac{n-k}{n} \leq \frac{c\sqrt{n}}{n}$, the condition of Lemma 3.3 holds. Thus, the expected running time is $2^{\Omega(\sqrt{n})}$, i.e., exponential. □

4 THE LEADINGONES PROBLEM

In this section, we first analyze the running time of the (1+1)-EA on LeadingOnes under bit-wise noise $(p, \frac{1}{n})$ and bit-wise noise $(1, q)$, respectively. Then, we transfer the analysis from bit-wise noise $(p, \frac{1}{n})$ to one-bit noise; the results are complementary to the known ones recently derived in [11].

4.1 Bit-wise Noise $(p, \frac{1}{n})$

For bit-wise noise $(p, \frac{1}{n})$, we prove in the following three theorems that the expected running time is polynomial if $p = O(\log n/n^2)$ and super-polynomial if $p = \omega(\log n/n)$. Their proofs are accomplished by applying additive drift analysis, the simplified drift theorem with self-loops and the simplified drift theorem, respectively.

THEOREM 4.1. *For the (1+1)-EA on LeadingOnes under bit-wise noise* $(p, \frac{1}{n})$*, the expected running time is polynomial if* $p = O(\log n/n^2)$.

PROOF. We use Lemma 2.5 to prove it. For any positive constant b, suppose that $p \leq b \log n/n^2$. Let $LO(x)$ denote the true number of leading 1-bits of a solution x. We first construct a distance function $V(x)$ as, for any x with $LO(x) = i$,

$$V(x) = \left(1 + \frac{c}{n}\right)^n - \left(1 + \frac{c}{n}\right)^i,$$

where $c = 4b \log n + 1$. It is easy to verify that $V(x \in \mathcal{X}^* = \{1^n\}) = 0$ and $V(x \notin \mathcal{X}^*) > 0$.

Then, we investigate $\mathbb{E}[\![V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = x]\!]$ for any x with $LO(x) < n$ (i.e., $x \notin \mathcal{X}^*$). Assume that currently $LO(x) = i$, where $0 \leq i \leq n-1$. We divide the drift into two parts: positive E^+ and negative E^-. That is,

$$\mathbb{E}[\![V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = x]\!] = E^+ - E^-.$$

For the positive drift, we need to consider that the number of leading 1-bits is increased. By mutation, we have

$$P(LO(x') \geq i+1) = \left(1 - \frac{1}{n}\right)^i \frac{1}{n}, \tag{1}$$

since it needs to flip the $(i+1)$-th bit (which must be 0) of x and keep the i leading 1-bits unchanged. For any x' with $LO(x') \geq i+1$, $f^n(x') < f^n(x)$ implies that $f^n(x') \leq i-1$ or $f^n(x) \geq i+1$. Note that,

$$P(f^n(x') \leq i-1) = p\left(1 - \left(1 - \frac{1}{n}\right)^i\right), \tag{2}$$

since at least one of the first i leading 1-bits of x' needs to be flipped by noise;

$$P(f^n(x) \geq i+1) = p\left(1 - \frac{1}{n}\right)^i \frac{1}{n}, \tag{3}$$

since it needs to flip the first 0-bit of x and keep the leading 1-bits unchanged by noise. By the union bound, we get

$$\begin{aligned} P(f^n(x') \geq f^n(x)) &= 1 - P(f^n(x') < f^n(x)) \\ &\geq 1 - p\left(1 - \left(1 - \frac{1}{n}\right)^{i+1}\right) \geq 1 - p\frac{i+1}{n} \geq \frac{1}{2}, \end{aligned} \tag{4}$$

where the last inequality is by $p = O(\log n/n^2)$. Furthermore, for any x' with $V(x') \geq i+1$,

$$V(x) - V(x') \geq \left(1 + \frac{c}{n}\right)^{i+1} - \left(1 + \frac{c}{n}\right)^i = \frac{c}{n}\left(1 + \frac{c}{n}\right)^i. \tag{5}$$

By combining Eqs. (1), (4) and (5), we have

$$E^+ \geq \left(1 - \frac{1}{n}\right)^i \frac{1}{n} \cdot \frac{1}{2} \cdot \frac{c}{n}\left(1 + \frac{c}{n}\right)^i \geq \frac{c}{6n^2}\left(1 + \frac{c}{n}\right)^i,$$

where the last inequality is by $(1 - \frac{1}{n})^i \geq (1 - \frac{1}{n})^{n-1} \geq \frac{1}{e} \geq \frac{1}{3}$.

For the negative drift, we need to consider that the number of leading 1-bits is decreased. By mutation, we have

$$P(LO(x') \leq i-1) = 1 - \left(1 - \frac{1}{n}\right)^i, \tag{6}$$

since it needs to flip at least one leading 1-bit of x. For any x' with $LO(x') \leq i-1$ (where $i \geq 1$), $f^n(x') \geq f^n(x)$ implies that $f^n(x') \geq i$ or $f^n(x) \leq i-1$. Note that,

$$P(f^n(x') \geq i) \leq p\left(1-\frac{1}{n}\right)^{i-1}\frac{1}{n}, \tag{7}$$

since for the first i bits of x', it needs to flip the 0-bits (whose number is at least 1) and keep the 1-bits unchanged by noise;

$$P(f^n(x) \leq i-1) = p\left(1-\left(1-\frac{1}{n}\right)^i\right), \tag{8}$$

since at least one leading 1-bit of x needs to be flipped by noise. By the union bound, we get

$$P(f^n(x') \geq f^n(x)) \leq p-p\left(1-\frac{2}{n}\right)\left(1-\frac{1}{n}\right)^{i-1} \leq p\frac{i+1}{n}. \tag{9}$$

Furthermore, for any x' with $LO(x') \leq i-1$,

$$V(x')-V(x) \leq \left(1+\frac{c}{n}\right)^i - 1. \tag{10}$$

By combining Eqs. (6), (9) and (10), we have

$$\begin{aligned} E^- &\leq \left(1-\left(1-\frac{1}{n}\right)^i\right)\cdot p\frac{i+1}{n}\cdot\left(\left(1+\frac{c}{n}\right)^i-1\right) \\ &\leq \left(1-\frac{1}{e}\right)\cdot p\cdot\left(1+\frac{c}{n}\right)^i \leq \frac{2p}{3}\left(1+\frac{c}{n}\right)^i. \end{aligned}$$

Thus, by subtracting E^- from E^+, we have

$$\begin{aligned} \mathbb{E}[\![V(\xi_t)-V(\xi_{t+1}) \mid \xi_t = x]\!] &\geq \left(1+\frac{c}{n}\right)^i\left(\frac{c}{6n^2}-\frac{2p}{3}\right) \quad (11) \\ &\geq \left(1+\frac{c}{n}\right)^i\left(\frac{4b\log n+1}{6n^2}-\frac{2b\log n}{3n^2}\right) \geq \frac{1}{6n^2}, \end{aligned}$$

where the second inequality is by $c = 4b\log n + 1$ and $p \leq b\log n/n^2$. Note that $V(x) \leq (1+\frac{c}{n})^n \leq e^c = e^{4b\log n+1} = en^{4b}$. By Lemma 2.5, we get

$$\mathbb{E}[\![\tau \mid \xi_0]\!] \leq 6n^2\cdot en^{4b} = O(n^{4b+2}),$$

i.e., the expected running time is polynomial. □

THEOREM 4.2. *For the (1+1)-EA on LeadingOnes under bit-wise noise $(p, \frac{1}{n})$, if $p = \omega(\log n/n)\cap o(1)$, the expected running time is super-polynomial.*

PROOF. We use Lemma 2.7 to prove it. Let $X_t = |x|_0$ be the number of 0-bits of the solution x after t iterations of the (1+1)-EA. Let c be any positive constant. We consider the interval $[0, c\log n]$, i.e., the parameters $a = 0$ (i.e., the global optimum) and $b = c\log n$ in Lemma 2.7.

Then, we analyze the drift $\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!]$ for $1 \leq i < c\log n$. As in the proof of Theorem 4.1, we divide the drift into two parts: positive E^+ and negative E^-. That is,

$$\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] = E^+ - E^-.$$

For the positive drift, we need to consider that the number of 0-bits is decreased. For mutation on x (where $|x|_0 = i$), let X and Y denote the number of flipped 0-bits and 1-bits, respectively. Then, $X \sim B(i, \frac{1}{n})$ and $Y \sim B(n-i, \frac{1}{n})$, where $B(\cdot,\cdot)$ is the binomial distribution. Let $P_{mut}(x, x')$ denote the probability of generating x' by mutating x. To estimate an upper bound on E^+, we assume that the offspring solution x' with $|x'|_0 < i$ is always accepted. Thus, we have

$$\begin{aligned} E^+ &\leq \sum_{x':|x'|_0<i} P_{mut}(x,x')(i-|x'|_0) = \sum_{k=1}^{i} k\cdot P(X-Y=k) \\ &= \sum\nolimits_{k=1}^{i} k\cdot\sum\nolimits_{j=k}^{i} P(X=j)\cdot P(Y=j-k) \\ &= \sum\nolimits_{j=1}^{i}\sum\nolimits_{k=1}^{j} k\cdot P(X=j)\cdot P(Y=j-k) \\ &\leq \sum\nolimits_{j=1}^{i} j\cdot P(X=j) = \frac{i}{n}. \end{aligned}$$

For the negative drift, we need to consider that the number of 0-bits is increased. We analyze the $n-i$ cases where only one 1-bit is flipped (i.e., $|x'|_0 = i+1$), which happens with probability $\frac{1}{n}(1-\frac{1}{n})^{n-1}$. Assume that $LO(x) = k \leq n-i$. If the j-th (where $1 \leq j \leq k$) leading 1-bit is flipped, the offspring solution x' will be accepted (i.e., $f^n(x') \geq f^n(x)$) if $f^n(x') \geq j-1$ and $f^n(x) \leq j-1$. Note that,

$$P(f^n(x') \geq j-1) = 1-p+p\left(1-\frac{1}{n}\right)^{j-1} \geq 1-p\frac{j-1}{n} \geq \frac{1}{2}, \tag{12}$$

where the equality is since it needs to keep the $j-1$ leading 1-bits of x' unchanged, and the last inequality is by $p = o(1)$;

$$\begin{aligned} P(f^n(x) \leq j-1) &= p\left(1-\left(1-\frac{1}{n}\right)^j\right) \quad (13) \\ &= p\left(1-\frac{1}{n}\right)^j\left(\left(1+\frac{1}{n-1}\right)^j-1\right) \geq \frac{p}{e}\cdot\frac{j}{n-1} \geq \frac{pj}{3n}, \end{aligned}$$

where the equality is since at least one of the first j leading 1-bits of x needs to be flipped by noise. Thus, we get

$$P(f^n(x') \geq f^n(x)) \geq \frac{pj}{6n}. \tag{14}$$

If one of the $n-i-k$ non-leading 1-bits is flipped, $LO(x') = LO(x) = k$. We can use the same analysis procedure as Eq. (4) in the proof of Theorem 4.1 to derive that

$$P(f^n(x') \geq f^n(x)) \geq 1-p\frac{k+1}{n} \geq \frac{1}{2}, \tag{15}$$

where the second inequality is by $p = o(1)$. Combining all the $n-i$ cases, we get

$$\begin{aligned} E^- &\geq \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\cdot\left(\sum_{j=1}^{k}\frac{pj}{6n}+\frac{n-i-k}{2}\right)\cdot(i+1-i) \quad (16) \\ &\geq \frac{1}{en}\left(\frac{pk(k+1)}{12n}+\frac{n-i-k}{2}\right) \geq \frac{pk^2}{36n^2}+\frac{n-i-k}{6n}. \end{aligned}$$

By subtracting E^- from E^+, we get

$$\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] \leq \frac{i}{n} - \frac{pk^2}{36n^2} - \frac{n-i-k}{6n}.$$

To investigate condition (1) of Lemma 2.7, we also need to analyze the probability $P(X_{t+1} \neq i \mid X_t = i)$. For $X_{t+1} \neq i$, it is necessary that at least one bit of x is flipped and the offspring x' is accepted. We consider two cases: (1) at least one of the k leading 1-bits of x is flipped; (2) the k leading 1-bits of x are not flipped and at least one of the last $n-k$ bits is flipped. For case (1), the mutation probability is $1-(1-\frac{1}{n})^k$

and the acceptance probability is at most $p\frac{k+1}{n}$ by Eq. (9). For case (2), the mutation probability is $(1-\frac{1}{n})^k(1-(1-\frac{1}{n})^{n-k}) \le \frac{n-k}{n}$ and the acceptance probability is at most 1. Thus,

$$P(X_{t+1} \neq i \mid X_t = i) \le p + \frac{n-k}{n}. \tag{17}$$

When $k < n - np$, we have

$$\begin{aligned}\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] &\le \frac{i}{n} - \frac{n-i-k}{6n} \qquad (18)\\ \le -\frac{n-k}{12n} - \frac{np/2-7c\log n}{6n} &\le -\frac{n-k}{12n} \le -\frac{1}{24}\left(p+\frac{n-k}{n}\right),\end{aligned}$$

where the second inequality is by $n - k > np$ and $i < c\log n$, the third inequality is by $p = \omega(\log n/n)$ and the last is by $n - k > np$. When $k \ge n - np$, we have

$$\begin{aligned}\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] &\le \frac{i}{n} - \frac{pk^2}{36n^2} \qquad (19)\\ \le \frac{c\log n}{n} - \frac{p}{144} &\le -\frac{p}{288} \le -\frac{1}{576}\left(p + \frac{n-k}{n}\right),\end{aligned}$$

where the second inequality is by $p = o(1)$ and $i < c\log n$, the third is by $p = \omega(\log n/n)$ and the last is by $n - k \le np$. Combining Eqs. (17), (18) and (19), we get that condition (1) of Lemma 2.7 holds with $\epsilon = \frac{1}{576}$.

For condition (2) of Lemma 2.7, we need to show $P(|X_{t+1}-X_t| \ge j \mid X_t = i) \le \frac{r(l)}{(1+\delta)^j} \cdot P(X_{t+1} \neq i \mid X_t = i)$ for $i \ge 1$. For $P(X_{t+1} \neq i \mid X_t = i)$, we analyze the n cases where only one bit is flipped. Using the similar analysis procedure as E^-, except that flipping any bit rather than only 1-bit is considered here, we easily get

$$P(X_{t+1} \neq i \mid X_t = i) \ge \frac{pk(k+1)}{36n^2} + \frac{n-k}{6n}. \tag{20}$$

For $|X_{t+1} - X_t| \ge j$, it is necessary that at least j bits of x are flipped and the offspring solution x' is accepted. We consider two cases: (1) at least one of the k leading 1-bits is flipped; (2) the k leading 1-bits are not flipped. For case (1), the mutation probability is at most $\frac{k}{n}\binom{n-1}{j-1}\frac{1}{n^{j-1}}$ and the acceptance probability is at most $p\frac{k+1}{n}$ by Eq. (9). For case (2), the mutation probability is at most $(1-\frac{1}{n})^k\binom{n-k}{j}\frac{1}{n^j}$ and the acceptance probability is at most 1. Thus, we have

$$\begin{aligned}&P(|X_{t+1} - X_t| \ge j \mid X_t = i) \qquad (21)\\ &\le \frac{k}{n}\binom{n-1}{j-1}\frac{1}{n^{j-1}} \cdot p\frac{k+1}{n} + \left(1-\frac{1}{n}\right)^k \binom{n-k}{j}\frac{1}{n^j}\\ &\le \frac{pk(k+1)}{n^2}\cdot\frac{4}{2^j} + \frac{n-k}{n}\cdot\frac{2}{2^j} \le \left(\frac{pk(k+1)}{36n^2}+\frac{n-k}{6n}\right)\cdot\frac{144}{2^j}.\end{aligned}$$

By combining Eq. (20) with Eq. (21), we get that condition (2) of Lemma 2.7 holds with $\delta = 1$ and $r(l) = 144$.

Note that $l = b - a = c\log n$. By Lemma 2.7, the expected running time is $2^{\Omega(c\log n)}$, where c is any positive constant. Thus, the expected running time is super-polynomial. □

THEOREM 4.3. *For the (1+1)-EA on LeadingOnes under bit-wise noise* $(p, \frac{1}{n})$*, the expected running time is exponential if* $p = \Omega(1)$.

PROOF. We use Lemma 2.6 to prove it. Let $X_t = i$ be the number of 0-bits of the solution x after t iterations of the (1+1)-EA. We consider the interval $i \in [0, n^{1/2}]$. To analyze the drift $\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] = E^+ - E^-$, we use the same analysis procedure as Theorem 4.2. For the positive drift, we have $E^+ \le \frac{i}{n} = o(1)$. For the negative drift, we re-analyze Eqs. (14) and (15). From Eqs. (12) and (13), we get that $P(f^n(x') \ge j-1) \ge p(1-\frac{j-1}{n})$ and $P(f^n(x) \le j-1) \ge \frac{pj}{3n}$. Thus, Eq. (14) becomes

$$P(f^n(x') \ge f^n(x)) \ge \frac{p^2 j}{3n}\left(1 - \frac{j-1}{n}\right). \tag{22}$$

For Eq. (15), we need to analyze the acceptance probability for $LO(x') = LO(x) = k$. Since it is sufficient to keep the first $(k+1)$ bits of x and x' unchanged in noise, Eq. (15) becomes

$$P(f^n(x') \ge f^n(x)) \ge p^2\left(1-\frac{1}{n}\right)^{2(k+1)} \ge p^2\left(1-\frac{k+1}{n}\right)^2. \tag{23}$$

By applying the above two inequalities to Eq. (16), we have

$$E^- \ge \frac{p^2}{en}\left(\sum_{j=1}^{k}\frac{j(n-j+1)}{3n^2} + \frac{(n-i-k)(n-1-k)^2}{n^2}\right) = \Omega(1),$$

where the equality is by $p = \Omega(1)$. Thus, $E^+ - E^- = -\Omega(1)$. That is, condition (1) of Lemma 2.6 holds.

Since it is necessary to flip at least j bits of x, we have

$$P(|X_{t+1} - X_t| \ge j \mid X_t \ge 1) \le \binom{n}{j}\frac{1}{n^j} \le \frac{1}{j!} \le 2\cdot\frac{1}{2^j},$$

which implies that condition (2) of Lemma 2.6 holds with $\delta = 1$ and $r(l) = 2$. Note that $l = n^{1/2}$. Thus, by Lemma 2.6, the expected running time is exponential. □

4.2 Bit-wise Noise $(1, q)$

For bit-wise noise $(1, q)$, we prove in the following three theorems that the expected running time is polynomial if $q = O(\log n/n^3)$ and super-polynomial if $q = \omega(\log n/n^2)$. The proof idea is similar to that for bit-wise noise $(p, \frac{1}{n})$. The main difference led by the change of noise is the probability of accepting the offspring solution, i.e., $P(f^n(x') \ge f^n(x))$.

THEOREM 4.4. *For the (1+1)-EA on LeadingOnes under bit-wise noise* $(1, q)$*, the expected running time is polynomial if* $q = O(\log n/n^3)$.

PROOF. The proof is very similar to that of Theorem 4.1. The change of noise only affects the probability of accepting the offspring solution in the analysis. For any positive constant b, suppose that $q \le b\log n/n^3$.

For the positive drift E^+, we need to re-analyze $P(f^n(x') \ge f^n(x))$ (i.e., Eq. (4) in the proof of Theorem 4.1) for the parent x with $LO(x) = i$ and the offspring x' with $LO(x') \ge i + 1$. By bit-wise noise $(1, q)$, Eqs. (2) and (3) change to

$$P(f^n(x') \le i-1) = 1-(1-q)^i; \quad P(f^n(x) \ge i+1) = (1-q)^i q.$$

Thus, by the union bound, Eq. (4) becomes

$$\begin{aligned}P(f^n(x') \ge f^n(x)) &\ge 1 - (1 - (1-q)^i + (1-q)^i q) \qquad (24)\\ &= (1-q)^{i+1} \ge 1 - q(i+1) \ge 1/2,\end{aligned}$$

where the last inequality is by $q = O(\log n/n^3)$.

For the negative drift E^-, we need to re-analyze $P(f^n(x') \geq f^n(x))$ (i.e., Eq. (9) in the proof of Theorem 4.1) for the parent x with $LO(x) = i$ (where $i \geq 1$) and the offspring x' with $LO(x') \leq i-1$. By bit-wise noise $(1, q)$, Eqs. (7) and (8) change to

$$P(f^n(x') \geq i) \leq q(1-q)^{i-1}, \quad P(f^n(x) \leq i-1) = 1-(1-q)^i.$$

Thus, by the union bound, Eq. (9) becomes

$$P(f^n(x') \geq f^n(x)) \leq q(1-q)^{i-1} + 1 - (1-q)^i \tag{25}$$
$$= 1-(1-q)^{i-1}(1-2q) \leq 1-(1-(i-1)q)(1-2q) \leq (i+1)q,$$

where the second inequality is by $(1-q)^{i-1} \geq 1-(i-1)q$ and $1-2q > 0$ for $q = O(\log n/n^3)$.

By applying Eq. (24) and Eq. (25) to E^+ and E^-, respectively, Eq. (11) changes to

$$\mathbb{E}[\![V(\xi_t) - V(\xi_{t+1}) \mid \xi_t = x]\!] \geq \left(1+\frac{c}{n}\right)^i \left(\frac{c}{6n^2} - \frac{2q(i+1)}{3}\right)$$
$$\geq \left(1+\frac{c}{n}\right)^i \left(\frac{4b\log n + 1}{6n^2} - \frac{2bn\log n}{3n^3}\right) \geq \frac{1}{6n^2}.$$

That is, the condition of Lemma 2.5 still holds with $\frac{1}{6n^2}$. Thus, the expected running time is polynomial. □

THEOREM 4.5. *For the (1+1)-EA on LeadingOnes under bit-wise noise* $(1, q)$, *if* $q = \omega(\log n/n^2) \cap o(1/n)$, *the expected running time is super-polynomial.*

PROOF. We use the same analysis process as Theorem 4.2. The only difference is the probability of accepting the offspring solution due to the change of noise. For the positive drift, we still have $E^+ \leq \frac{i}{n}$, since we assume that x' is always accepted in the proof of Theorem 4.2.

For the negative drift, we need to re-analyze $P(f^n(x') \geq f^n(x))$ for the parent solution x with $LO(x) = k$ and the offspring solution x' with $LO(x') = j-1$ (where $1 \leq j \leq k+1$). For $j \leq k$, to derive a lower bound on $P(f^n(x') \geq f^n(x))$, we consider the j cases where $f^n(x) = l$ and $f^n(x') \geq l$ for $0 \leq l \leq j-1$. Since $P(f^n(x) = l) = (1-q)^l q$ and $P(f^n(x') \geq l) = (1-q)^l$, Eq. (14) changes to

$$P(f^n(x') \geq f^n(x)) \geq \sum_{l=0}^{j-1} (1-q)^l q \cdot (1-q)^l \geq \frac{1-(1-q)^{2j}}{2} \tag{26}$$
$$= \frac{1}{2}(1-q)^{2j}\left(\left(1+\frac{q}{1-q}\right)^{2j} - 1\right) \geq (1-q)^{2j}\frac{qj}{1-q} \geq \frac{qj}{2},$$

where the last inequality is by $(1-q)^{2j} \geq 1-2qj \geq 1/2$ since $q = o(1/n)$. For $j = k+1$ (i.e., $LO(x') = LO(x) = k$), we can use the same analysis as Eq. (24) to derive a lower bound $1/2$, since the last inequality of Eq. (24) still holds with $p = o(1/n)$. Thus, Eq. (15) also holds here, i.e.,

$$P(f^n(x') \geq f^n(x)) \geq \frac{1}{2}. \tag{27}$$

By applying Eqs. (26) and (27) to E^-, Eq. (16) changes to

$$E^- \geq \frac{qk^2}{12n} + \frac{n-i-k}{6n}.$$

Thus, we have

$$\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] = E^+ - E^- \leq \frac{i}{n} - \frac{qk^2}{12n} - \frac{n-i-k}{6n}.$$

For the upper bound analysis of $P(X_{t+1} \neq i \mid X_t = i)$ in the proof of Theorem 4.2, we only need to replace the acceptance probability $p\frac{k+1}{n}$ in the case of $LO(x') < LO(x)$ with $(k+1)q$ (i.e., Eq. (25)). Thus, Eq. (17) changes to

$$P(X_{t+1} \neq i \mid X_t = i) \leq (k+1)q + \frac{n-k}{n} \leq nq + \frac{n-k}{n}.$$

To compare $\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!]$ with $P(X_{t+1} \neq i \mid X_t = i)$, we consider two cases: $k < n - n^2 q$ and $k \geq n - n^2 q$. By using $q = \omega(\log n/n^2)$ and applying the same analysis procedure as Eqs. (18) and (19), we can derive that condition (1) of Lemma 2.7 holds with $\epsilon = \frac{1}{192}$.

For the lower bound analysis of $P(X_{t+1} \neq i \mid X_t = i)$, by applying Eqs. (26) and (27), Eq. (20) changes to

$$P(X_{t+1} \neq i \mid X_t = i) \geq \frac{qk(k+1)}{12n} + \frac{n-k}{6n}.$$

For the analysis of $|X_{t+1} - X_t| \geq j$, by replacing the acceptance probability $p\frac{k+1}{n}$ in the case of $LO(x') < LO(x)$ with $(k+1)q$, Eq. (21) changes to

$$P(|X_{t+1} - X_t| \geq j \mid X_t = i) \leq \frac{qk(k+1)}{n} \cdot \frac{4}{2^j} + \frac{n-k}{n} \cdot \frac{2}{2^j}$$
$$\leq \left(\frac{qk(k+1)}{12n} + \frac{n-k}{6n}\right) \cdot \frac{48}{2^j}.$$

That is, condition (2) of Lemma 2.7 holds with $\delta = 1, r(l) = 48$. Thus, the expected running time is super-polynomial. □

THEOREM 4.6. *For the (1+1)-EA on LeadingOnes under bit-wise noise* $(1, q)$, *the expected running time is exponential if* $q = \Omega(1/n)$.

PROOF. We use Lemma 2.6 to prove it. Let $X_t = i$ be the number of 0-bits of the solution x after t iterations of the (1+1)-EA. We consider the interval $i \in [0, n^{1/2}]$. To analyze the drift $\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!]$, we use the same analysis procedure as the proof of Theorem 4.2.

We first consider $q = \Omega(1/n) \cap o(1)$. We need to analyze the probability $P(f^n(x') \geq f^n(x))$, where the offspring solution x' is generated by flipping only one 1-bit of x. Let $LO(x) = k$. For the case where the j-th (where $1 \leq j \leq k$) leading 1-bit is flipped, as the analysis of Eq. (26), we get

$$P(f^n(x') \geq f^n(x)) \geq \frac{1-(1-q)^{2j}}{2} \geq (1-q)^{2j}\frac{qj}{1-q}.$$

If $(1-q)^{2j} < \frac{1}{2}$, $\frac{1-(1-q)^{2j}}{2} \geq \frac{1}{4}$; otherwise, $(1-q)^{2j}\frac{qj}{1-q} \geq \frac{qj}{2}$. Thus, we have

$$P(f^n(x') \geq f^n(x)) \geq \min\{1/4, qj/2\}.$$

For the case that flips one non-leading 1-bit (i.e., $LO(x') = LO(x) = k$), to derive a lower bound on $P(f^n(x') \geq f^n(x))$, we consider $f^n(x) = l$ and $f^n(x') \geq l$ for $0 \leq l \leq k$. Thus,

$$P(f^n(x') \geq f^n(x)) \geq \sum_{l=0}^{k-1} (1-q)^l q \cdot (1-q)^l + (1-q)^{k+1} \cdot (1-q)^k$$
$$\geq \frac{1-(1-q)^{2k}}{2} + (1-q)^{2k+1} = \frac{1}{2} + (1-q)^{2k}\left(\frac{1}{2} - q\right) \geq \frac{1}{2},$$

where the last inequality is by $q = o(1)$. By applying the above two inequalities to Eq. (16), we get

$$E^- \geq \frac{1}{en}\left(\sum_{j=1}^{k}\min\left\{\frac{1}{4},\frac{qj}{2}\right\}+\frac{n-i-k}{2}\right).$$

If $k \geq \frac{n}{2}$, $\sum_{j=1}^{k}\min\{\frac{1}{4},\frac{qj}{2}\} = \Omega(n)$ since $q = \Omega(1/n)$. If $k < \frac{n}{2}$, $\frac{n-i-k}{2} = \Omega(n)$ since $i \leq \sqrt{n}$. Thus, $E^- = \Omega(1)$.

For $q = \Omega(1)$, we use the trivial lower bound q for the probability of accepting the offspring solution x', since it is sufficient to flip the first leading 1-bit of x by noise. Then,

$$E^- \geq \frac{1}{en}(kq+(n-i-k)q) = \frac{(n-i)q}{en} = \Omega(1).$$

Thus, for $q = \Omega(1/n)$, we have

$$\mathbb{E}[\![X_t - X_{t+1} \mid X_t = i]\!] = E^+ - E^- \leq \frac{i}{n} - \Omega(1) = -\Omega(1).$$

That is, condition (1) of Lemma 2.6 holds. Its condition (2) trivially holds with $\delta = 1$ and $r(l) = 2$. Thus, the expected running time is exponential. □

4.3 One-bit Noise

For the (1+1)-EA on LeadingOnes under one-bit noise, it has been known that the running time is polynomial if $p \leq \frac{1}{6en^2}$ and exponential if $p = \frac{1}{2}$ [11]. We extend this result by proving in Theorem 4.7 that the running time is polynomial if $p = O(\log n/n^2)$ and super-polynomial if $p = \omega(\log n/n)$. The proof can be accomplished as same as that of Theorems 4.1, 4.2 and 4.3 for bit-wise noise $(p, \frac{1}{n})$. This is because although the probabilities $P(f^n(x') \geq f^n(x))$ of accepting the offspring solution are different, their bounds used in the proofs for bit-wise noise $(p, \frac{1}{n})$ still hold for one-bit noise.

THEOREM 4.7. *For the (1+1)-EA on LeadingOnes under one-bit noise, the expected running time is polynomial if* $p = O(\log n/n^2)$*, super-polynomial if* $p = \omega(\log n/n) \cap o(1)$ *and exponential if* $p = \Omega(1)$.

PROOF. We re-analyze $P(f^n(x') \geq f^n(x))$ for one-bit noise, and show that the bounds on $P(f^n(x') \geq f^n(x))$ used in the proofs for bit-wise noise $(p, \frac{1}{n})$ still hold for one-bit noise.

For the proof of Theorem 4.1, Eqs. (2) and (3) change to

$$P(f^n(x') \leq i-1) = p\frac{i}{n}, \qquad P(f^n(x) \geq i+1) = p\frac{1}{n},$$

and thus Eq. (4) still holds; Eqs. (7) and (8) change to

$$P(f^n(x') \geq i) \leq p\frac{1}{n}, \qquad P(f^n(x) \leq i-1) = p\frac{i}{n},$$

and thus Eq. (9) still holds.

For the proof of Theorem 4.2, Eqs. (12) and (13) change to

$$P(f^n(x') \geq j-1) = 1 - p\frac{j-1}{n}, \quad P(f^n(x) \leq j-1) = p\frac{j}{n},$$

and thus Eq. (14) still holds.

For the proof of Theorem 4.3, Eq. (22) still holds by the above two equalities; Eq. (23) still holds since the probability of keeping the first $(k+1)$ bits of a solution unchanged in one-bit noise is $1 - p\frac{k+1}{n} \geq p(1 - \frac{k+1}{n})$. □

5 CONCLUSION

In this paper, we theoretically study the (1+1)-EA solving OneMax and LeadingOnes under bit-wise noise. We derive the ranges of noise parameters for the running time being polynomial and super-polynomial, respectively. The previously known ranges for the (1+1)-EA solving LeadingOnes under one-bit noise are also improved. In the future, we shall improve the currently derived bounds on LeadingOnes, as they do not cover the whole range of noise parameters.

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