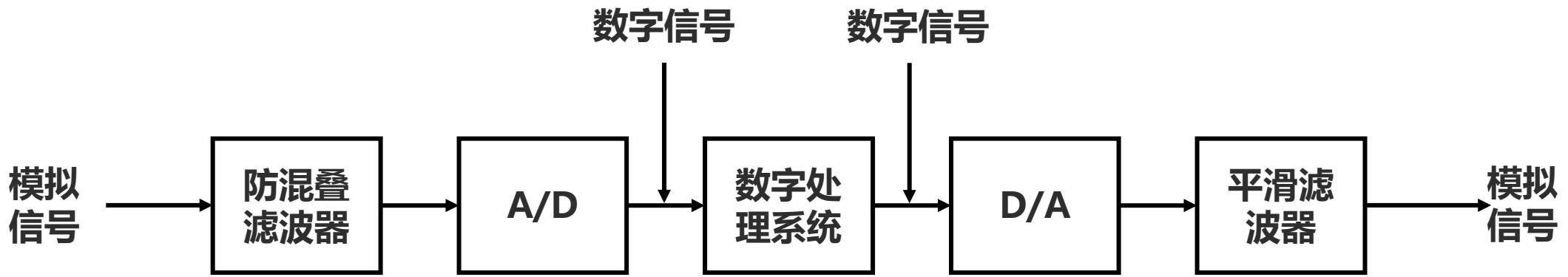
NJUAI 南京大学 人工智能学院
SCHOOL OF ARTIFICIAL INTELLIGENCE, NANJING UNIVERSITY

06 信号的采样

模拟信号和数字信号的互相转换

信号处理与分析的典型过程

- 数字信号的处理（**A:Analog, D:Digital**）



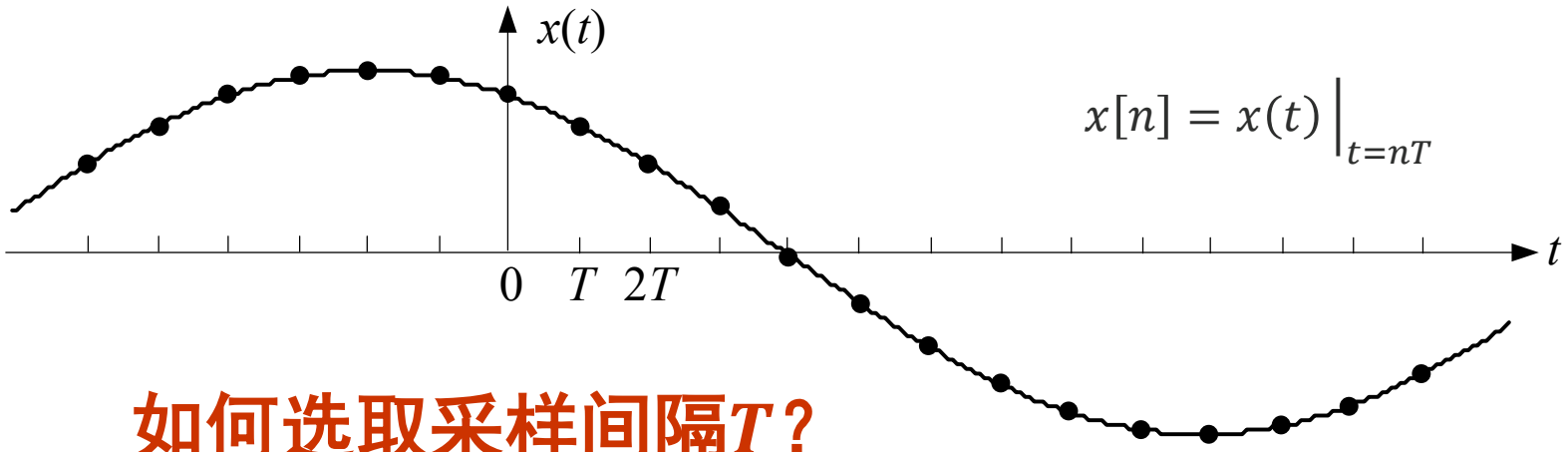
- 如何将模拟信号转化为数字信号?

- 如何将数字信号转化为模拟信号?

如何进行信号采样

- 图片信号

- 时序信号



如何选取采样间隔T？

为什么进行信号采样

- (1) 信号稳定性好：数据用二进制表示，受外界影响小。
- (2) 信号可靠性高：存储无损耗，传输抗干扰。
- (3) 信号处理简便：信号压缩，信号编码，信号加密等
- (4) 系统精度高： 可通过增加字长提高系统的精度。
- (5) 系统灵活性强：改变系统的系数使系统完成不同功能。

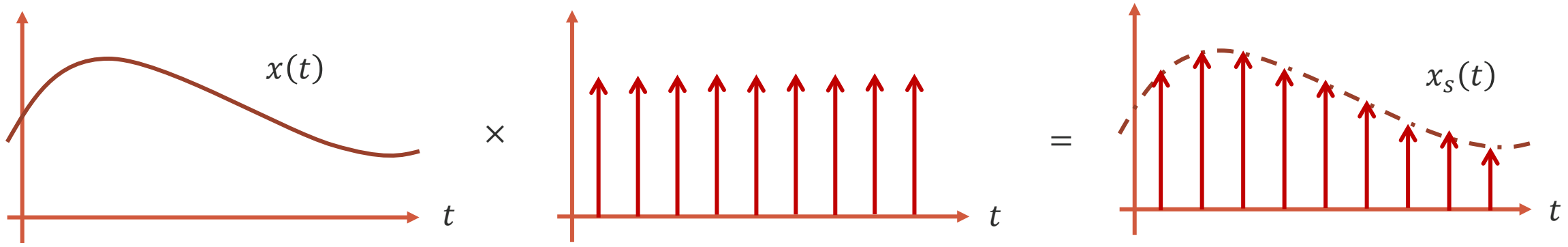
用数字方式处理模拟信号

信号的时域采样

- 信号$x(t)$使用脉冲序列

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

采样得到$x_s(t)$



$$x_s(t) = x(t) \cdot \delta_{T_s}(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

时域采样的频域分析

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- 已知

$$x_s(t) = x(t) \cdot \delta_{T_s}(t)$$

- 设

$$\mathcal{F}[x(t)] = X(j\omega)$$

- 且

$$\mathcal{F}\left[\delta_{T_s}(t)\right] = ?$$

周期信号的傅里叶变换

时域采样的频域分析

- 已知

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

求解$\mathcal{F}\left[\delta_{T_s}(t)\right]$

$\omega_s = \frac{2\pi}{T_s}$

- 针对周期信号首先计算**傅里叶级数**

$$X_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta_{T_s}(t)\, e^{-jn\omega_s t}\mathrm{d}t = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-jn\omega_s t}\mathrm{d}t = \frac{1}{T_s}$$

傅里叶变换为

$$X(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega - n\omega_s) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

时域采样的频域分析

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- 已知$x_s(t) = x(t) \cdot \delta_{T_s}(t)$, 设$\mathcal{F}[x(t)] = X(j\omega)$，且

$$\mathcal{F}\left[\delta_{T_s}(t)\right] = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

周期信号的傅里叶变换

$$\omega_s = \frac{2\pi}{T_s}$$

- 由频域卷积特性

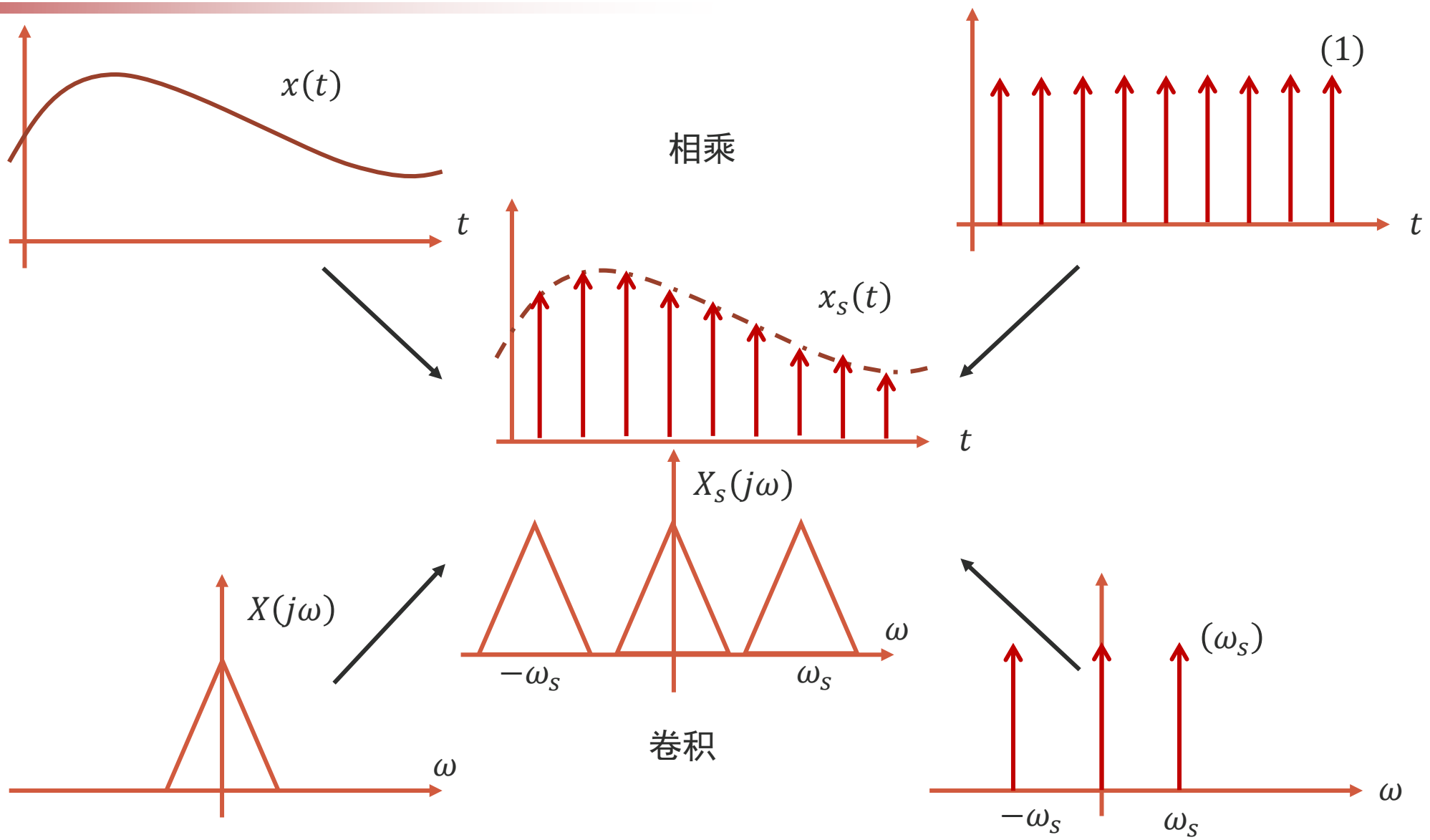
$$\mathcal{F}[x_s(t)] = \frac{1}{2\pi}\left[X(j\omega) * \mathcal{F}\left[\delta_{T_s}(t)\right]\right] = \frac{1}{2\pi}\left[X(j\omega) * \omega_s \sum_{-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(j(\omega - n\omega_s)\right)$$

- **时域**对信号做**离散化**，**频域**表现为原始时域信号频谱$X(j\omega)$的**周期延拓（重复）**，时域的离散化导致了频域的周期性

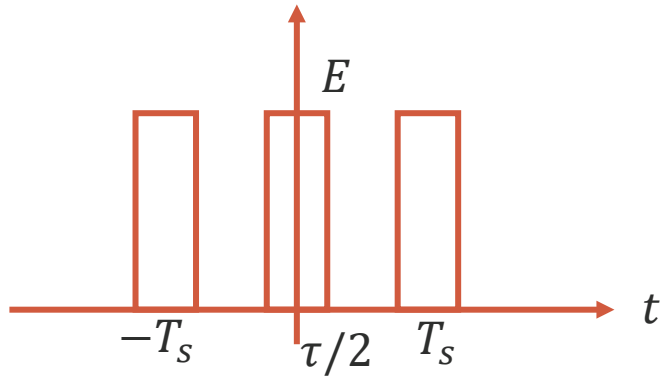
时域采样的频域分析

- 时域
- 频域



信号的时域采样

采样信号为周期矩形信号$p(t)$，求采样后信号的频谱



- 根据$x_s(t) = x(t) \cdot p(t)$，且$p(t)$的傅里叶级数为

$$P_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} p(t)e^{-jn\omega_s t}\mathrm{d}t = \frac{E\tau}{T_s}Sa\left(\frac{n\omega_s\tau}{2}\right)$$

因此

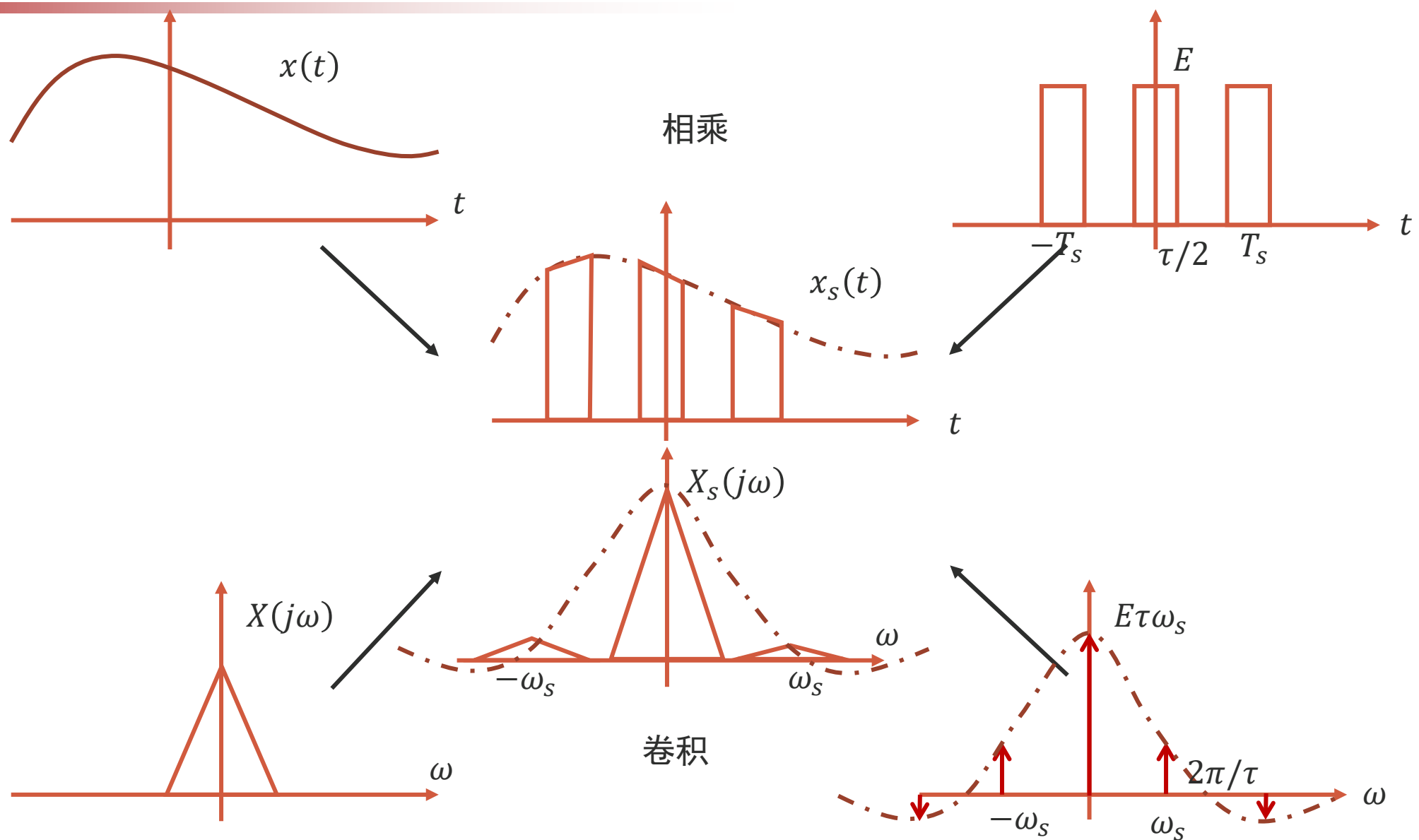
$$\mathcal{F}[p(t)] = 2\pi\sum_{n=-\infty}^{\infty} P_n\delta(\omega - n\omega_s)$$

- 所以

$$\mathcal{F}[x_s(t)] = \sum_{n=-\infty}^{\infty} P_n X\big(j(\omega - n\omega_s)\big)$$

时域采样的频域分析

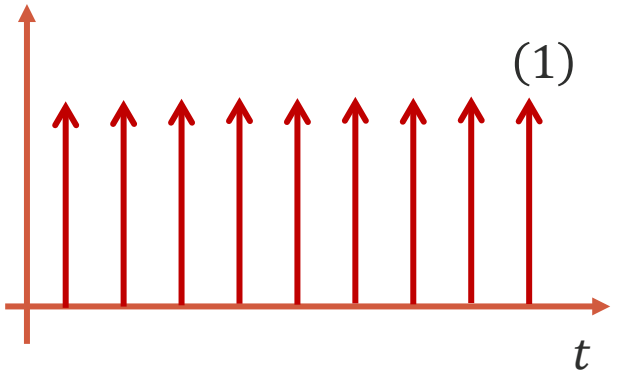
- 时域
- 频域



信号的时域采样

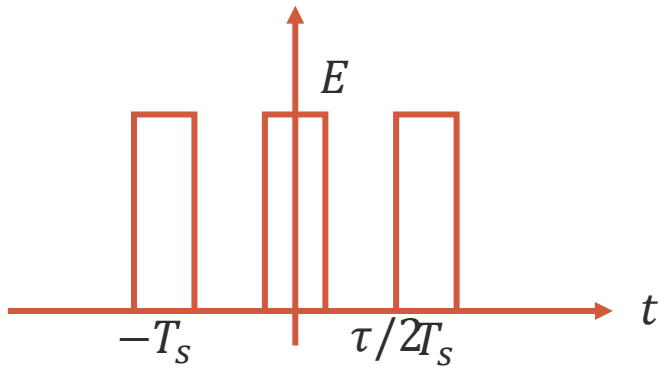
- 【理想采样】采样信号为周期冲激信号$\delta_{T_s}(t)$，采样后信号的频谱

$$\mathcal{F}[x_s(t)] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\big(j(\omega - n\omega_s)\big)$$



- 采样信号为足够窄的周期矩形信号$p(t)$，采样后信号的频谱

$$\mathcal{F}[x_s(t)] = \sum_{n=-\infty}^{\infty} P_n X\big(j(\omega - n\omega_s)\big)$$



信号的频域采样

- 频域信号$X(j\omega)$使用频域脉冲序列

$$\delta_{\omega_s}(\omega)=\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

采样得到$X_s(j\omega)$，因此

$$X_s(j\omega)=X(j\omega)\cdot\delta_{\omega_s}(\omega)$$

- 设$\mathcal{F}^{-1}[X(j\omega)]=x(t)$，$\mathcal{F}^{-1}[X_s(j\omega)]=x_s(t)$，由于

$$\mathcal{F}[\delta_{T_s}(t)]=\omega_s\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)=\omega_s\delta_{\omega_s}(\omega)$$

所以$\mathcal{F}^{-1}[\delta_{\omega_s}(\omega)]=\frac{1}{\omega_s}\delta_{T_s}(t)$

- 设

$$\mathcal{F}^{-1}[X(j\omega)]=x(t)$$
$$\mathcal{F}^{-1}[X_s(j\omega)]=x_s(t),$$
$$\mathcal{F}^{-1}[\delta_{\omega_s}(\omega)]=\frac{1}{\omega_s}\delta_{T_s}(t)$$

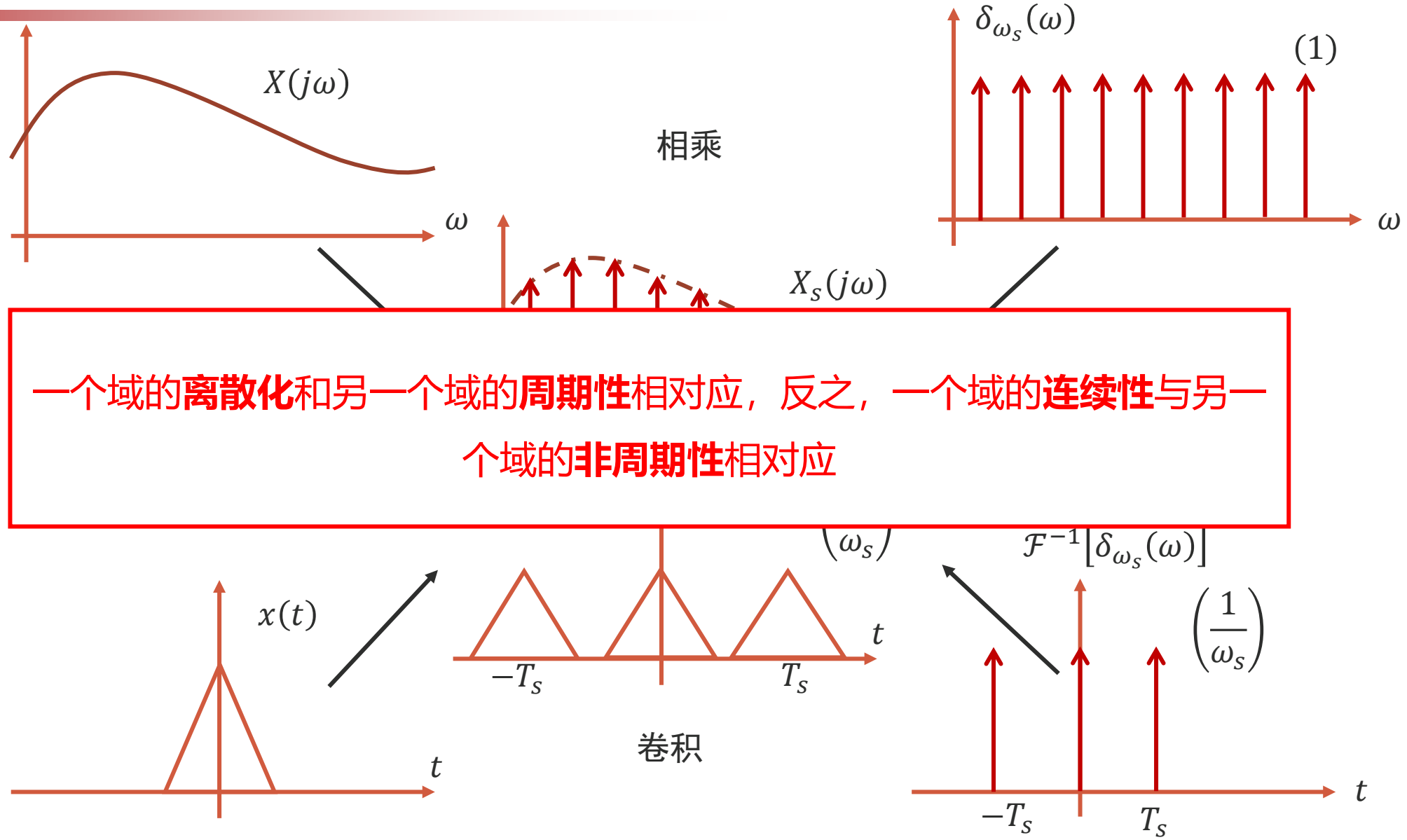
- 由卷积定理

$$x_s(t)=x(t)*\mathcal{F}^{-1}[\delta_{\omega_s}(\omega)]$$
$$=x(t)*\left[\frac{1}{\omega_s}\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right]$$
$$=\frac{1}{\omega_s}\sum_{n=-\infty}^{\infty}x(t-nT_s)$$

频域的**离散化**对应时域信号的**周期延拓**

信号的频域采样

- 频域
- 时域



一个域的**离散化**和另一个域的**周期性**相对应，反之，一个域的**连续性**与另一个域的**非周期性**相对应

时域采样定理

- 采样定理：若**连续**信号$x(t)$是一个**频带受限**信号（若$|\omega| > \omega_m$则$X(j\omega) = 0$，$\omega_m = 2\pi f_m$），$x(t)$ 的等间隔样本值$x_s(t)$，用$x_s(t)$**唯一表示**$x(t)$的条件是

$$T_s < \frac{1}{2f_m}, \text{即} \omega_s > 2\omega_m$$

$f_s = 2f_m$ 为最小采样频率，称为Nyquist Rate.

- 条件
  - 等间隔采样，频带受限信号
  - 唯一恢复条件（采样频率）
  - 恢复方法（低通滤波器）

时域采样定理的历程

Harry Nyquist

1928

美国物理学家，于1928年发表《Certain Topics in Telegraph Transmission Theory》，提出“2B independent pulse samples could be sent through a system of bandwidth B”，但没有给出如何采样重构原始信号的具体方法

Karl Küpfmüller

德国电气工程师，同年在德语论文中给出类似结论，并讨论冲激响应为sinc函数的带宽受限滤波器的情况

Vladimir Kotelnikov

1933

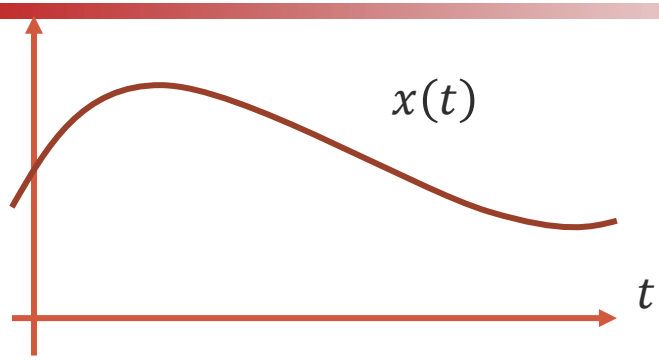
苏联物理学家，于1933年用公式较严谨地证明了采样定理。1999年Eduard Rhein Foundation授予Kotelnikov基础研究奖“for the first theoretically exact formulation of the sampling theorem”

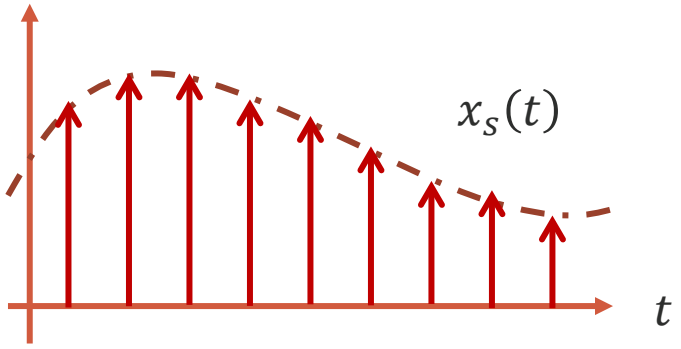
Claude Shannon

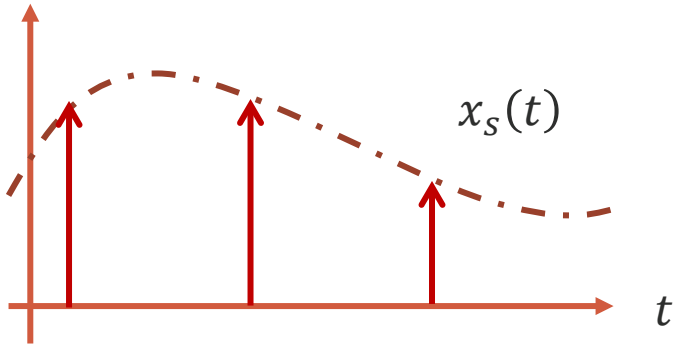
1948

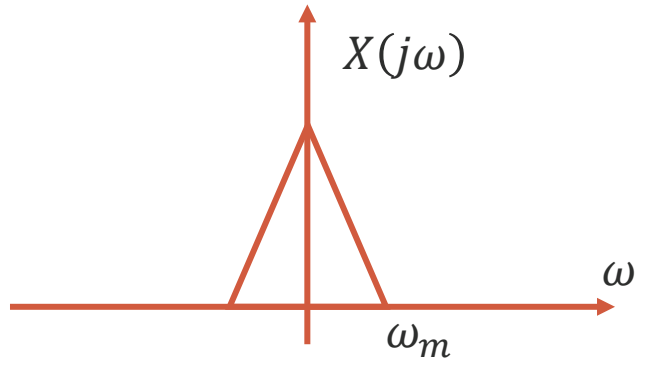
美国数学家，1948年在《Bell System Technical Journal》上连载《A Mathematical Theory of Communication》，给出从采样值恢复原始信号的公式并严格证明。采样定理为论文中的“定理13”。于1949年发表《Communication in the Presence of noise》，奠定现代信息论的基础。

信号的时域采样

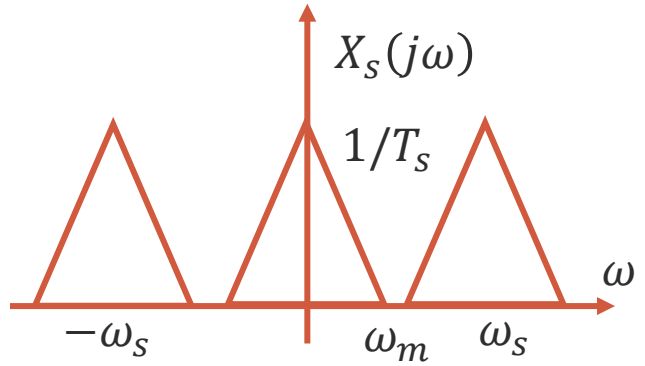




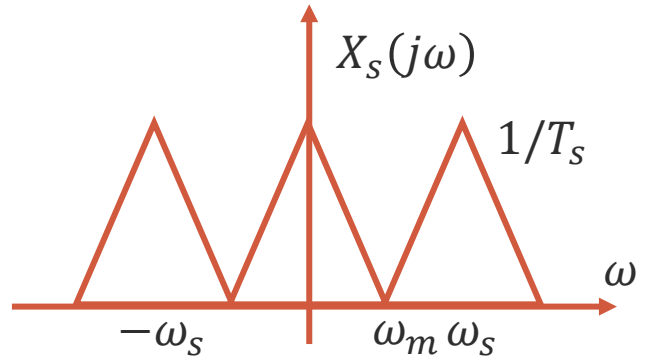




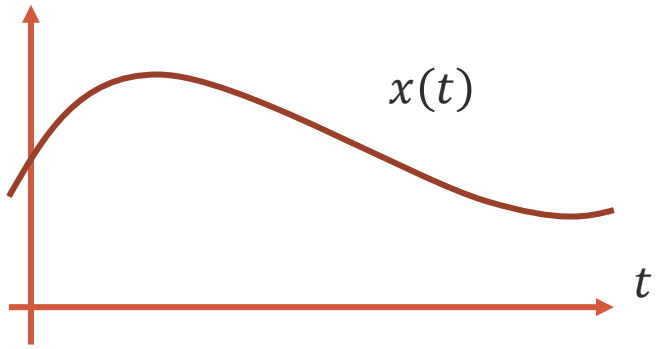
$\omega_s > 2\omega_m$

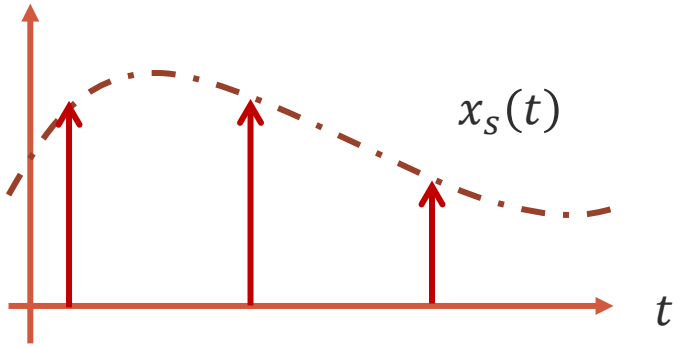


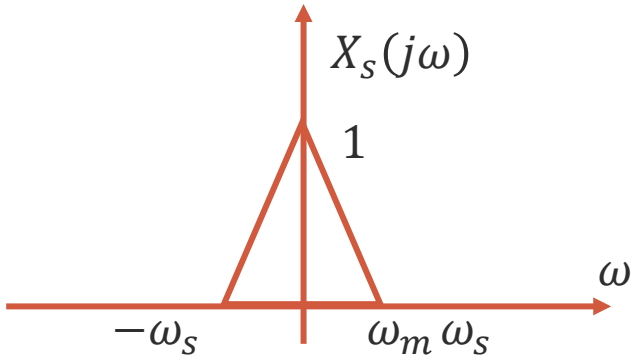
$\omega_s = 2\omega_m$



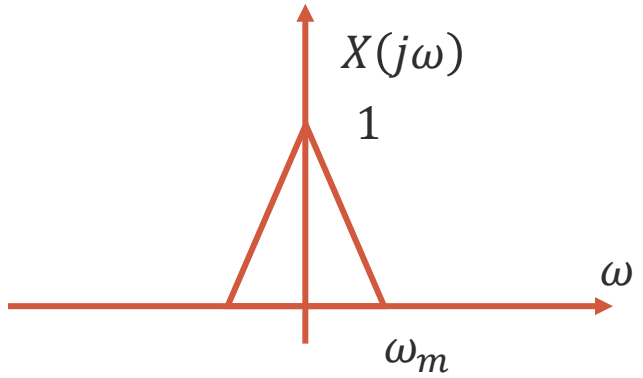
信号时域采样的恢复

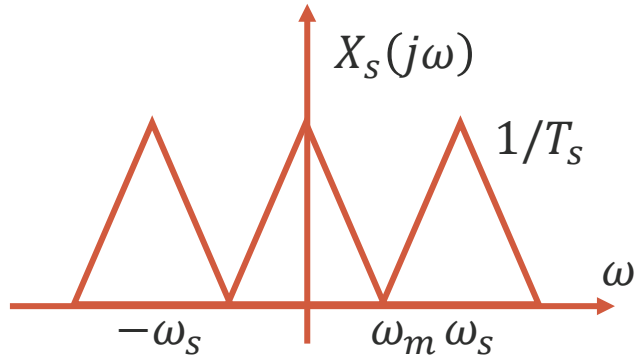


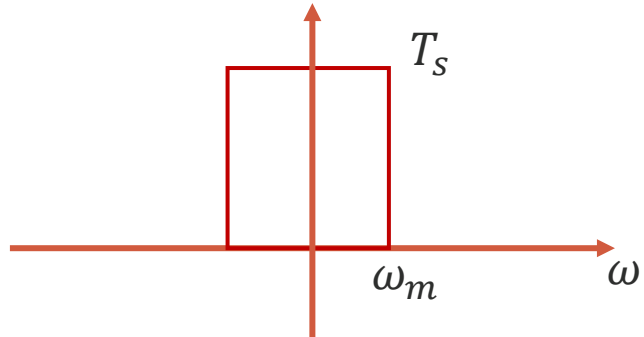




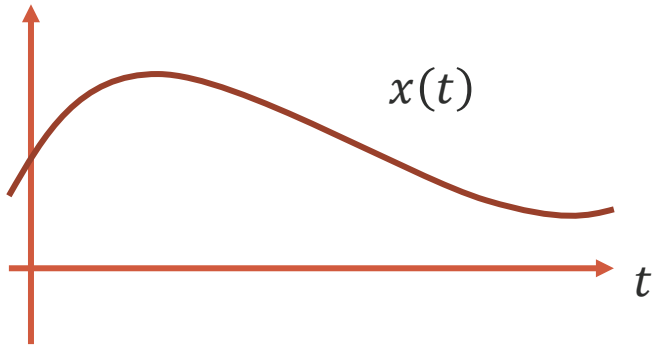
相乘

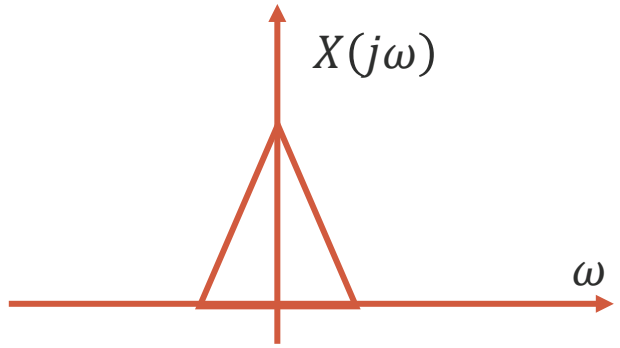


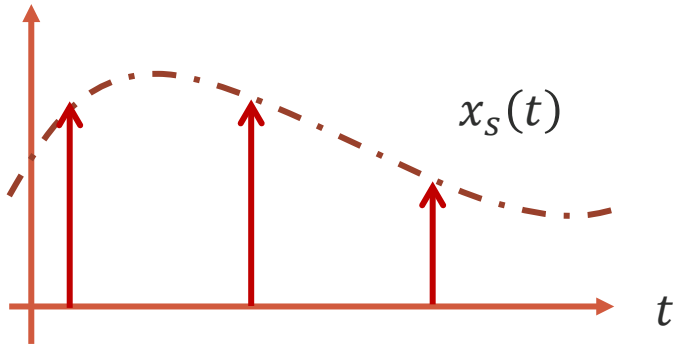




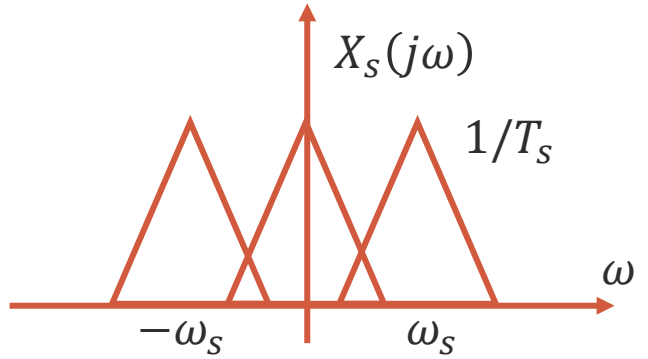
信号的时域采样







$\omega_s < 2\omega_m$



产生混叠 (aliasing)

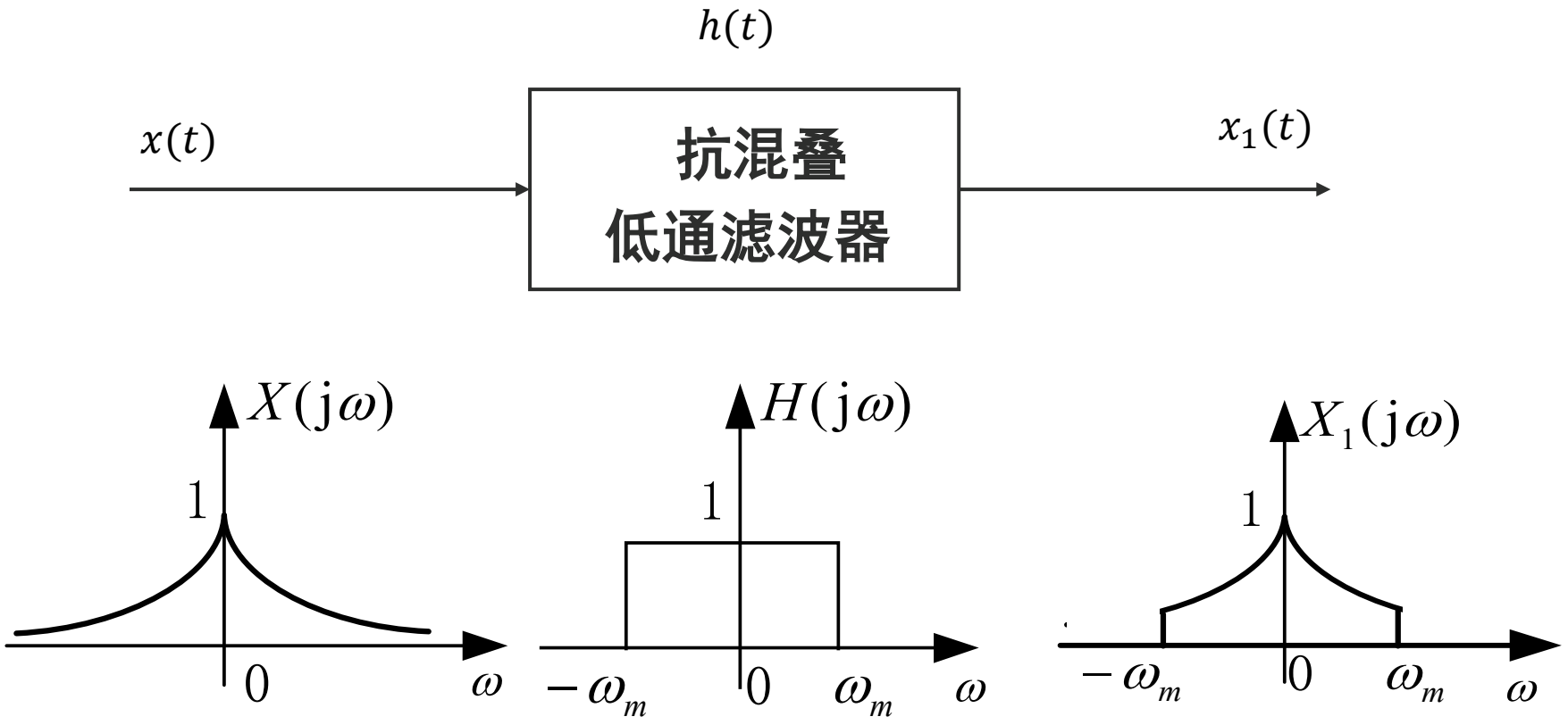
信号的时域采样

已知实信号$x(t)$的最高频率为f_m (Hz)，试计算对各信号$x(2t)$，$x(t)*x(2t)$，$x(t)\cdot x(2t)$采样不混叠的最小采样频率。

- 对信号$x(2t)$采样时，最小采样频率为$4f_m$ (Hz)
- 对$x(t)*x(2t)$采样时，最小采样频率为$2f_m$ (Hz)
- 对$x(t)\cdot x(2t)$采样时，最小采样频率为$6f_m$ (Hz)

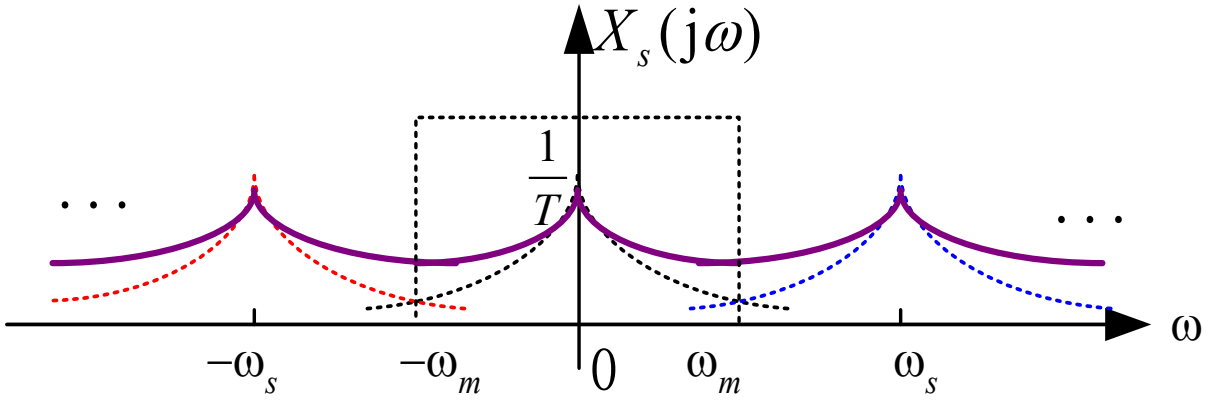
采样定理的实际应用

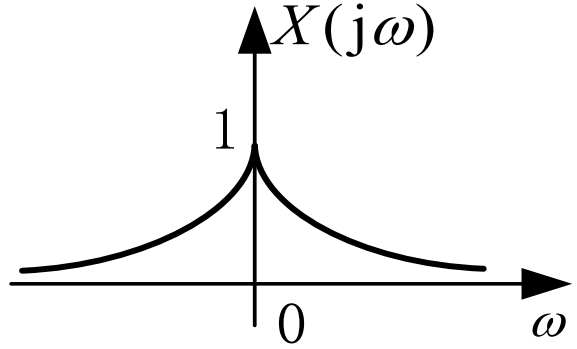
- 许多实际工程信号不满足**带限**条件

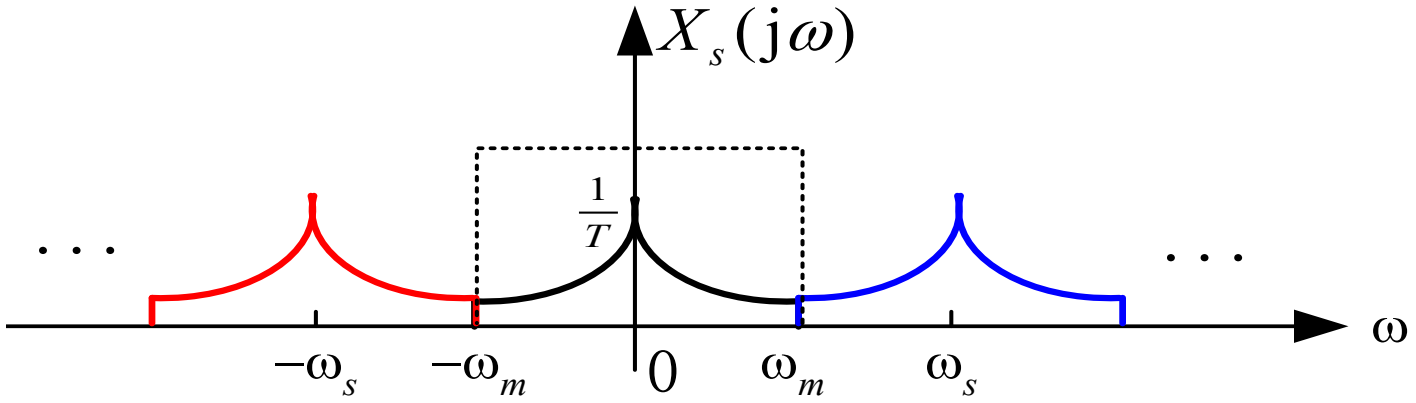


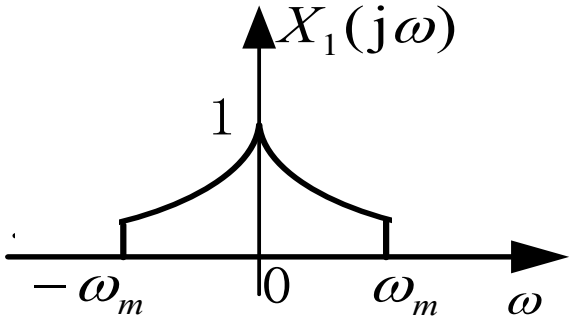
采样定理的实际应用

- 混叠误差与截断误差比较





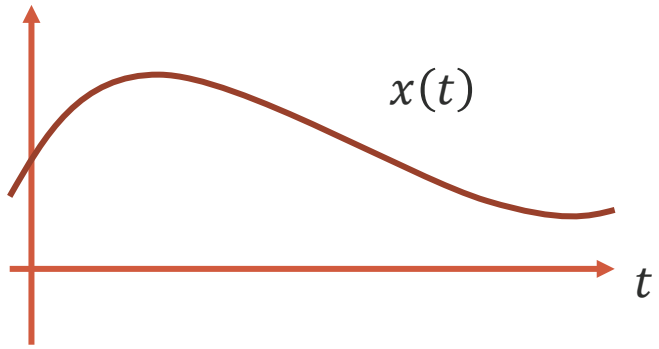


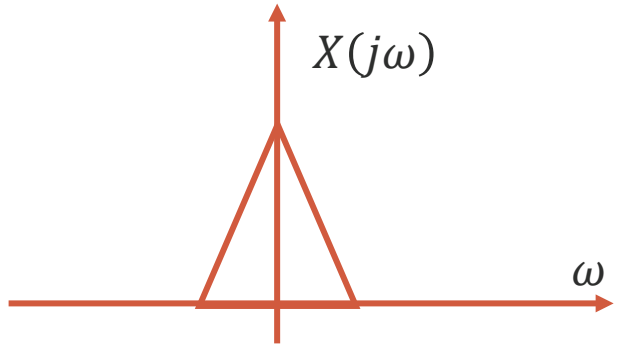


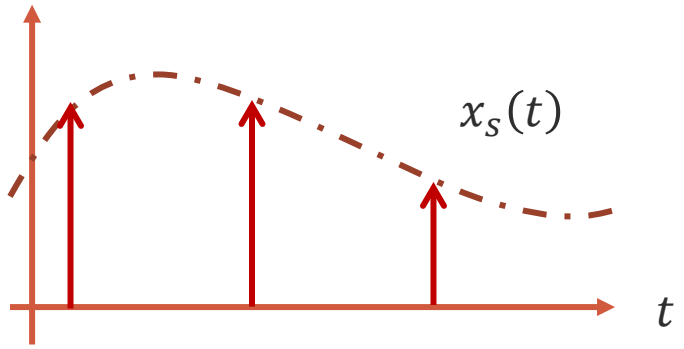
为避免频谱混叠，通常把采样频率设为$\omega_s = (3 \sim 4)\omega_m$或更高

信号的时域采样（欠采样）

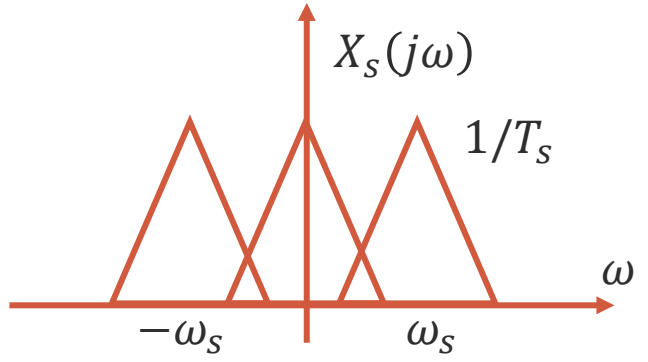
- 不满足采样定理条件时，信号重构时可能会干扰原始频谱







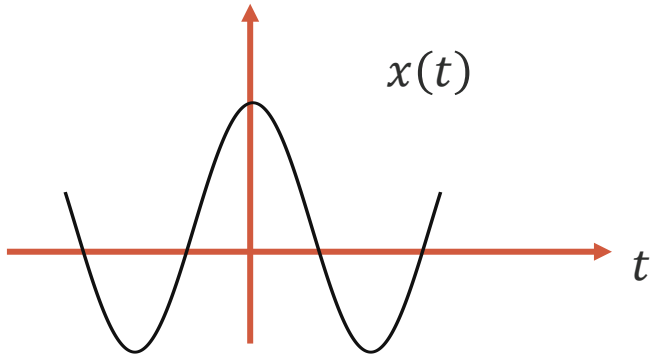
$\omega_s < 2\omega_m$

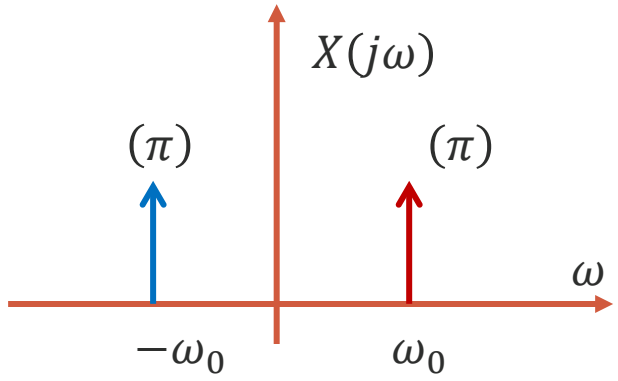


产生混叠 (aliasing)

信号的时域采样（欠采样）

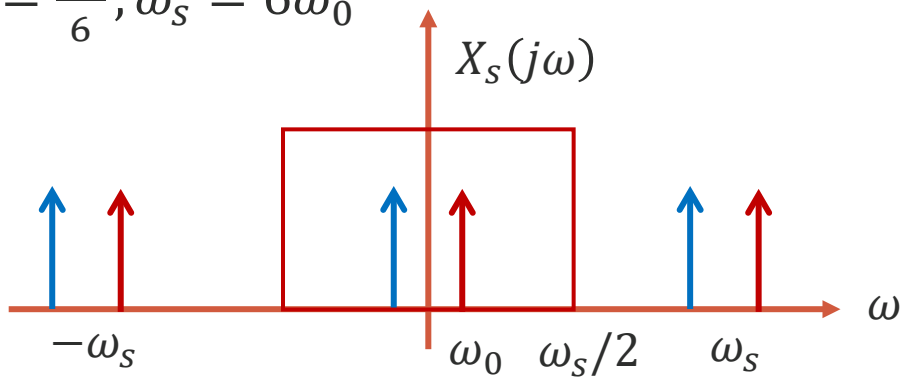
- 设$x(t)=\cos(\omega_0 t)$



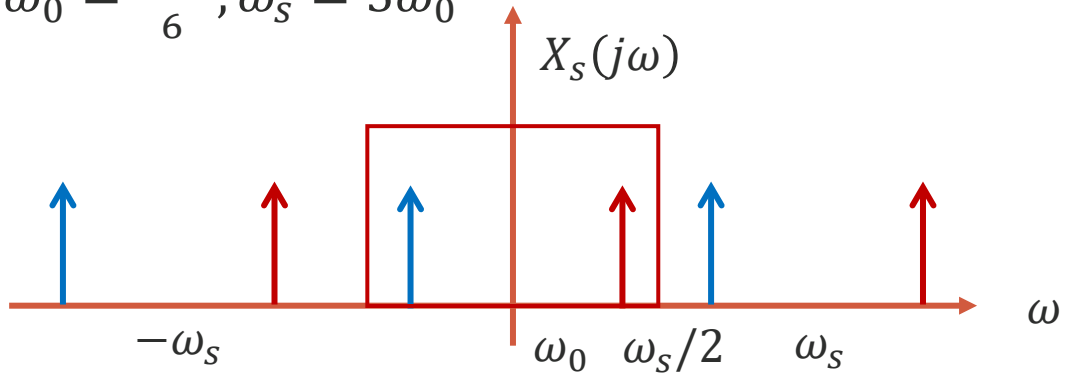


- **固定ω_s**，考察不同ω_0与ω_s的关系时的情况

$\omega_0=\frac{\omega_s}{6},\omega_s=6\omega_0$

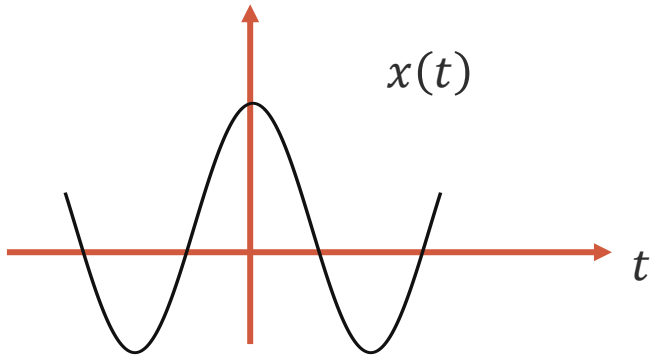


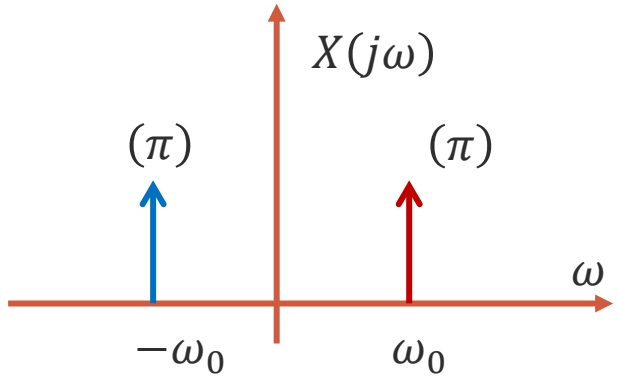
$\omega_0=\frac{2\omega_s}{6},\omega_s=3\omega_0$



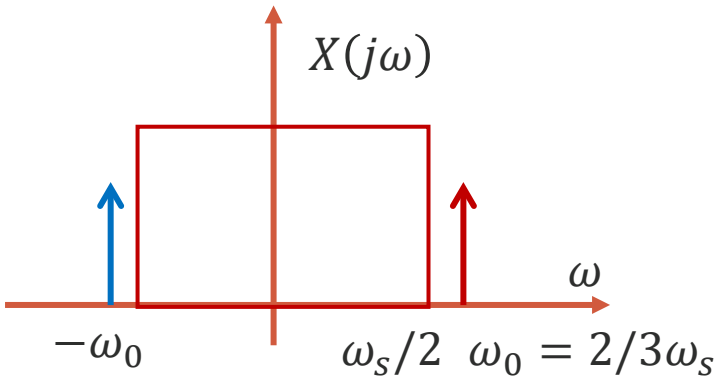
信号的时域采样（欠采样）

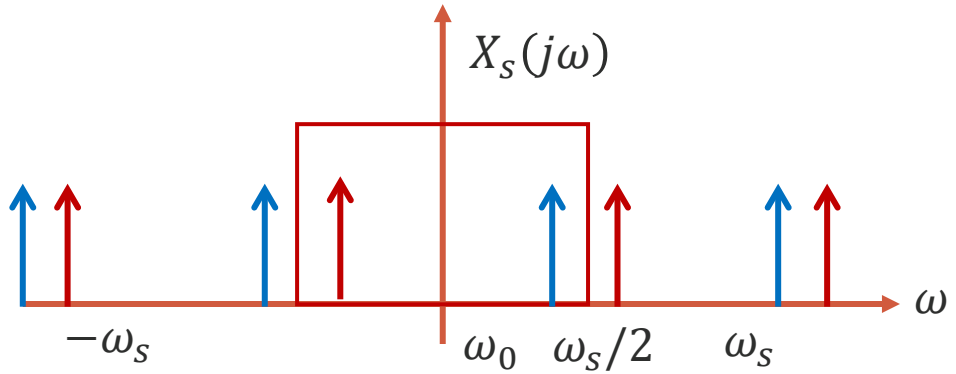
- 设$x(t) = \cos(\omega_0 t)$





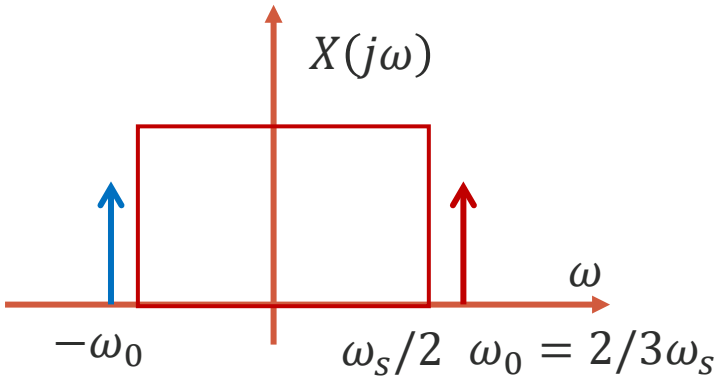
- **固定ω_s**，考察不同ω_0与ω_s的关系时的情况
  - $\omega_0 = \frac{4\omega_s}{6}, \omega_s = 1.5\omega_0$ （发生混叠）

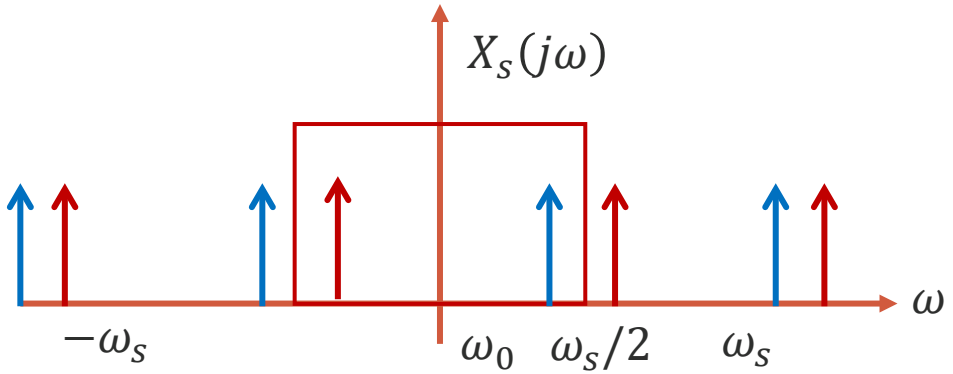




信号的时域采样（欠采样）

- 设$x(t)=\cos(\omega_0 t)$，固定ω_s，考察不同ω_0与ω_s的关系时的情况
  - $\omega_0=\frac{4\omega_s}{6},\omega_s=1.5\omega_0$ （发生混叠）

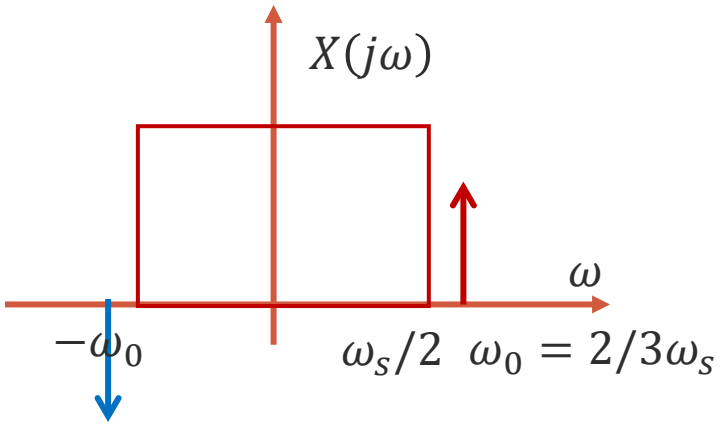


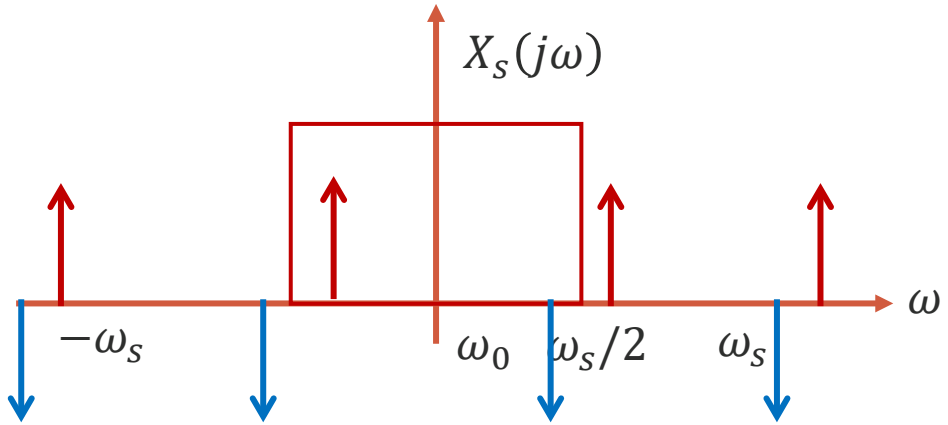


- 原始频率ω_0被混叠为低频率$\omega_s-\omega_0$
  - 当$\frac{\omega_s}{2}<\omega_0<\omega_s$时，随$\omega_0$相对$\omega_s$的增加，输出频率$\omega_s-\omega_0$会减小
  - $\omega_s=\omega_0$时，重建后的信号为常数（每个周期只采样一次）

信号的时域采样（欠采样）

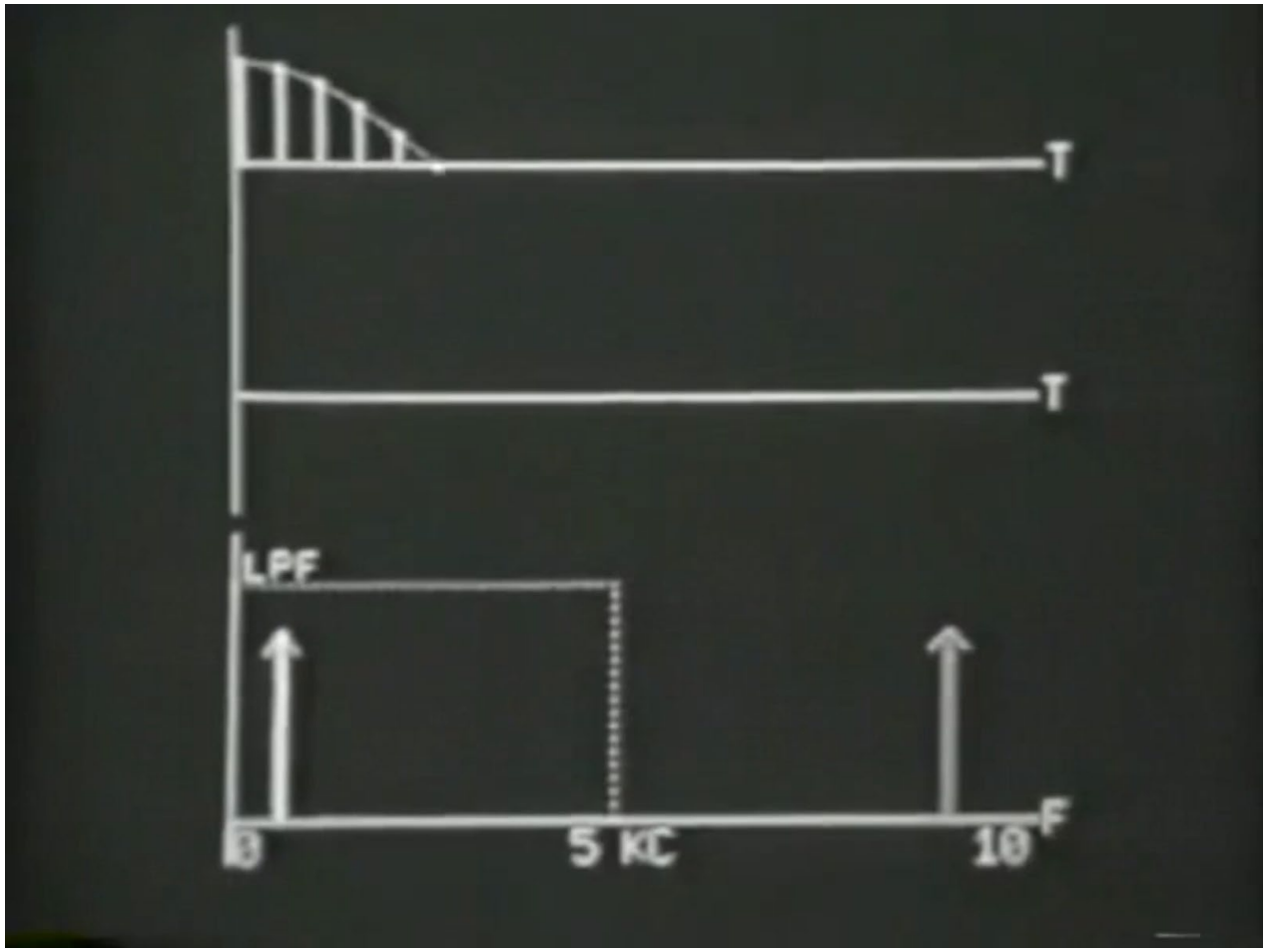
- 设$x(t)=\sin(\omega_0 t)$，固定ω_s，考察不同ω_0与ω_s的关系时的情况
  - $\omega_0=\frac{4\omega_s}{6},\omega_s=1.5\omega_0$ （发生混叠）





- 产生相位倒置（phase reversal）

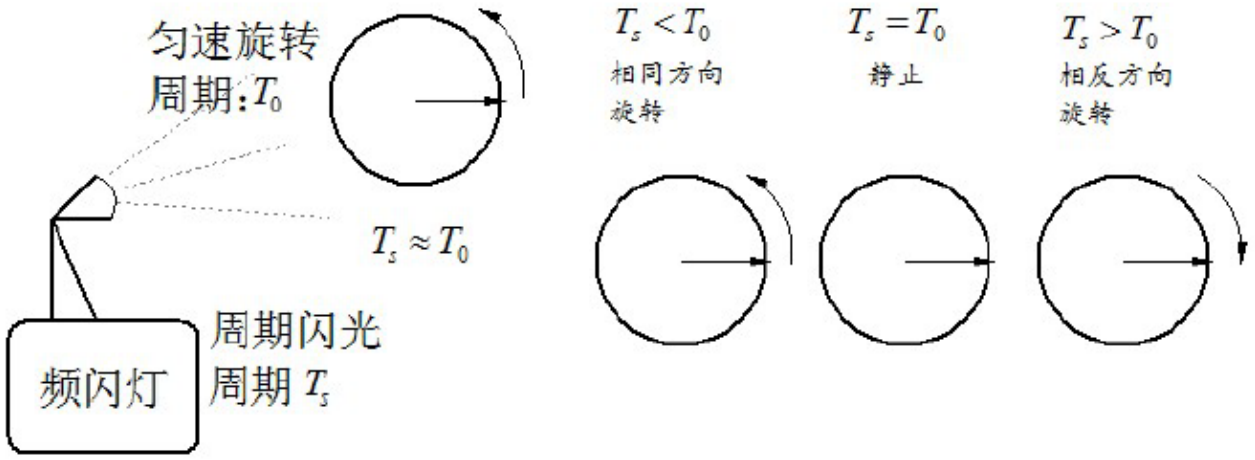
信号的时域采样（欠采样）

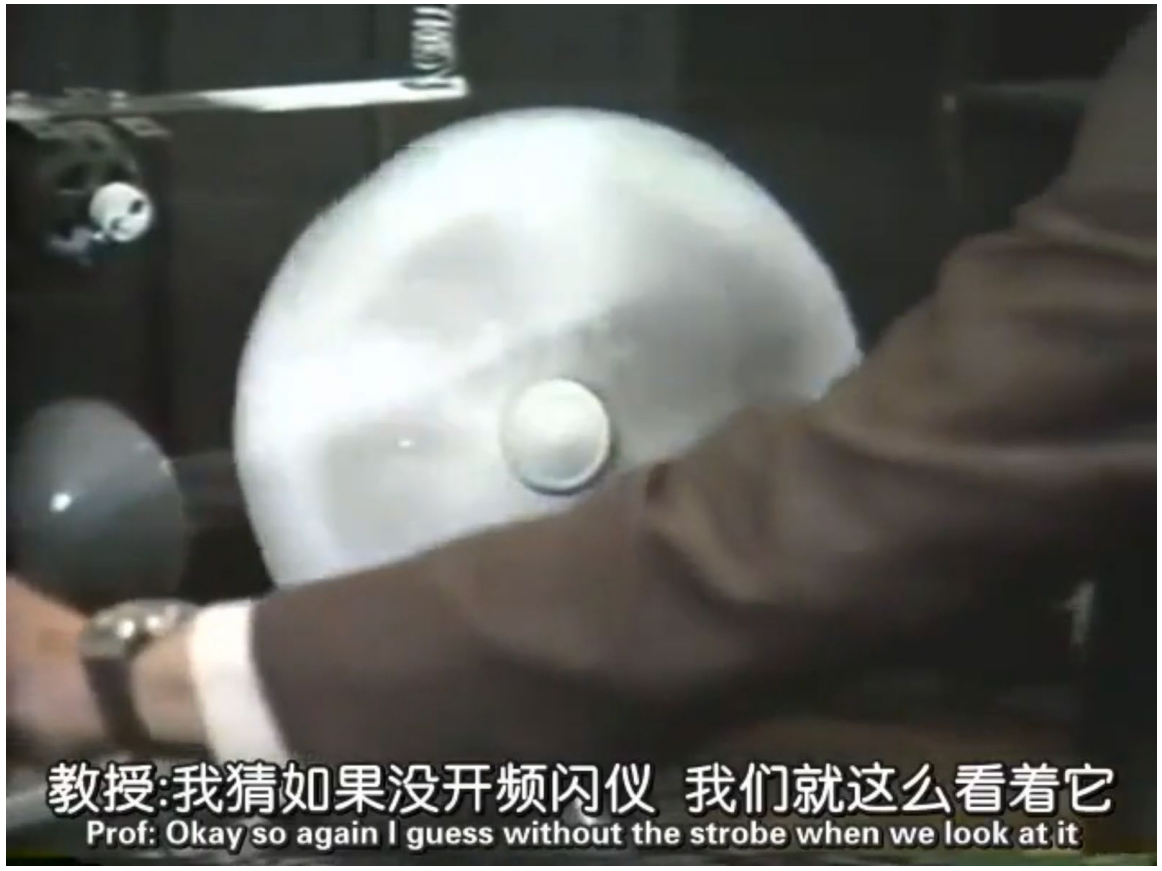


- 车轮运转

欠采样示例

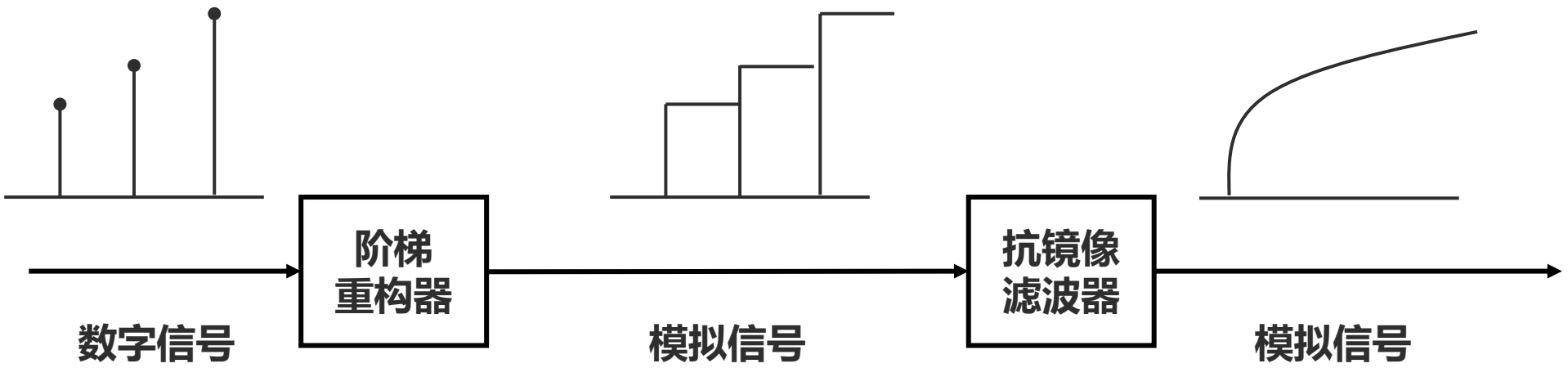
- 频率混叠对低频的影响，原本是高频信号，采样后会变成低频序列，干扰原始信号的低频频谱





从数字信号到模拟信号

- D/A的工作流程



保持信号，使当前时刻的样本值保持到下一个时刻，使信号更加光滑（滤除高频部分）

再次通过低通滤波器，进一步滤除高频分量